The solution of a system of differential equations occurring in the theory of radio-active transformations. By H. BATEMAN, M.A., Trinity College.

[*Read* 21 February 1910.]

1. It has been shown by Prof. Rutherford * that the amounts of the primary substance and the different products in a given quantity of radio-active matter vary according to the system of differential equations,

$$\left.\begin{aligned} \frac{dP}{dt} &= -\lambda_1 P \\ \frac{dQ}{dt} &= \lambda_1 P - \lambda_2 Q \\ \frac{dR}{dt} &= \lambda_2 Q - \lambda_3 R \\ \frac{dS}{dt} &= \lambda_3 R - \lambda_4 T \\ &\dots\dots\dots\dots \end{aligned}\right\} \dots\dots\dots\dots (1),$$

where $P, Q, R, S, T, \dots$ denote the number of atoms of the primary substance and successive products which are present at time t.

Prof. Rutherford has worked out the various cases in which there are only two products in addition to the primary substance, and it looks at first sight as if the results may be extended to any number of products without much labour.

Unfortunately the straightforward method is unsymmetrical and laborious, and as the results of the calculations are needed in some of the researches which are being carried on in radio-activity the author has thought it worth while to publish a simple and symmetrical method of obtaining the required formulae.

2. Let us introduce a set of auxiliary quantities $p(x)$, $q(x)$, $r(x)$, ... depending on a variable x and connected with the quantities $P(t)$, $Q(t)$, $R(t)$, ... by the equations,

$$p(x) = \int_0^\infty e^{-xt} P(t)\,dt, \quad q(x) = \int_0^\infty e^{-xt} Q(t)\,dt \dots\dots (2).$$

It is easily seen that

$$\int_0^\infty e^{-xt} \frac{dP}{dt}\,dt = -P(0) + x\int_0^\infty e^{-xt} P(t)\,dt \dots\dots\dots (3),$$

$$= -P_0 + xp,$$

* *Radio-activity*, 2nd edition, p. 331.

where p is written for $p(x)$, and P_0 for $P(0)$, the initial value of $P(t)$.

Multiplying equations (1) by e^{-xt}, and integrating from 0 to ∞ with regard to t we obtain the system of equations

$$\left.\begin{aligned} xp - P_0 &= -\lambda_1 p \\ xq - Q_0 &= \lambda_1 p - \lambda_2 q \\ xr - R_0 &= \lambda_2 q - \lambda_3 r \\ xs - S_0 &= \lambda_3 r - \lambda_4 s \end{aligned}\right\} \qquad (4),$$

from which the values of p, q, r may be obtained at once.

If $Q_0 = R_0 = S_0 = \ldots = 0$, i.e. if there is only one substance present initially, we have

$$p = \frac{P_0}{x+\lambda_1}, \quad q = \frac{\lambda_1 P_0}{(x+\lambda_1)(x+\lambda_2)}, \quad r = \frac{\lambda_1\lambda_2 P_0}{(x+\lambda_1)(x+\lambda_2)(x+\lambda_3)},$$

and for the nth product

$$\nu(x) = \frac{\lambda_1\lambda_2 \ldots \lambda_{n-1} P_0}{(x+\lambda_1)(x+\lambda_2)\ldots(x+\lambda_n)} \qquad (5).$$

Putting this into partial fractions, we have

$$\nu(x) = \frac{c_1}{x+\lambda_1} + \frac{c_2}{x+\lambda_2} + \ldots \frac{c_n}{x+\lambda_n},$$

where

$$\left.\begin{aligned} c_1 &= \frac{\lambda_1\lambda_2 \ldots \lambda_{n-1} P_0}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\ldots(\lambda_n-\lambda_1)} \\ c_2 &= \frac{\lambda_1\lambda_2 \ldots \lambda_{n-1} P_0}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\ldots(\lambda_n-\lambda_2)} \\ &\ldots\ldots \text{ etc.} \end{aligned}\right\} \qquad (6).$$

To obtain the corresponding function $N(t)$ we must solve the integral equation

$$\nu(x) = \int_0^\infty e^{-xt} N(t)\, dt.$$

Now it has been shown by Lerch* that there is only *one continuous function* $N(t)$ which will yield a given function $\nu(x)$; hence if we can find a function which satisfies this condition it will be the solution of our problem. It is clear, however, that

$$\frac{1}{x+\lambda} = \int_0^\infty e^{-xt} \cdot e^{-\lambda t}\, dt;$$

hence the above value of $\nu(x)$ is obtained by taking

$$N(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} + \ldots c_n e^{-\lambda_n t} \qquad (7),$$

where the constants have the values given by (6).

* *Acta Mathematica*, 1903, p. 339.

In the case when $Q(0)$, $R(0)$, ... are not zero, we have

$$\left.\begin{aligned} p&=\frac{P_0}{x+\lambda_1},\quad q=\frac{\lambda_1 P_0}{(x+\lambda_1)(x+\lambda_2)}+\frac{Q_0}{x+\lambda_2} \\ r&=\frac{\lambda_1\lambda_2 P_0}{(x+\lambda_1)(x+\lambda_2)(x+\lambda_3)}+\frac{\lambda_2 Q_0}{(x+\lambda_2)(x+\lambda_3)}+\frac{R_0}{(x+\lambda_3)} \\ &\text{etc.}\end{aligned}\right\}\quad\ldots(8),$$

and we may obtain the values of P, Q, R by expressing these quantities in partial fractions as before.

The complete solution for the case of a primary substance P and three products Q, R, S is

$$P=P_0e^{-\lambda_1 t},\quad Q=\frac{\lambda_1}{\lambda_2-\lambda_1}P_0e^{-\lambda_1 t}+\left(\frac{\lambda_1 P_0}{\lambda_1-\lambda_2}+Q_0\right)e^{-\lambda_2 t},$$

$$\begin{aligned} R=\frac{\lambda_1\lambda_2 P_0}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 t}&+\left[\frac{\lambda_1\lambda_2 P_0}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}+\frac{\lambda_2 Q_0}{\lambda_3-\lambda_2}\right]e^{-\lambda_2 t} \\ &+\left[\frac{\lambda_1\lambda_2 P_0}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}+\frac{\lambda_2 Q_0}{\lambda_2-\lambda_3}+R_0\right]e^{-\lambda_3 t},\end{aligned}$$

$$\begin{aligned} S=&\frac{\lambda_1\lambda_2\lambda_3 P_0}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}e^{-\lambda_1 t} \\ &+\left[\frac{\lambda_1\lambda_2\lambda_3 P_0}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}+\frac{\lambda_2\lambda_3 Q_0}{(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\right]e^{-\lambda_2 t} \\ &+\left[\frac{\lambda_1\lambda_2\lambda_3 P_0}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}+\frac{\lambda_2\lambda_3 Q_0}{(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}\right. \\ &\left.+\frac{\lambda_3 R_0}{\lambda_4-\lambda_3}\right]e^{-\lambda_3 t}+\left[\frac{\lambda_1\lambda_2\lambda_3 P_0}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\right. \\ &\left.+\frac{\lambda_2\lambda_3 Q_0}{(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}+\frac{\lambda_3 R_0}{\lambda_3-\lambda_4}+S_0\right]e^{-\lambda_4 t}.\end{aligned}$$

The solution may evidently be obtained by superposing the solutions of the cases in which the initial values of P, Q, R, S are given by

(1) $P(0)=P_0,\quad Q(0)=0,\quad R(0)=0,\quad S(0)=0.$
(2) $P(0)=0,\quad Q(0)=Q_0,\quad R(0)=0,\quad S(0)=0.$
(3) $P(0)=0,\quad Q(0)=0,\quad R(0)=R_0,\quad S(0)=0.$
(4) $P(0)=0,\quad Q(0)=0,\quad R(0)=0,\quad S(0)=S_0.$

The method is perfectly general, and the corresponding formulae for the case of $n-1$ products may be written down at once by using (6).

The general formula covers all the four cases (*Radio-activity*, pp. 331—337). For instance in Case 2 when initially there is radio-active equilibrium, we have

$$n_0=\lambda_1 P_0=\lambda_2 Q_0=\lambda_3 R_0=\lambda_4 S_0.$$

The solutions are then

$$P = \frac{n_0}{\lambda_1} e^{-\lambda_1 t}, \quad Q = \frac{n_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{n_0 \lambda_1}{\lambda_2 (\lambda_1 - \lambda_2)} e^{-\lambda_2 t},$$

$$R = \frac{\lambda_2 n_0}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 t} + \frac{\lambda_1 n_0}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2 n_0}{\lambda_3 (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t}$$

$$S = \frac{\lambda_2 \lambda_3 n_0}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_3 n_0}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2 n_0}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} e^{-\lambda_3 t} + \frac{\lambda_1 \lambda_2 \lambda_3 n_0}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} e^{-\lambda_4 t}.$$

The solution for the case of $n - 1$ products is given by

$$N = \Sigma c_r e^{-\lambda_r t},$$

where the constants c_r are obtained by expressing

$$\frac{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_{n-1}}{x (x + \lambda_1)(x + \lambda_2) \ldots (x + \lambda_n)}$$

in partial fractions.

The method by which the solution of the system of differential equations has been obtained is really of very wide application and may be employed to solve problems depending on a partial differential equation of the form

$$\frac{\partial V}{\partial t} = F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \ldots\right) V,$$

provided the initial value of V is known.

For if we put

$$u(s) = \int_0^\infty e^{-st} V(t)\, dt,$$

$$su(s) - V_0 = \int_0^\infty e^{-st} \frac{\partial V}{\partial t} dt,$$

it appears that $u(s)$ satisfies the partial differential equation

$$F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \ldots\right) U + su + V_0 = 0 \quad \ldots\ldots\ldots(9).$$

Further, if V satisfies some linear boundary condition which is independent of t the function u will generally satisfy the same boundary condition. This function (u) must be obtained from the differential equation (10) which is simpler than (9), inasmuch as it depends upon fewer independent variables.

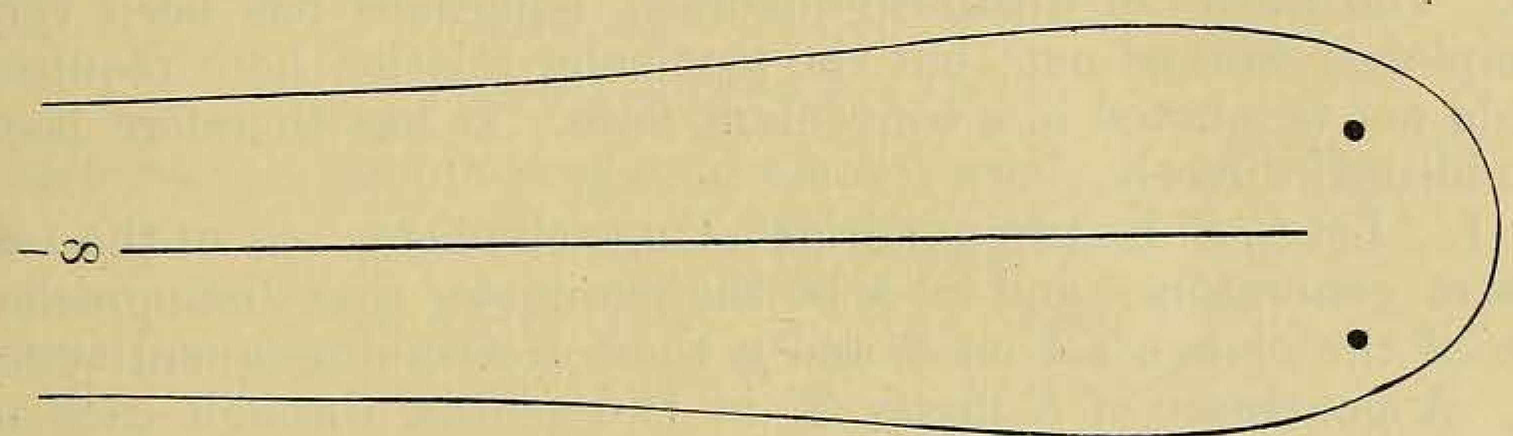
In many cases the solution of the integral equation

$$u(s) = \int_0^\infty e^{-st} V(t)\, dt$$

may be calculated by means of the inversion formula *

$$V(t) = \frac{1}{2\pi i} \int_c e^{t\zeta} u(\zeta)\, d\zeta,$$

where c is a contour which starts at $-\infty$ at a point below the real axis, surrounds all the singularities of the function $u(\zeta)$ and returns to $-\infty$ at a point above the real axis, as in the figure.



The conditions to be satisfied by $u(\zeta)$ in order that this inversion formula may be applicable have not yet been expressed in a concise form.

The formula may be used to obtain the solution of a problem in the conduction of heat when we require a solution of

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t},$$

which satisfies the boundary conditions

$$V = 0 \text{ when } x = 0 \text{ and } x = a,$$

$$V = f(x) \text{ when } t = 0.$$

The solution found in this way is identical with the one given in Carslaw's *Fourier's Series and Integrals*, p. 383.

* A particular case of this formula has been given by Pincherle, *Bologna Memoirs*, 10 (8), 1887.