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Quantum Electrodynamical Corrections to the Fine Structure of Helium*

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Order $\alpha^6 mc^2$ corrections to the fine structure splitting of the deepest triplet P state ($2^3P_{0,1,2}$) of the He^4 atom have been investigated. The investigation is based on the covariant Bethe–Salpeter equation including external potential to take account of the nuclear Coulomb field. All order $\alpha^6 mc^2$ corrections which arise from Feynman diagrams involving the exchange of one, two, and three photons, as well as radiative corrections to the electron magnetic moment have been found. The results are presented in a form suitable for computerized numerical evaluation.

1. INTRODUCTION

Analysis of the helium atom has played an important historic role in the verification of both basic quantum theory and the more detailed properties of electrons [1]. In particular, theoretical analysis and measurement of the fine structure splittings of the deepest lying triplet P state ($2^3P_{J=0,1,2}$), Fig. 1, provided important

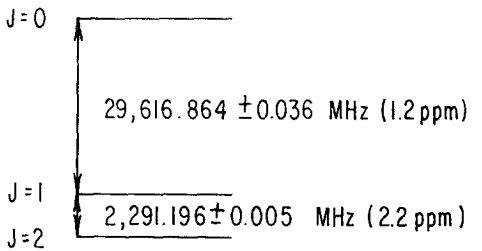

FIG. 1. Triplet P fine structure splitting.

confirmation of the developing theory of electron–electron interaction [2]. At present the theory of QED is regarded as very well established, but it is still true that analysis of the 2^3P fine structure can provide a precise test of the theory. In fact the splittings have now been measured so accurately [3] that combination with a theoretical calculation of comparable accuracy could be used to provide a

determination of the coupling constant $\alpha = e^2/\hbar c$ to one part per million or better. The most accurate value of α at present [4], $\sim$1.5 ppm, is derived from experiments on the Josephson effect, which measures $e/\hbar$, combined with the precisely known values of other fundamental constants. It would be highly desirable to have a purely QED determination of at least comparable accuracy, and especially one which is only weakly dependent on renormalization theory. Thus, while similar accuracy appears to be achievable by measurement of the anomalous magnetic moment, in order to interpret that measurement as a sensitive test of renormalization theory a value of α much less dependent on renormalization theory is required. In this paper we present the necessary theoretical analysis of the helium atom. A brief discussion of this work along with the most current theoretical values and comparison with experiment has been published elsewhere [5].

A short sketch of the simplest theory of the fine structure of the helium atom [1] will provide orientation as well as a basis for discussing higher order corrections. In lowest approximation the helium atom is described by the two electron Schroedinger equation (in atomic units)

$$\left(\frac{\bar{p}_1^{\,2}}{2m} + \frac{\bar{p}_2^{\,2}}{2m} + \frac{\alpha}{r} - \frac{Z\alpha}{r_1} - \frac{Z\alpha}{r_2}\right)\eta_0(\bar{r}_1\bar{r}_2) = W_0\eta_0(\bar{r}_1\bar{r}_2), \qquad (1.1)$$

$$\bar{r} = \bar{r}_1 - \bar{r}_2\,, \qquad \bar{p}_{1,2} = \bar{\nabla}_{1,2}/i.$$

The η_0 are eigenfunctions of the total orbital angular momentum squared $\bar{L}^2 = (\bar{L}_1 + \bar{L}_2)^2$. We now construct a "Pauli-type" wave function φ_0 from η_0 and two component spinors for each electron to be an eigenfunction of J^2, J_z, L^2, S^2. The triplet P states have $L = 1$, $S = 1$, $J = 0, 1, 2$. It is worth pointing out that the triplet states are spatially antisymmetric, and, thus, $\varphi_0(\bar{r}_1\bar{r}_2)_{r\to 0} \to 0$, considerably simplifying the higher order calculation. The spatial parts, to a fair approximation, can be thought of as antisymmetrized products of $1S$ and $2P$ hydrogenic functions.

To obtain the lowest order corrections to W_0 one calculates the expectation with φ_0 of the fine structure operator [1], obtained first almost completely from classical considerations,

$$H_{FS} = H_{\text{Coulomb}} + H_{\text{retardation}}\,, \qquad (1.2)$$

where

$$\begin{aligned} H_{\text{Coulomb}} = &-\frac{\bar{p}_1^{\,4}}{8m^3} - \frac{\bar{p}_2^{\,4}}{8m^3} + \pi Z\alpha(\delta^3(\bar{r}_1) + \delta^3(\bar{r}_2)) \\ &- 2\pi\alpha\delta^3(\bar{r}) + \frac{Z\alpha}{4m^2}\,\bar{\sigma}_1 \cdot \left(\frac{\bar{r}_1}{r_1^{\,3}} \times \bar{p}_1\right) + \frac{Z\alpha}{4m^2}\,\bar{\sigma}_2 \cdot \left(\frac{\bar{r}_2}{r_2^{\,3}} \times \bar{p}_2\right) \\ &- \frac{\alpha}{4m^2}\,\bar{\sigma}_1 \cdot \left(\frac{\bar{r}}{r^3} \times \bar{p}_1\right) + \frac{\alpha}{4m^2}\,\bar{\sigma}_2 \cdot \left(\frac{\bar{r}}{r^3} \times \bar{p}_2\right), \end{aligned} \qquad (1.3)$$

$$H_{\text{retardation}} = -\frac{\alpha}{2m^2}\frac{1}{r}(\bar{p}_1 \cdot \bar{p}_2 - \hat{r} \cdot (\hat{r} \cdot \bar{p}_1)\,\bar{p}_2)$$
$$+ \frac{\alpha}{2m^2}\bar{\sigma}_1 \cdot \left(\frac{\bar{r}}{r^3} \times \bar{p}_2\right) - \frac{\alpha}{2m^2}\bar{\sigma}_2 \cdot \left(\frac{\bar{r}}{r^3} \times \bar{p}_1\right)$$
$$+ \frac{\alpha}{4m^2}\left[-\frac{8\pi}{3}\bar{\sigma}_1 \cdot \bar{\sigma}_2\delta^3(\bar{r}) + \frac{1}{r^3}(\bar{\sigma}_1 \cdot \bar{\sigma}_2 - 3\bar{\sigma}_1 \cdot \hat{r}\bar{\sigma}_2 \cdot \hat{r})\right]. \quad (1.4)$$

The p^4 terms represent relativistic increase of mass. The spin parts of H_{Coulomb} represent the spin orbit interaction of an electron with the Coulomb fields of the nucleus and the other electron. The contact parts of H_{Coulomb} have no classical analogy. The "retardation" terms come from the retarded magnetic and spin interactions between electrons, i.e., "orbit–orbit," "spin–other orbit," "spin–spin." The spin dependent terms of H_{FS} are diagonal in J and in first order yield the 2^3P splittings accurate to a few percent.

In about 1930, following its successful application to the hydrogen spectrum, the Dirac equation was generalized by Breit [2] to the approximate, and non-covariant, "Breit equation"

$$(H_1 + H_2 + \alpha/r + B(\bar{r}_1\bar{r}_2))\,\Phi_B(\bar{r}_1\bar{r}_2) = E_B\Phi_B(\bar{r}_1\bar{r}_2),$$
$$H_{1,2} = \bar{\alpha}_{1,2} \cdot \bar{p}_{1,2} + \beta_{1,2}m - Z\alpha/r_{1,2}\,, \quad (1.5)$$
$$B(\bar{r}_1\bar{r}_2) = (\alpha/2r)(\bar{\alpha}_1 \cdot \bar{\alpha}_2 + \bar{\alpha}_1 \cdot \hat{r}\bar{\alpha}_2 \cdot \hat{r}).$$

The wave function is now a sixteen component object; $H_{1,2}$ are external potential Dirac Hamiltonians. The "Breit operator" $B(\bar{r}_1\bar{r}_2)$ may be obtained simply by replacing $\bar{v}_1$, $\bar{v}_2$ by $\bar{\alpha}_1$, $\bar{\alpha}_2$ in the classical Darwin magnetic Hamiltonian, or one can derive it from second quantized radiation theory. Breit showed that by reducing this equation to "large components," as had been done with the Dirac equation, one obtains a fine structure operator in agreement with the semiclassical treatment, including the contact terms, which give no fine structure, and an extra term which must be discarded to obtain agreement with experiment. The wrong term was demonstrated later [6] to correspond to a contribution from negative energy intermediate states, which in hole theory must be discarded. These facts indicate that one must be careful in using the Breit equation and that it cannot be completely correct. We have mentioned the Breit equation here mainly for historical completeness, and also to indicate the general form of a correct relativistic equation. Use of the fully covariant Bethe–Salpeter equation, derived from the Feynman form of QED by Bethe and Salpeter [7] and from field theory by Gell-Mann and Low [8], which uses Feynman propagators, i.e., consistent with hole theory, is regarded at present as the rigorous starting point for the treatment of two body atomic bound state problems. As shown by Salpeter [9], Sucher [10], and others, the Bethe–Salpeter equation can be cast into a form similar to the Breit equation.

In general energy level splittings of low electron number atoms are calculated as a power series expansion in α times mc^2, the electron rest energy. Thus, the Schroedinger energy W_0 is of order $\alpha^2 mc^2$, the fine structure expansion begins in order $\alpha^4 mc^2$, and the Lamb shift expansion begins in order $(\alpha^5 + \alpha^5 \log \alpha)\, mc^2$. In this paper we propose to analyze the ^3P splitting through order $\alpha^6 mc^2$ by means of the Bethe–Salpeter equation. Bethe–Salpeter bound state techniques have been used by a number of authors; for example, by Salpeter [9] to investigate nuclear recoil effects in the Lamb shift, by Karplus and Klein [11], Fulton and Karplus [12], Fulton and Martin [13] to calculate the fine structure of positronium; by Newcomb and Salpeter [14] and Arnowitt [15] to calculate nuclear recoil effects in the hyperfine structure splitting in hydrogen; and by Sucher [10, 16] and Araki [17] to calculate all terms in the expansion of an arbitrary energy level of the helium atom through orders $(\alpha^5 + \alpha^5 \log \alpha)\, mc^2$. To date two calculations of order $\alpha^6 mc^2$ using the Bethe–Salpeter equation have appeared. Sternheim has calculated nuclear recoil corrections to the hydrogen hyperfine structure [18] "ratio" $R = (8\nu_{2s}/\nu_{1s} - 1)$ (ν_{1s}, ν_{2s} are the hyperfine structure splittings of the $1s$ and $2s$ states). This calculation is mathematically similar to the present one and the terms that they have in common are in agreement. Also calculations of the $\alpha^6 \log \alpha mc^2$ coefficient in the expansion of the ground state hyperfine structure splittings in positronium [19] and muonium [20] have appeared. In addition, a calculation of the helium atom ^3P splitting to order $\alpha^6 mc^2$, essentially an extension of the above mentioned techniques of Breit, has been carried out by Kim [21]. While it is difficult to compare the results of this work with ours, certain parts do appear to be in disagreement. We should note here also that our calculation most nearly follows that of Sucher, and much of his procedure and notation has been employed.

It has been proposed by Schwartz [22] that the complete evaluation of the ^3P splitting to $\sim$1 ppm should be divisible into four well defined tasks. One first must calculate the expectation value with the Schroedinger–Pauli wave functions of the well known fine structure operators, of order $\alpha^4 mc^2$; to include the $\alpha^5 mc^2$ term one simply modifies this calculation by multiplying the electron magnetic moment by $(1 + \alpha/2\pi)$, the familiar correction first calculated by Schwinger [23]. Schwartz [22] and Schiff *et al.* [24] have extended the numerical accuracy of this calculation to about 1 ppm. The second task would be the evaluation of the $\alpha^4 mc^2$ fine structure operators in second order, again with Schroedinger–Pauli wave functions, a contribution of order $\alpha^6 mc^2$. This task has been carried out by Hambro [25]. Third, operators of intrinsic order $\alpha^6 mc^2$ must be derived from a more general theory. The final task is the evaluation of these operators, again with Schroedinger–Pauli wave functions. This has been carried out by Daley [26]. In this paper we shall justify this procedure and derive the necessary $\alpha^6 mc^2$ fine structure operator. We have not, however, derived any results of order $\alpha^7 mc^2$,

and it is conceivable that these could contribute at the 1 ppm level. Finally, there are numerically important corrections of order $\alpha^4(m/M)\,mc^2$ due to the motion of the nucleus. These have been derived from a three-body treatment analogous to the old-fashioned methods mentioned previously and reported elsewhere [27].

In outline this paper proceeds as follows. In Section 2 the Bethe–Salpeter equation is presented. Introduction of the external potential, gauge invariance, and renormalization are briefly discussed, and references on these topics are given. In Section 3 the instantaneous Coulomb interaction is separated from all other interaction effects, and the perturbation procedure for calculating these in terms of an "equal times" wave function is developed. In Section 4 the "equal times equation" with pure Coulomb interaction is transformed to a Pauli–Schroedinger type equation which reduces to the Schroedinger equation in roughest approximation. A procedure is developed for expressing the matrix element of any Dirac type matrix operator as the matrix element of a corresponding Pauli type matrix operator with the transformed wave function. We find in this section the following energy shifts with respect to the $2mc^2$ rest energy: the Schroedinger energy $W_0 \sim \alpha^2 mc^2$, the Coulomb fine structure operators of the order $\alpha^4 mc^2$, and additional fine structure operators of order $\alpha^6 mc^2$. In Section 5 the effect of single transverse photon exchange is calculated. This gives rise to the $\alpha^4 mc^2$ "retardation" fine structure as well as numerous correction terms of order $\alpha^6 mc^2$. Terms coming from the exchange of two transverse photons are treated in Section 6. In Section 7 the radiative corrections are discussed. The rigorous treatment of these has not been completed. We present instead a phenomenological treatment based on the assumption that radiative corrections to electron interaction may be taken into account by ascribing to the electron a modified charge form factor and modified static magnetic moment. Up to this point results are presented as expectation values of operators with momentum space wave functions. Since numerical evaluation is carried out in coordinate space all these results are Fourier transformed in Section 8.

It is worthwhile to mention at this point certain simplifying features of this calculation. One notes [16] that the $\alpha^5 mc^2$ energy level expression for an arbitrary level is quite complicated, but that the bulk of the terms do not contribute to the ^3P splitting. There are two reasons for this: terms not explicitly spin dependent contribute equally to the $J = 0, 1, 2$ levels and, thus, do not contribute to the splitting; and many terms are "contact" terms, that is are of the general form $\langle \varphi_0 \mid \delta^3(\bar{r}_1 - \bar{r}_2) \mid \varphi_0 \rangle$ and, thus, give zero with triplet state wave functions because, since these are spatially antisymmetric, they vanish linearly with $\bar{r} = \bar{r}_1 - \bar{r}_2$. These two factors also simplify the $\alpha^6 mc^2$ splitting calculation. In the first place one may drop spin independent operators of this order. Secondly, all operators which contribute to the $\alpha^6 mc^2$ fine structure splitting can be evaluated in nonrelativistic approximation. This is due to the fact that the fine structure occurs

only for triplet states (and, therefore, spatially antisymmetric) of nonvanishing orbital angular momentum. (This is in contrast to the situation which exists for operators of similar structure which affect the energy levels of the spatially symmetric singlet states.) These nonrelativistic approximations simplify the computation by orders of magnitude and are necessary for the simplicity of the coordinate space representation of these operators.

2. External Potential Bethe–Salpeter Equation

The starting point of the calculation is a renormalized, "mixed gauge," Bethe–Salpeter with external potential included to account for the Coulomb potential of the nucleus, which is regarded as fixed.[1] Defining the Feynman two body kernel as

$$K(x_3x_4x_1x_2) = \langle \Psi_0 \mid T[\psi(x_3)\,\psi(x_4)\,\bar{\psi}(x_1)\,\bar{\psi}(x_2)] \mid \Psi_0 \rangle \tag{2.1}$$

i.e., the vacuum expectation value of the time ordered product of the indicated fermion field operators, one finds following Bethe and Salpeter [7] and Gell-Mann and Low [8] that it satisfies the integral equation

$$K(x_3x_4x_1x_2) = K'_{1V}(x_3x_1)\,K'_{2V}(x_4x_2) - i\int dx_5\,dx_6\,dx_7\,dx_8$$
$$\times\, K'_{1V}(x_3x_5)\,K'_{2V}(x_4x_6)\,\bar{G}(x_5x_6x_7x_8)\,K(x_7x_8x_1x_2), \tag{2.2}$$

where $\bar{G}$ is the "irreducible interaction function," which will be described shortly, and

$$K'_{1V}(x_3x_1) = \langle \Psi_0 \mid T[\psi(x_3)\,\bar{\psi}(x_1)] \mid \Psi_0 \rangle, \tag{2.3}$$

with a corresponding definition for K'_{2V}. It will be noted that the discussion of Refs. [7] and [8] has been extended in the following manner: (a) all radiative corrections to the single particle propagators are included in K'_{1V}, K'_{2V}, (b) the effect of the external potential has also been included in K'_{1V}, K'_{2V} by virtue of the fact that the Hamiltonian determining ψ includes interaction with the external field. Justification for these extensions may be found in the work of Furry [28] and Mathews and Salam [29].

[1] Sections 2 and 3 through Eq. (3.46) consist entirely of material taken from Ref. [10]. We reproduce it here because it is essential to understanding the rest of this paper and not readily available elsewhere. We also hope that it may be of value to others to call attention in this way to a very elegant and useful method.

The wave function corresponding to a two electron state is defined as

$$\psi(x_3x_4) = \langle \Psi_B \mid \bar{\psi}(x_3)\, \bar{\psi}(x_4) \mid \Psi_0 \rangle, \tag{2.4}$$

and for Ψ_B a bound state is found to satisfy

$$\psi(x_3x_4) = -i \int dx_5\, dx_6\, dx_7\, dx_8\, K'_{1V}(x_3x_5)\, K'_{2V}(x_4x_6)\, \bar{G}(x_5x_6x_7x_8)\, \psi(x_7x_8), \tag{2.5}$$

the equation we use as a starting point for this work.

With the neglect of radiative corrections K'_{1V}, K'_{2V} become the usual external potential propagators defined with respect to the external potential Dirac equation [30], i.e.,

$$K_{1V}(x_3x_1) = \begin{cases} \sum\limits_{E_n>0} \Phi_n(\bar{x}_3)\, \bar{\Phi}_n(\bar{x}_1)\, e^{-iE_n(t_3-t_1)} & t_3 > t_1\,, \\ -\sum\limits_{E_n<0} \Phi_n(\bar{x}_3)\, \bar{\Phi}_n(\bar{x}_1)\, e^{-iE_n(t_3-t_1)} & t_3 < t_1\,, \end{cases} \tag{2.6}$$

where

$$E_n\Phi_n(\bar{x}_1) = \left[\bar{\alpha}_1 \cdot \frac{\bar{\nabla}_1}{i} + \beta_1 m + V(\bar{x}_1)\right] \Phi_n(\bar{x}_1) \tag{2.7}$$

is the coordinate space Dirac equation, and for this problem V will be taken to be a Coulomb potential

$$V(\bar{x}_1) = -Ze^2/|\, \bar{x}_1\, |. \tag{2.8}$$

K_{1V} satisfies the equation

$$[i(\partial/\partial t_1) - H_1]\, K_{1V}(x_1x_3) = i\beta_1\delta^4(x_1 - x_3). \tag{2.9}$$

Corresponding definitions are always taken to hold for particle 2 operators.

The complete single electron propagator is given by

$$K'_{1V} = K_{1V} + K_{1V}\Sigma_{1V}K_{1V} + K_{1V}\Sigma_{1V}K_{1V}\Sigma_{1V}K_{1V} + \cdots, \tag{2.10}$$

where Σ_{1V} is the sum of all "proper one electron self energy parts"; the mass renormalization, second order and fourth order contributions to Σ_{1V} are represented by the diagrams of Fig. 2(a,b,c). The double line for an electron propagator denotes propagation in the external field. We remark that like K_{1V}, Σ_{1V} is an integral operator whose kernel is a c number matrix function of two space-time variables, i.e., a "two point function."

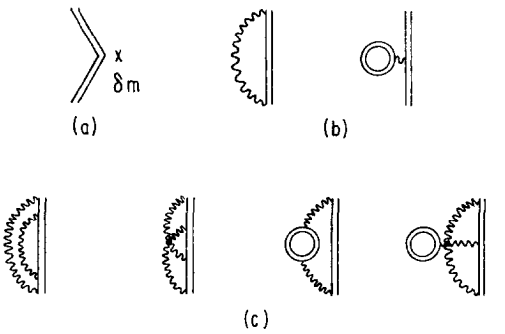

FIG. 2. Proper one electron self-energy parts. The double line indicates propogation in the external field.

The irreducible interaction operator $\bar{G}$ is defined as the sum of operators $\bar{G}^{(j)}$ associated with the irreducible Feynman diagram (j) in which the two main electron lines are in some way connected (a graph is called reducible if it can be split into two simpler graphs by drawing a line which cuts no photon lines and each of the main electron lines only once). Examples of diagrams contributing to $\bar{G}$ are illustrated in Fig. 3. The basic interaction operator $\bar{G}_0$, corresponding to the single photon exchange graph of Fig. 3(a), is given by

$$\bar{G}_0(x_1x_2x_3x_4) = e^2\gamma_{\mu 1}\gamma_{\mu 2}\delta_+[(x_1 - x_2)^2]\,\delta^4(x_1 - x_3)\,\delta^4(x_2 - x_4) \qquad (2.11)$$

with the δ_+ function representing propagation of the photon [31].

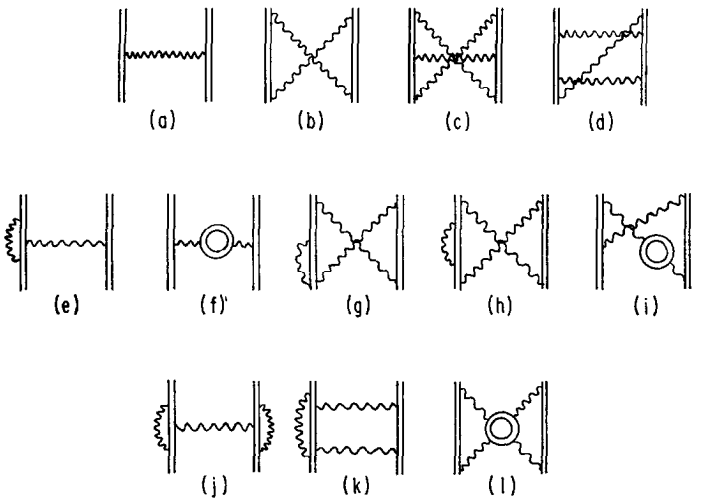

FIG. 3. Examples of interaction diagrams.

Diagrams containing electron or photon self energy parts and vertex corrections give rise to divergent expressions and require renormalization. As discussed by Mathews and Salam [29], the Bethe–Salpeter equation may be renormalized by replacing the electron bare mass and charge by their physical values and replacing K'_{1V}, K'_{2V}, $\gamma_{\mu 1}$, $\gamma_{\mu 2}$, and the photon propagator D_F by K^*_{1V}, K^*_{2V}, $\Gamma^*_{\mu 1}$, $\Gamma^*_{\mu 2}$, and $D_F{}^*$, respectively, where

$$K^*_{1V} = K_{1V} + K_{1V}\Sigma^*_{1V}K_{1V} + \cdots, \tag{2.12}$$

$$\Gamma^*_{\mu 1} = \gamma_{\mu 1} + \Lambda^*_{\mu 1}, \tag{2.13}$$

$$D_F{}^* = D_F + D_F\Pi^* D_F + \cdots, \tag{2.14}$$

and Σ^*_{1V}, Π^*, $\Lambda^*_{\mu 1}$ are the renormalized electron self energy, photon self energy and vertex correction operators. For the explicit calculation of these to second order in e, excluding external potentials, see Karplus and Kroll [32] and Jauch and Rohrlich [33].

We wish to make some further manipulation and for this it is convenient to write Eq. (2.5) in the operator form

$$\psi = -iK'_{1V}K'_{2V}\bar{G}\psi. \tag{2.15}$$

Operating with $(K'_{1V})^{-1}(K'_{2V})^{-1}$ and noting that Eq. (2.10) implies that

$$(K'_{1V})^{-1} = K_{1V}^{-1} - \Sigma_{1V}, \tag{2.16}$$

we find

$$K_{1V}^{-1}K_{2V}^{-1}\psi = -i[\bar{G} + K_{1V}^{-1}\Sigma_{2V} + K_{2V}^{-1}\Sigma_{1V} - \Sigma_{1V}\Sigma_{2V}]\,\psi. \tag{2.17}$$

Radiative corrections without photon exchange are thus treated on the same footing as interaction diagrams. Reference to Eq. (2.9) shows that the coordinate space representation of K_{1V}^{-1} is

$$K_{1V}^{-1} = -i\beta_1(i(\partial/\partial t_1) - H_1), \tag{2.18}$$

and thus multiplying Eq. (2.17) by $-\beta_1\beta_2$ we find finally

$$\begin{aligned}(i(\partial/\partial t_1) - H_1)(i(\partial/\partial t_2) - H_2)\,\psi &= i\beta_1\beta_2[\bar{G} + K_{1V}^{-1}\Sigma_{2V} + K_{2V}^{-1}\Sigma_{1V} - \Sigma_{1V}\Sigma_{2V}]\,\psi \\ &\equiv i\beta_1\beta_2 G'\psi.\end{aligned} \tag{2.19}$$

Since the external field is time independent in the frame of reference in which we are working, we look for solutions of the form

$$\psi(x_1x_2) = e^{-iET}\psi(\bar{x}_1\bar{x}_2 t), \tag{2.20}$$

where

$$T = \tfrac{1}{2}(t_1 + t_2), \qquad t = t_1 - t_2 \tag{2.21}$$

with t the so-called relative time variable. Such a solution corresponds to a state of total energy E. Substituting into Eq. (2.19) and introducing the variables

$$T' = \tfrac{1}{2}(t_3 + t_4), \qquad t' = t_3 - t_4\,, \tag{2.22}$$

we find

$$\begin{aligned}(E/2 + i(\partial/\partial t) - H_1)(E/2 - i(\partial/\partial t) - H_2)\,\psi \\ = i\beta_1\beta_2 \int G'(\bar{x}_1\bar{x}_2 t \bar{x}_3\bar{x}_4 t')\,\psi(\bar{x}_3\bar{x}_4 t')\,d\bar{x}_3\,d\bar{x}_4\,dt',\end{aligned} \tag{2.23}$$

where

$$G'(\bar{x}_1\bar{x}_2 t\bar{x}_3\bar{x}_4 t') = \int G'(x_1x_2x_3x_4)\,e^{iE(T-T')}\,dT' \tag{2.24}$$

is independent of T since, if the external potential is time independent, it may be shown that $G'(x_1x_2x_3x_4)$ depends on T and T' only through the combination $T - T'$.

For much of the following it is convenient to work in momentum space. Defining the momentum space wave function by

$$\psi(\bar{x}_1\bar{x}_2 t) = 1/(2\pi)^{7/2} \int e^{i(\bar{p}_1\cdot\bar{x}_1 + \bar{p}_2\cdot\bar{x}_2 - \epsilon t)}\psi(\bar{p}_1\bar{p}_2\epsilon)\,d\bar{p}_1\,d\bar{p}_2\,d\epsilon \tag{2.25}$$

we find that $\psi(\bar{p}_1\bar{p}_2\epsilon)$ satisfies

$$\mathscr{F}\psi = \mathscr{G}\psi, \tag{2.26}$$

where

$$\mathscr{F} = \mathscr{F}_1\mathscr{F}_2\,, \tag{2.27}$$

$$\mathscr{F}_1 = E/2 + \epsilon - H_1\,, \tag{2.28a}$$

$$\mathscr{F}_2 = E/2 - \epsilon - H_2\,, \tag{2.28b}$$

$$H_1 = \bar{\alpha}_1\cdot\bar{p}_1 + \beta_1 m + V_1 \equiv H_{01} + V_1\,, \tag{2.29a}$$

$$H_2 = \bar{\alpha}_2\cdot\bar{p}_2 + \beta_2 m + V_2 \equiv H_{02} + V_2\,. \tag{2.29b}$$

H_1, H_2 are the usual momentum space Dirac Hamiltonians. V_1, V_2, $\mathscr{G}$ and

S_1, S_2, the external potential electron propagators, are now integral operators defined by

$$V_1 f(\bar{p}_1 \bar{p}_2 \epsilon) = -\frac{Z\alpha}{2\pi^2} \int \frac{d\bar{k}}{\bar{k}^2} f(\bar{p}_1 - \bar{k}, \bar{p}_2, \epsilon), \tag{2.30a}$$

$$V_2 f(\bar{p}_1 \bar{p}_2 \epsilon) = -\frac{Z\alpha}{2\pi^2} \int \frac{d\bar{k}}{\bar{k}^2} f(\bar{p}_1, \bar{p}_2 + \bar{k}, \epsilon), \tag{2.30b}$$

$$\mathcal{G} f(\bar{p}_1 \bar{p}_2 \epsilon) = \int \mathcal{G}(\bar{p}_1 \bar{p}_2 \epsilon \bar{p}_1' \bar{p}_2' \epsilon') f(\bar{p}_1' \bar{p}_2' \epsilon') \, d\bar{p}_1' \, d\bar{p}_2' \, d\epsilon', \tag{2.31}$$

$$S(p_1) = \mathcal{F}_1^{-1} = (E/2 + \epsilon - H(\bar{p}_1))^{-1} = \frac{1}{\not{p}_1 - m - \not{V}_1} \beta_1 \, ^2 \tag{2.32a}$$

$$S(p_2) = \mathcal{F}_2^{-1} = (E/2 - \epsilon - H(\bar{p}_2))^{-1} = \frac{1}{\not{p}_2 - m - \not{V}_2} \beta_2, \tag{2.32b}$$

$$p_1 = (E/2 + \epsilon, \bar{p}_1), \tag{2.33a}$$

$$p_2 = (E/2 - \epsilon, \bar{p}_2). \tag{2.33b}$$

The momentum space form of the Dirac equation, Eq. (2.7), is

$$E_n \psi_n(\bar{p}_1) = (\bar{\alpha}_1 \cdot \bar{p}_1 + \beta_1 m + V_1) \psi_n(\bar{p}_1); \tag{2.34}$$

multiplying by β_1 gives the familiar alternate form

$$(\not{p}_1 - m - \not{V}_1) \psi_n(\bar{p}_1) = 0. \tag{2.35}$$

To carry out renormalization it is necessary to work in covariant gauge; however, use of Coulomb gauge in some diagrams is convenient for the development of a rapidly convergent perturbation treatment. Sucher [10, 34] has given a general discussion of the possibilities for mixing gauges in the calculation of bound state energy levels. We note that the most general diagram, due to fermion conservation, contains two unbroken electron lines entering and leaving, plus an arbitrary number of electron closed loops, all of which we shall call "electron lines." The general rule that must be followed is that the members of the set of photon lines connecting a given pair of electron lines or a line with itself must be expressed in the same gauge but we are free to choose different gauges for different sets. Furthermore, the choice of gauges must be kept the same for different orders; that is diagrams differing only by the numbers of photons in the various sets must

[2] In equations like (2.32a) the argument p_1 in $S(p_1)$ is to be regarded as an operator. For a fuller notational discussion see Appendix II.

still use the same gauge for corresponding sets. For example, if the photon line in the diagram of Fig. 4(a) is expressed in Coulomb gauge the photon lines in Fig. 4(b,c,d) must also be expressed in Coulomb gauge. For the purpose of this

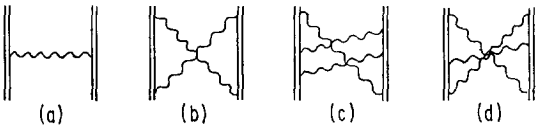

FIG. 4. Diagrams illustrating gauge requirements.

calculation it is, thus, permissible to use Coulomb gauge for the photons connecting the two electron lines and a covariant gauge for photon lines beginning and ending on the same electron, i.e., vertex and self-energy diagrams, which require renormalization.

Thus, for interaction lines we make the substitution for the momentum space interaction function

$$G_0(k) = e^2(\gamma_{\mu 1}\gamma_{\mu 2}/k_\mu{}^2) \to -e^2[\beta_1\beta_2/\bar{k}^2 + (\beta_1\alpha_1{}^i)(\beta_2\alpha_2{}^i)/k_\mu{}^2], \qquad k = (\omega, \bar{k}), \qquad k_\mu{}^2 = \omega^2 - \bar{k}^2. \tag{2.36}$$

In these equations $i = 1, 2$ denotes components of $\bar{\alpha}_1$, $\bar{\alpha}_2$ in two orthogonal directions both orthogonal to $\bar{k}$, and the expressions are summed on i. The first term represents the instantaneous Coulomb interaction and the second the retarded transverse interaction.

Finally, we make the separation

$$\mathcal{G} = \mathcal{G}_I + \mathcal{G}^{(\mathrm{rad})} \tag{2.37}$$

corresponding to the separation of the total interaction $\mathcal{G}$ into pure exchange diagrams, e.g. Fig. 3(a,b,c,d), and radiative diagrams. The simplest diagrams, providing the largest contribution to the energy levels, are those describing the exchange of a single Coulomb or transverse photon:

$$\mathcal{G}_c\psi = \frac{\alpha}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^2}\,\frac{d\omega}{-2\pi i}\,\psi(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}, \epsilon - \omega) \tag{2.38}$$

$$\mathcal{G}_T\psi = \frac{\alpha}{2\pi^2}\int \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta}\,\frac{d\omega}{-2\pi i}\,\psi(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}, \epsilon - \omega). \tag{2.39}$$

It turns out that for our fine structure calculation we may approximate $\mathcal{G}_I$ by

$$\mathcal{G}_I \to \mathcal{G}_c + \mathcal{G}_T + \mathcal{G}_{T\times c} + \mathcal{G}_{T\times c^2} + \mathcal{G}_{T\times T} \tag{2.40}$$

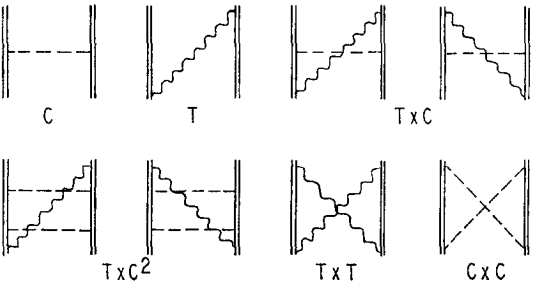

FIG. 5. Interaction diagrams contributing in order $\alpha^6 m$.

corresponding to the diagrams of Fig. 5. Rules for constructing the momentum space interaction operators are given in Appendix II. We note here that although $\mathcal{G}_{c\times c}$ appears to be of the order of interest, its contribution to the splitting vanishes.

3. Equal Times Equation and Perturbation Theory

We present, following Sucher [10], the method used in calculating the eigenvalue E of the Bethe–Salpeter equation, which we write for reference in the form

$$\mathcal{F}\psi(\bar{p}_1\bar{p}\epsilon) = \mathcal{G}\psi(\bar{p}_1\bar{p}_2\epsilon), \tag{3.1}$$

$$\mathcal{F} = \mathcal{F}_1\mathcal{F}_2 = (E/2 + \epsilon - H_1(\bar{p}_1))(E/2 - \epsilon - H_2(\bar{p}_2)), \tag{3.2}$$

$$\mathcal{G} = \mathcal{G}_I + \mathcal{G}^{(\mathrm{rad})}. \tag{3.3}$$

The standard method of solving Eq. (3.1) involves using as a zero order interaction the instantaneous part of the single photon exchange diagram, writing

$$\mathcal{G}_I + \mathcal{G}^{(\mathrm{rad})} = \mathcal{G}_c + \mathcal{G}_\Delta, \tag{3.4}$$

where

$$\mathcal{G}_\Delta = \mathcal{G}_T + \mathcal{G}_{T\times c} + \mathcal{G}_{T\times c^2} + \mathcal{G}_{T\times T} + \cdots + \mathcal{G}^{(\mathrm{rad})}. \tag{3.5}$$

We also rearrange (3.1) to read

$$\psi = (\mathcal{F} - \mathcal{G}_\Delta)^{-1}\mathcal{G}_c\psi. \tag{3.6}$$

An "equal times" function is now defined by

$$\Phi(\bar{p}_1\bar{p}_2) = \int_{-\infty}^{\infty} d\epsilon\ \psi(\bar{p}_1\bar{p}_2\epsilon). \tag{3.7}$$

Using the definition of $\mathscr{G}_c$ we see that

$$\mathscr{G}_c\psi = (1/-2\pi i)\, I_c\Phi, \tag{3.8}$$

where I_c is the Coulomb interaction operator in momentum space

$$I_c\Phi(\bar{p}_1\bar{p}_2) = (\alpha/2\pi^2)\int (d\bar{k}/\bar{k}^2)\,\Phi(\bar{p}_1 - \bar{k}\,\bar{p}_2 + \bar{k}). \tag{3.9}$$

Substituting (3.8) into (3.6) and integrating the resultant equation over ϵ we find

$$\Phi = \int (d\epsilon/-2\pi i)(\mathscr{F} - \mathscr{G}_\Delta)^{-1}\, I_c\Phi, \tag{3.10}$$

or, using the operator identity,

$$(A - B)^{-1} = A^{-1} + A^{-1}B(A - B)^{-1}, \tag{3.11}$$

$$\Phi = \int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}I_c\Phi + \int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}\mathscr{G}_\Delta(\mathscr{F} - \mathscr{G}_\Delta)^{-1}\, I_c\Phi. \tag{3.12}$$

Next we define single particle bound state positive and negative energy projection operators by

$$\mathscr{L}_\pm(\bar{p}) = \tfrac{1}{2}[1 \pm H(\bar{p})\,\mathscr{E}^{-1}(\bar{p})], \tag{3.13}$$

where

$$\mathscr{E}(\bar{p}) = |\,H(\bar{p})|, \tag{3.14}$$

the absolute value of H, i.e.,

$$\mathscr{E}(\bar{p})\,\Phi_n(\bar{p}) = |\,E_n\,|\,\Phi_n(\bar{p}). \tag{3.15}$$

In the limit $V \to 0$ $\mathscr{L}_\pm$ reduce to the familiar free particle projectors

$$\Lambda_\pm(\bar{p}) = \tfrac{1}{2}[1 \pm H_0(\bar{p})/E_p], \tag{3.16a}$$

where

$$H_0(\bar{p}) = \bar{\alpha}\cdot\bar{p} + \beta m. \tag{3.16b}$$

For explicit calculation we use the formulation

$$\mathscr{E} = +(H^2)^{1/2}, \tag{3.17}$$

where "+" means we take that branch of the square root with the property

$$\lim \mathscr{E}_{V\to 0} = E_p = +(\bar{p}^2 + m^2)^{1/2}. \tag{3.18}$$

Using the relations

$$\mathscr{E}^2 = H^2, \tag{3.19a}$$

$$[\mathscr{E}, H] = 0, \tag{3.19b}$$

it is found that

$$H\mathscr{L}_\pm = \pm\mathscr{E}\mathscr{L}_\pm . \tag{3.20}$$

These are used to exhibit the poles of the propagators S_1, S_2

$$\begin{aligned} S_1 &= \frac{1}{E/2 + \epsilon - \mathscr{E}_1 + i\delta}\,\mathscr{L}_{1+} + \frac{1}{E/2 + \epsilon + \mathscr{E}_1 - i\delta}\,\mathscr{L}_{1-} \\ &\equiv S_{1+}\mathscr{L}_{1+} + S_{1-}\mathscr{L}_{1-} , \end{aligned} \tag{3.21a}$$

$$\begin{aligned} S_2 &= \frac{1}{E/2 - \epsilon - \mathscr{E}_2 + i\delta}\,\mathscr{L}_{2+} + \frac{1}{E/2 - \epsilon + \mathscr{E}_2 - i\delta}\,\mathscr{L}_{2-} \\ &\equiv S_{2+}\mathscr{L}_{2+} + S_{2-}\mathscr{L}_{2-} , \end{aligned} \tag{3.21b}$$

where $\delta \to 0+$. (The Feynman prescription is to add a negative infinitesimal imaginary part to the mass, which is equivalent to adding it to $\mathscr{E}$.) Use is made of these relations to perform explicitly the ϵ integration of the first term of the right side of Eq. (3.12). Also the formulas

$$\int \frac{d\epsilon}{-2\pi i}\,\frac{1}{\epsilon - A + i\delta}\,\frac{1}{\epsilon + B - i\delta} = \frac{1}{A + B} , \tag{3.22a}$$

$$\int \frac{d\epsilon}{-2\pi i}\,\frac{1}{\epsilon - A + i\delta}\,\frac{1}{\epsilon + B + i\delta} = 0 \tag{3.22b}$$

are employed. The first of these holds only if the Hermitean operators A, B commute, but the second is true generally. We find

$$\int \frac{d\epsilon}{-2\pi i}\,\mathscr{F}^{-1} = \mathscr{L}_{++}\,\frac{1}{E - \mathscr{E}_1 - \mathscr{E}_2} - \mathscr{L}_{--}\,\frac{1}{E + \mathscr{E}_1 + \mathscr{E}_2} , \tag{3.23}$$

which can be rewritten, by virtue of Eq. (3.20),

$$\int \frac{d\epsilon}{-2\pi i}\,\mathscr{F}^{-1} = D^{-1}(\mathscr{L}_{++} - \mathscr{L}_{--}), \tag{3.24}$$

where

$$D = E - H_1 - H_2 . \tag{3.25}$$

Using Eq. (3.24) in Eq. (3.12) and multiplying on the left by D we find finally

$$E\Phi = [H_1 + H_2 + (\mathscr{L}_{++} - \mathscr{L}_{--})\, I_c]\, \Phi + \left[\int \frac{d\epsilon}{-2\pi i}\, D\mathscr{F}^{-1}\mathscr{G}_\Delta(\mathscr{F} - \mathscr{G}_\Delta)^{-1}\, I_c\right] \Phi. \tag{3.26}$$

This equation, called the equal times equation, is an exact consequence of the Bethe–Salpeter equation. The "relative energy" ϵ now occurs in the last term only as a parameter, but not as a variable. Since the operator in the last term depends explicitly on E, Eq. (3.26) is a homogeneous, linear integral equation but nonlinear eigenvalue equation for the determination of E. We remark that the above procedure differs from that of Salpeter [9, 12], which has been used in most published level shift calculations.

As a first step in the perturbation treatment the equal times equation is rewritten in the form

$$(H_c + H_\Delta)\Phi = E\Phi, \tag{3.27}$$

where

$$H_c = H_1 + H_2 + \mathscr{L}_{++} I_c \mathscr{L}_{++} \tag{3.28}$$

and

$$H_\Delta = \mathscr{L}_{++} I_c (1 - \mathscr{L}_{++}) - \mathscr{L}_{--} I_c + \int \frac{d\epsilon}{-2\pi i}\, D\mathscr{F}^{-1}\mathscr{G}_\Delta(\mathscr{F} - \mathscr{G}_\Delta)^{-1}\, I_c\,. \tag{3.29}$$

The first two terms of H_Δ represent pair effects induced by Coulomb exchange. The $\mathscr{L}_{++} I_c \mathscr{L}_{++}$ term of H_c is the no pair part of the Coulomb interaction and must be included in the zero order problem since it gives rise to the I_c term in the non-relativistic Schroedinger equation. When we speak of "pair" processes we refer to the creation of pairs with Dirac–Coulomb wave functions, corresponding to the use of bound state single particle propagators.

Certain mathematical advantages are gained by the above decomposition of $(\mathscr{L}_{++} - \mathscr{L}_{--})\, I_c$. Setting aside H_Δ for the time being we write

$$(E - H_1 - H_2 - \mathscr{L}_{++} I_c \mathscr{L}_{++})\Phi = 0 \tag{3.30}$$

and call this the Coulomb ladder equation (CLE). The fact that

$$[\mathscr{L}_{++}, H_c] = [\mathscr{L}_{+-}, H_c] = [\mathscr{L}_{-+}, H_c] = [\mathscr{L}_{--}, H_c] = 0 \tag{3.31}$$

means that we can choose solutions to CLE that are eigenfunctions of the projectors $\mathscr{L}_{\pm\pm}$. In general any sixteen component equation can be written as four coupled equations for the projections

$$\Phi_{\pm\pm} = \mathscr{L}_{\pm\pm}\Phi. \tag{3.32}$$

Due to the commutation relations (3.31) these are in fact decoupled and, furthermore, the Φ_{+-}, Φ_{-+}, Φ_{--} equations do not contain I_c:

$$\mathscr{L}_{++}(E - H_1 - H_2 - \mathscr{L}_{++} I_c \mathscr{L}_{++})\,\Phi_{++} = 0, \tag{3.33a}$$

$$\mathscr{L}_{+-}(E - H_1 - H_2)\,\Phi_{+-} = 0, \tag{3.33b}$$

$$\mathscr{L}_{-+}(E - H_1 - H_2)\,\Phi_{-+} = 0, \tag{3.33c}$$

$$\mathscr{L}_{--}(E - H_1 - H_2)\,\Phi_{--} = 0. \tag{3.33d}$$

We are particularly interested in the solutions of (3.33a) (to be referred to as Φ_c), since the eigenvalues, E_c, of low momentum states will be $\sim 2m$. The solutions of (3.33b)–(3.33d) can be taken to be products of Dirac hydrogenic eigenfunctions since there is no electron–electron coupling in these equations. Thus the spectrum of CLE involving negative energy projections is very simple. Solutions to the positive energy equation (3.33a) can be regarded as Hartree type superpositions of products of positive energy Dirac hydrogenic eigenfunctions. Summarizing, our unperturbed wave function is determined by the equation

$$(E_c - H_c)\,\Phi_c \tag{3.34}$$

subject to the subsidiary conditions

$$\mathscr{L}_{++}\Phi_c = \Phi_c\,, \tag{3.35a}$$

$$\langle \Phi_c \mid \Phi_c \rangle < \infty. \tag{3.35b}$$

To calculate the effect of H_Δ use is made of the exact Brillouin–Wigner perturbation formula

$$\Delta E = E - E_c = \langle \Phi_c \mid H_\Delta [1 - \Gamma(E, \Phi_c)\, H_\Delta]^{-1} \mid \Phi_c \rangle. \tag{3.36}$$

This formula applies even if H_Δ contains E. The Green's function $\Gamma(E, \Phi_c)$ is given by

$$\begin{aligned} \Gamma(E, \Phi_c) &= \sum_n \frac{|\,\Phi_n\rangle\langle\Phi_n\,|}{E - E_n} - \frac{|\,\Phi_c\rangle\langle\Phi_c\,|}{E - E_c} \\ &\equiv \sum_n{}' \frac{|\,\Phi_n\rangle\langle\Phi_n\,|}{E - E_n} \end{aligned} \tag{3.37}$$

where the unprimed sum over n includes *all* solutions of CLE. The simplicity of the negative energy spectrum of CLE can be used to simplify $\Gamma(E, \Phi_c)$. Writing Γ in operator form

$$\Gamma(E, \Phi_c) = (E - H_c)^{-1}[1 - |\,\Phi_c\rangle\langle\Phi_c\,|] \tag{3.38}$$

it is seen that an equivalent form implied by Eqs. (3.33) and the completeness relation

$$1 = \mathscr{L}_{++} + \mathscr{L}_{+-} + \mathscr{L}_{-+} + \mathscr{L}_{--} \tag{3.39}$$

is

$$\Gamma(E, \Phi_c) = (E - H_c)^{-1}[\mathscr{L}_{++} - |\Phi_c\rangle\langle\Phi_c|] + D^{-1}(1 - \mathscr{L}_{++}). \tag{3.40}$$

A consequence of this is the relation

$$(1 - \mathscr{L}_{++})\,\Gamma(E, \Phi_c) = (1 - \mathscr{L}_{++})\,D^{-1}. \tag{3.41}$$

The level shift is expanded as

$$\Delta E = \Delta E^{(1)} + \Delta E^{(2)} + \cdots, \tag{3.42}$$

where

$$\Delta E^{(1)} = \langle\Phi_c | H_\Delta | \Phi_c\rangle, \tag{3.43a}$$

$$\Delta E^{(2)} = \langle\Phi_c | H_\Delta \Gamma H_\Delta | \Phi_c\rangle, \tag{3.43b}$$

$$\Delta E^{(3)} = \langle\Phi_c | H_\Delta \Gamma H_\Delta \Gamma H_\Delta | \Phi_c\rangle, \tag{3.43c}$$

etc. Using the fact that $\mathscr{L}_{++}\Phi_c = \Phi_c$ we find that

$$\Delta E^{(1)} = \int (d\epsilon/-2\pi i)\,\langle\Phi_c | D\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c | \Phi_c\rangle, \tag{3.44}$$

where the definition

$$J = \mathscr{G}_\Delta[1 - \mathscr{F}^{-1}\mathscr{G}_\Delta]^{-1} = \mathscr{G}_\Delta + \mathscr{G}_\Delta\mathscr{F}^{-1}\mathscr{G}_\Delta + \cdots \tag{3.45}$$

has been introduced. Note that $\int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}J\mathscr{F}^{-1}$ is an integral operator in 3-momentum space, all energy integrations being performed within the operator itself. Using again $\mathscr{L}_{++}\Phi_c = \Phi_c$ and also Eq. (3.41) we find

$$\begin{aligned}\Delta E^{(2)} &= \langle\Phi_c | I_c[-D]^{-1}\,\mathscr{L}_{--}I_c | \Phi_c\rangle \\ &\quad + \langle\Phi_c | (1 - \mathscr{L}_{++}) \int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}J\mathscr{F}^{-1} | \Phi_c\rangle \\ &\quad + \langle\Phi_c | I_c\mathscr{L}_{++} \int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c[-D]^{-1}\,\mathscr{L}_{--}I_c | \Phi_c\rangle \\ &\quad + \langle\Phi_c | D \int (d\epsilon/-2\pi i)\,\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c\Gamma D \int (d\epsilon'/-2\pi i)\,\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c | \Phi_c\rangle \\ &\equiv \Delta E_a^{(2)} + \Delta E_b^{(2)} + \Delta E_c^{(2)} + \Delta E_d^{(2)}.\end{aligned} \tag{3.46}$$

It has been found that neither $\Delta E^{(3)}$, $\Delta E_c^{(2)}$, nor $\Delta E_a^{(2)}$ contribute in order $\alpha^6 m$ to the ^3P splitting. $\Delta E_a^{(2)}$ is a high energy double pair effect, and the first two are essentially three photon terms which do not contribute to the order of interest.

Consider

$$\Delta E^{(1)} = \int (d\epsilon/-2\pi i) \langle \Phi_c \mid D\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c \mid \Phi_c \rangle. \tag{3.47}$$

We have

$$\mathscr{F}^{-1} = \mathscr{F}_1^{-1}\mathscr{F}_2^{-1} = (\mathscr{F}_1^{-1} + \mathscr{F}_2^{-1})(\mathscr{F}_1 + \mathscr{F}_2)^{-1}, \tag{3.48a}$$

or

$$\mathscr{F}^{-1} = (S_1 + S_2)\, D^{-1} = D^{-1}(S_1 + S_2), \tag{3.48b}$$

and, thus,

$$\Delta E^{(1)} = \int (d\epsilon/-2\pi i) \langle \Phi_c \mid (S_{1+} + S_{2+})\, J(S_1 + S_2)\, D^{-1} I_c \mid \Phi_c \rangle. \tag{3.49}$$

We decompose D^{-1} as

$$D^{-1} = \frac{1}{D} - \frac{\Delta E}{D_c D} \cong \frac{1}{D_c} - \frac{\Delta E}{D_c^2} \tag{3.50a}$$

with

$$D_c = E_c - H_1 - H_2\,. \tag{3.50b}$$

It will become apparent as the calculation progresses that we can now let $E \to E_c$ everywhere it occurs, i.e. in propagators, D factors, and in $\mathscr{G}_\Delta$ itself, and from now on this will be understood. In many cases the approximation $E = 2m$ is in fact adequate, and when this is not the case the approximation $E = 2m + W_0$ will suffice. The one exception is the explicit factor D^{-1} appearing in Eq. (3.49) for which Eq. (3.50a) must be used; however, here it is sufficient to let $J \to \mathscr{G}_T$, the single transverse photon diagram. The term of $\Delta E^{(1)}$ containing the $1/D_c$ part of (3.50a) can be combined, by making use of CLE in the form

$$(D_c - \mathscr{L}_{++}I_c)\, \Phi_c = 0 \tag{3.51}$$

with $\Delta E_b^{(2)}$, Eq. (3.46), yielding the simple form for the sum

$$\int (d\epsilon/-2\pi i) \langle \Phi_c \mid I_c\mathscr{F}^{-1}J\mathscr{F}^{-1}I_c \mid \Phi_c \rangle. \tag{3.52}$$

A word about J is now in order. To calculate J it must be expanded according to Eq. (3.45). Terms of higher order than first in $\mathscr{G}_\Delta$ may be interpreted as arising

from reducible diagrams; for example, the term $\mathcal{G}_{T\times T}\mathcal{F}^{-1}\mathcal{G}_T$ corresponds to Fig. 6. The largest contributor to $\mathcal{G}_\Delta$ is $\mathcal{G}_T$. In certain parts of ΔE, we can let $\mathcal{G}_\Delta \to \mathcal{G}_T$, and, thus, it is useful to separate $\mathcal{G}_\Delta$ into two pieces

$$\mathcal{G}_\Delta = \mathcal{G}_T + \Delta\mathcal{G}. \tag{3.53}$$

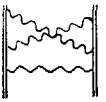

FIG. 6. Reducible diagram.

We can now display the terms that contribute to the ^3P splitting in order $\alpha^6 m$:

$$\begin{aligned}\Delta E = \int (d\epsilon/-2\pi i)\,\langle\Phi_c \mid I_c[&\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1} + \mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1} \\ &+ \mathcal{F}^{-1}\Delta\mathcal{G}\mathcal{F}^{-1}]\, I_c \mid \Phi_c\rangle + \langle\Phi_c \mid \left[D_c \int (d\epsilon/-2\pi i)\,\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1} I_c\right] \\ &\times \Gamma\left[D_c \int (d\epsilon'/-2\pi i)\,\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}\right] I_c \mid \Phi_c\rangle \\ &- \Delta E \int (d\epsilon/-2\pi i)\,\langle\Phi_c \mid (S_{1+} + S_{2+})\,\mathcal{G}_T(S_1 + S_2)(1/D_c^2)\, I_c \mid \Phi_c\rangle.\end{aligned} \tag{3.54}$$

Also it has been found that $\Delta\mathcal{G}$ can be approximated as

$$\Delta\mathcal{G} \to \mathcal{G}_{T\times c} + \mathcal{G}_{T\times c^2} + \mathcal{G}_{T\times T} + \mathcal{G}^{(\mathrm{rad})}. \tag{3.55}$$

The diagrams $\mathcal{G}_T$, $\mathcal{G}_{T\times c}$, $\mathcal{G}_{T\times c^2}$ are treated in Section 5; $\mathcal{G}_{T\times T}$, $\mathcal{G}_T\mathcal{F}^{-1}\mathcal{G}_T \equiv \mathcal{G}_{T\cdot T}$ and the last two terms of Eq. (3.54) are treated in Section 6, and $\mathcal{G}^{(\mathrm{rad})}$ is discussed in Section 7.

4. COULOMB LADDER EQUATION

The object of this section is the calculation of the eigenvalue E_c and eigenfunction Φ_c of the Coulomb ladder equation

$$(E_c - H_1 - H_2 - \mathcal{L}_{++}I_c\mathcal{L}_{++})\,\Phi_c = (E_c - H_c)\,\Phi_c = 0. \tag{4.1a}$$

We will be looking in particular for solutions that satisfy

$$\mathcal{L}_{++}\Phi_c = \Phi_c\,. \tag{4.1b}$$

It will be shown that in lowest order Eq. (4.1) is approximated by the nonrelativistic Schroedinger equation

$$(W_0 - H_0)\,\varphi_0 = 0, \tag{4.2a}$$

$$H_0 = \bar{p}_1^{\,2}/2m + \bar{p}_2^{\,2}/2m + V_1 + V_2 + I_c\,, \tag{4.2b}$$

which, of course, contains no fine structure. We shall be interested only the spatially antisymmetric solutions of Eq. (4.2a). The binding energy $E_c - 2m$ contains, in addition to W_0, a portion of the fine structure. This includes some of the familiar contributions of order $\alpha^4 m$ as well as some of the $\alpha^6 m$ terms that we seek.

The function Φ_c is a sixteen component object and the operators we work with are direct products of 4×4 matrices. On the other hand Eq. (4.2a,b) conventionally refers to a single component wave function. The formal treatment of Eq. (4.1) is facilitated by regarding φ_0 as a 16 component object as well. The first four components correspond to the conventional Pauli spinor representation and are determined by requiring that φ_0 be an eigenstate of J^2, J_z, and S^2 as well as L^2. The remaining 12 components are taken to be zero. Hence, φ_0 satisfies

$$((1 + \beta_1)/2)((1 + \beta_2)/2)\,\varphi_0 = \varphi_0 \tag{4.3a}$$

or

$$\varphi_0 = \varphi_0^{++}, \tag{4.3b}$$

where we are introducing the notation convention

$$\Phi^{\pm\pm} \equiv ((1 \pm \beta_1)/2)\,((1 \pm \beta_2)/2)\,\Phi, \tag{4.4}$$

the convention being now understood that superscripts refer to a spinor component projection, and subscripts refer to positive–negative energy projections (Eq. 3.32). It is worth pointing out that any 16×16 component matrix operator A can be decomposed into 16 components as follows:

$$A = \sum_{i,j,k,l=\pm,\pm,\pm,\pm} A^{ijkl} \tag{4.5a}$$

with

$$A^{\pm\pm\pm\pm} = ((1 \pm \beta_1)/2)\,((1 \pm \beta_2)/2)\,A\,((1 \pm \beta_1)/2)\,((1 \pm \beta_2)/2). \tag{4.5b}$$

It is clear that if Φ satisfies

$$\Phi = \Phi^{++}$$

then

$$\langle \Phi \mid A \mid \Phi \rangle = \langle \Phi \mid A^{++++} \mid \Phi \rangle. \tag{4.6}$$

All of our results will ultimately be expressed in terms of expectation values and matrix elements based upon solutions of Eq. (4.2). Reduction to these forms will be accomplished by means of a suitable generalization of the Foldy–Wouthysen [35] transformation. Historically, the Breit large–small component reduction [2] was the first method to be applied to this sort of problem. It becomes quite cumbersome, however, for the high order of accuracy required here. A method developed by Sucher [10, 36] for the treatment of the Lamb shift is also suitable for this problem, and we have verified, as a check, that it gives the same result as the F–W method. The F–W method has in our view some minor technical advantages and seems better suited to systematic exposition.

We illustrate our generalization of the F–W transformation first with a fictitious CLE in which there are no external potentials,

$$(E - H_{01} - H_{02} - \Lambda_{++} I_c \Lambda_{++})\Phi = 0. \tag{4.7}$$

The product of exact, unitary, free particle F–W transformations for each particle

$$e^{iS_0} = U_0 = \frac{1}{(2E_p(E_p + m))^{1/2}} [\beta\vec{\alpha} \cdot \vec{p} + (E_p + m)], \tag{4.8a}$$

where

$$e^{-iS_0} = U_0^{-1} = U_0^{\dagger} = \frac{1}{(2E_p(E_p + m))^{1/2}} [\vec{\alpha} \cdot \vec{p}\beta + (E_p + m)] \tag{4.8b}$$

is applied. U_0 has been constructed to transform H_0 to an "even" Dirac operator. Equation (4.7) then becomes

$$\{E - \beta_1 E_{p_1} - \beta_2 E_{p_2} - [U_{01}U_{02}\Lambda_{1+}\Lambda_{2+}U_{01}^{-1}U_{02}^{-1}][U_{01}U_{02}I_cU_{01}^{-1}U_{02}^{-1}]$$
$$\times [U_{01}U_{02}\Lambda_{1+}\Lambda_{2+}U_{01}^{-1}U_{02}^{-1}]\}\,[U_{01}U_{02}\Phi \equiv \varphi]. \tag{4.9}$$

The projectors transform simply as

$$U_0\Lambda_{\pm}U_0^{-1} = (1 \pm \beta)/2, \tag{4.10}$$

implying that the Coulomb operator does not connect "large" and "small" components. Thus, CLE breaks up into four uncoupled four component equations corresponding to the energy projections Λ_{++}, Λ_{+-}, Λ_{-+}, Λ_{--}:

$$(E - E_{p_1} - E_{p_2} - [U_{01}U_{02}I_cU_{01}^{-1}U_{02}^{-1}]^{++++})\,\varphi^{++} = 0, \tag{4.11a}$$
$$(E - E_{p_1} + E_{p_2})\,\varphi^{+-} = 0, \tag{4.11b}$$
$$(E + E_{p_1} - E_{p_2})\,\varphi^{-+} = 0, \tag{4.11c}$$
$$(E + E_{p_1} + E_{p_2})\,\varphi^{--} = 0. \tag{4.11d}$$

This set is the transformed version of Eq. (3.33a)–(3.33d). The transformed Coulomb operator is given exactly by

$$
\begin{aligned}
&[U_{01}U_{02}I_cU_{01}^{-1}U_{01}^{-1}]^{++++} \\
&= \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2} I_c \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2} \\
&\quad + \left[\frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2} I_c \left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2}\frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\right] \\
&\quad + [1 \rightleftarrows 2] + \frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\frac{\bar{\sigma}_2\cdot\bar{p}_2}{(2E_{p_2}(E_{p_2}+m))^{1/2}} \\
&\quad \times I_c \frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\frac{\bar{\sigma}_2\cdot\bar{p}_2}{(2E_{p_2}(E_{p_2}+m))^{1/2}} \equiv \mathscr{I}_c\,. \qquad (4.12)
\end{aligned}
$$

(Here we have written only the $(++++)$ component of the matrix, its other components are, of course, zero.) The $++$ equation reduces to the Schroedinger equation for p_1, p_2 small.

The F–W transformation may also be used to reexpress the matrix element of an arbitrary operator between two positive energy states in the 16 component representation in reduced form as the matrix element of a corresponding Pauli type operator in the F–W representation. We have for a positive energy wave function

$$\Phi = \Lambda_{++}\Phi = \Phi_{++} = U_{01}^{-1}U_{02}^{-1}\varphi, \qquad (4.13)$$

where

$$\varphi^{++} = \varphi, \qquad (4.14)$$

which implies

$$\langle\Phi_1 \mid M \mid \Phi_2\rangle = \langle\varphi_1 \mid U_{01}U_{02}MU_{01}^{-1}U_{02}^{-1} \mid \varphi_2\rangle \qquad (4.15)$$

and, using Eq. (4.6),

$$\langle\Phi_1 \mid M \mid \Phi_2\rangle = \langle\varphi_1 \mid [U_{01}U_{02}MU_{01}^{-1}U_{02}^{-1}]^{++++} \mid \varphi_2\rangle. \qquad (4.16)$$

This method will be employed in later sections for reduction of operator expectation values.

We wish to extend this whole procedure to take account of external potentials. Foldy and Wouthysen developed a nonrelativistic procedure which yields a transformed Hamiltonian and projectors as expansions in p and V and products of these. Of course, higher order terms become increasingly singular. For triplet φ_0 this procedure does, however, yield a convergent result which agrees with what

we shall find. One would prefer an expansion in nondivergent operators, that is roughly speaking an expansion of H in powers of "V/E_p" about the zero order term βE_p. This enables one to examine corrections to the nonrelativistic result, and provides a method for treating the singlet φ_0 case and proceeding to higher order for the triplet case. Such a generalization of the F–W procedure has been obtained and is derived in Appendix III. The main result of this analysis is a modification of U_0

$$U(\bar{p}) \cong U_0(\bar{p})(1 + W_1(\bar{p}) + \tfrac{1}{2}W_1^2(\bar{p})), \tag{4.17}$$

$$W_1(\bar{p}) = \frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}}\frac{V'}{2m}\left(\frac{E_p+m}{2E_p}\right)^{1/2} - \left(\frac{E_p+m}{2E_p}\right)^{1/2}\frac{V'}{2m}\frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}}, \tag{4.18}$$

where the kernel of the integral operator V' is given by

$$V'(\bar{p}\bar{p}') = \frac{2m}{E_p+E_{p'}}V(\bar{p}\bar{p}') = \frac{2m}{E_p+E_{p'}}\left(-\frac{Z\alpha}{2\pi^2}\right)\frac{1}{|\bar{p}-\bar{p}'|^2}. \tag{4.19}$$

Applying $U(\bar{p})$ to the external potential Hamiltonian one finds

$$\begin{aligned} H' &= U(\bar{p})\,HU^{\dagger}(\bar{p}) \\ &= \beta E_p + \left(\frac{E_p+m}{2E_p}\right)^{1/2}V\left(\frac{E_p+m}{2E_p}\right)^{1/2} + \frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}}V\frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}} \\ &\quad + \beta[W_1E_pW_1 + \tfrac{1}{2}W_1^2E_p + \tfrac{1}{2}E_pW_1^2] + O_2 + \epsilon_3, \end{aligned} \tag{4.20}$$

where O_2 and ϵ_3 are "odd" and "even" Dirac operators, respectively. It is observed that performing an additional transformation U_2 of order V^2 to remove O_2 will not change the even part of H' to order V^2, and so the first four terms of Eq. (4.20) give H' accurate to order V^2, that is the remaining even and odd parts are of order V^3. This means also that large small coupling terms in Eq. (4.11a)–(4.11d) are of order V^3. These are neglected.

It is demonstrated in Appendix III that to the order we are working in V it is legitimate to put

$$\mathscr{L}_{\pm}(\bar{p}) = \tfrac{1}{2}(1 \pm \beta). \tag{4.21}$$

Thus putting together Eqs. (4.12), (4.17), (4.18), (4.20), (4.21) we find for the generalization of Eq. (4.11a)

$$E_c\varphi^{++} = ([H_1^1]^{++} + [H_2^1]^{++} + J_c)\,\varphi^{++} = H_{\text{red}}\varphi^{++}. \tag{4.22}$$

H_{red} is the "reduced" Hamiltonian), with

$$[H_1']^{++} = E_p + \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} V_1 \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} + \frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} V_1 \frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} + W_1 E_{p_1} W_1 + \tfrac{1}{2}\{W_1^2, E_{p_1}\}, \tag{4.23}$$

and

$$J_c = \mathscr{J}_c + \left[\left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2} M_c^{(1)} \left(\frac{E_{p_2}+m}{2E_{p_2}}\right)^{1/2} + \frac{\bar{\sigma}_2\cdot\bar{p}_2}{(2E_{p_2}(E_{p_2}+m))^{1/2}} M_c^{(1)} \frac{\bar{\sigma}_2\cdot\bar{p}_2}{(2E_{p_2}(E_{p_2}+m))^{1/2}}\right] + [1 \rightleftarrows 2], \tag{4.24a}$$

where

$$\begin{aligned} M_c^{(1)} = {} & \left[\frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} \frac{V_1'}{2m} \left(\frac{E_{p_1}+m}{2E_{p_1}}\right) I_c \frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} + \text{h.c.}\right] \\ & - \left[\left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} \frac{V_1'}{2m} \frac{\bar{\sigma}_1\cdot\bar{p}_1}{2E_{p_1}} I_c \frac{\sigma_1\cdot p_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} + \text{h.c.}\right] \\ & - \left[\frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}} \frac{V_1'}{2m} \frac{\bar{\sigma}_1\cdot\bar{p}_1}{2E_{p_1}} I_c \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} + \text{h.c.}\right] \\ & + \left[\left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} \frac{V_1'}{2m} \frac{\bar{p}_1^{\,2}}{2E_{p_1}(E_{p_1}+m)} I_c \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2} + \text{h.c.}\right] \end{aligned} \tag{4.24b}$$

(h.c. denotes Hermitian conjugate and $[1 \rightleftarrows 2]$ denotes a corresponding operator with particle indices interchanged). Terms of order $V_1 \cdot V_2 \cdot I_c$ are negligible and have been dropped.

We wish to calculate the $J = 0, 1, 2$ splittings in E_c to order $\alpha^6 m$ and to find an approximation to Φ_c sufficiently accurate to evaluate the $\alpha^6 m$ contributions of the diagrams discussed in Section 3. The technique used is nonrelativistic expansion of H_{red} in a series of operators obtained by expanding all E_p factors as $E_p = m + \bar{p}^2/2m + \cdots$. Regarding p to contribute a factor αm and V and I_c factors $\alpha^2 m$, their so-called "nominal order," we then have an expansion in powers of α (for each of the terms powers of α can be scaled out of the operator). Carrying out this procedure, we obtain

$$H_{\text{red}} = 2m + H_0 + \Delta H_c^{(4)} + \Delta H_c^{(6)} + \cdots. \tag{4.25}$$

Equation (4.22) is to be regarded as solved (to whatever numerical accuracy is necessary) in zeroth order with $2m + H_0$ taken as the zeroth order Hamiltonian. The effects of $\Delta H_c^{(4)}$ and $\Delta H_c^{(6)}$ are to be taken into account by applying perturbation theory to Eq. (4.22).

$\Delta H_c^{(4)}$ is defined as the correction to H_{red} of nominal order $\alpha^4 m$ and is given by

$$\begin{aligned}\Delta H_c^{(4)} = &-\frac{\bar{p}_1^4}{8m^3} - \frac{\bar{p}_2^4}{8m^3} + \frac{1}{4m^2}\,\bar{\sigma}_1 \cdot \bar{p}_1(V_1 + I_c)\,\bar{\sigma}_1 \cdot \bar{p}_1 \\ &+ \frac{1}{4m^2}\,\bar{\sigma}_2 \cdot \bar{p}_2(V_2 + I_c)\,\bar{\sigma}_2 \cdot \bar{p}_2 - \frac{1}{8m^2}\{\bar{p}_1^2, V_1 + I_c\} \\ &- \frac{1}{8m^2}\{\bar{p}_2^2, V_2 + I_c\} \equiv H_{\text{Coulomb}}\,. \end{aligned} \tag{4.26}$$

It contributes precisely the "Coulomb fine structure," Eq. (1.3), in first order perturbation theory and an $\alpha^6 m$ correction in second order. This latter is a part of a set of $\alpha^6 m$ corrections which will be lumped together under the name "second order fine structure."

$\Delta H_c^{(6)}$ is defined as the correction to H_{red} of nominal order $\alpha^6 m$. It contributes an $\alpha^6 m$ correction in first order and can be neglected in higher order. Therefore only the spin dependent part

$$\begin{aligned}\Delta H_c^{(6)} \to [\Delta H_c^{(6)}]_{\text{fine structure}} = &-\frac{3}{32m^4}\{\bar{\sigma}_1 \cdot \bar{p}_1(V_1 + I_c), \bar{\sigma}_1 \cdot \bar{p}_1\,\bar{p}_1^2\} + [1 \rightleftarrows 2] \\ &- \frac{1}{32m^4}\{\bar{\sigma}_1 \cdot \bar{p}_1 I_c \bar{\sigma}_1 \cdot \bar{p}_1\,, \bar{p}_2^2\} + [1 \rightleftarrows 2] \\ &+ \frac{1}{16m^4}\,\bar{\sigma}_1 \cdot \bar{p}_1\,\bar{\sigma}_2 \cdot \bar{p}_2 I_c \bar{\sigma}_1 \cdot \bar{p}_1\,\bar{\sigma}_2 \cdot \bar{p}_2 \end{aligned} \tag{4.27}$$

can contribute to the splitting (the $V \cdot I_c$ and $V \cdot V$ terms turn out spin independent).

We have, thus, arrived at a simple theoretical expression for calculating the splittings. Defining

$$\Delta W_c = W_c - W_0 \tag{4.28a}$$

with

$$W_c = E_c - 2m \tag{4.28b}$$

we have

$$\begin{aligned}\Delta W_c = &\langle \varphi_0 \mid \Delta H_c^{(4)} \mid \varphi_0 \rangle + \langle \varphi_0 \mid \Delta H_c^{(6)} \mid \varphi_0 \rangle \\ &+ \sum_n{}' \frac{\langle \varphi_0 \mid \Delta H_c^{(4)} \mid \varphi_n \rangle \langle \varphi_n \mid \Delta H_c^{(4)} \mid \varphi_0 \rangle}{W_0 - W_n}\,. \end{aligned} \tag{4.29}$$

The wave function, corrected to first order is given by

$$\varphi^{++} \cong \varphi_0 + \varphi^{(1)} \tag{4.30}$$

with

$$\varphi^{(1)} = \sum_n{}' \frac{|\varphi_n\rangle\langle\varphi_n| \Delta H_c^{(4)} | \varphi_0\rangle}{W_0 - W_n}. \tag{4.31}$$

One must inquire as to whether this nonrelativistic expansion is valid. The general rule that applies is that if an operator's nonrelativistic approximation has a convergent expectation value then this value is accurate to the nominal order of the operator, and the correction terms are of higher order. However, if the nonrelativistic approximation diverges, the true order is lower than the nominal order. We know that $\Delta H_c^{(4)}$ is convergent for both triplet and singlet φ_0. One finds however that $\Delta H_c^{(6)}$ is divergent for singlets, and this is connected with the well known fact that singlets have an $\alpha^5 m$ term in their energy level expansion. These $\alpha^5 m$ terms arise from high momentum portions of the momentum space integrals and turn out to be of the general form of a numerical factor of order unity times $\alpha^2 \times \langle\varphi_0 | \delta^3(\bar{r}_1 - \bar{r}_2)| \varphi_0\rangle$, i.e., they are "contact interactions" and, thus, vanish for triplets. This is in turn connected with the fact that $\Delta H_c^{(6)}$ has a convergent expectation value with triplet φ_0, there being sufficient extra powers of momenta in wave function denominators to make the integrals converge. Also in Section 8 arguments will be given showing that the coordinate space representation of $\langle\varphi_0 | \Delta H_c^{(6)} | \varphi_0\rangle$ converges. We shall find that the identical situation occurs for all operators evaluated in this paper. Similar arguments can be made for second order perturbation sums as well.

Finally, we display $\langle\varphi_0 | \Delta H_c^{(4)} | \varphi_0\rangle$ and $\langle\varphi_0 | \Delta H_c^{(6)} | \varphi_0\rangle$ as integrals over momentum space wave functions. Spin dependent and spin independent parts are separated by means of the "Dirac relation"

$$\bar{\sigma} \cdot \bar{A}\bar{\sigma} \cdot \bar{B} = \bar{A} \cdot \bar{B} + i\bar{\sigma} \cdot (\bar{A} \times \bar{B}). \tag{4.32}$$

We have

$$\begin{aligned}
&\langle\varphi_0 | \Delta H_c^{(4)} | \varphi_0\rangle \\
&= -\frac{1}{8m^3} \langle\varphi_0(\bar{p}_1\bar{p}_2) | \bar{p}_1^4 + \bar{p}_2^4 | \varphi_0(\bar{p}_1\bar{p}_2)\rangle \\
&\quad - \frac{1}{4m^2}\left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{\bar{k}^2} \langle\varphi_0(\bar{p}_1\bar{p}_2) | \bar{k}^2 | \varphi_0(\bar{p}_1 - \bar{k}\,\bar{p}_2 + \bar{k})\rangle \\
&\quad - \frac{1}{8m^2}\left(\frac{-Z\alpha}{2\pi^2}\right) \int \frac{d\bar{k}_1}{\bar{k}_1^2} \langle\varphi_0(\bar{p}_1\bar{p}_2) | \bar{k}_1^2 | \varphi_0(\bar{p}_1 - \bar{k}_1\,\bar{p}_2)\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8m^2}\left(\frac{-Z\alpha}{2\pi^2}\right)\int \frac{d\bar{k}_2}{\bar{k}_2^{\,2}}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{k}_2^{\,2}\mid \varphi_0(\bar{p}_1\bar{p}_2+\bar{k}_2)\rangle\\
&+ \frac{i}{4m^2}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_1\cdot[\bar{k}\times(\bar{p}_1-\bar{k})]\mid \varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle\\
&- \frac{i}{4m^2}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_2\cdot[\bar{k}\times(\bar{p}_2+\bar{k})]\mid \varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle\\
&+ \frac{i}{4m^2}\left(\frac{-Z\alpha}{2\pi^2}\right)\int \frac{d\bar{k}_1}{\bar{k}_1^{\,2}}\langle\varphi_0(p_1p_2)\mid \bar{\sigma}_1\cdot[\bar{k}_1\times(\bar{p}_1-\bar{k}_1)]\mid \varphi_0(\bar{p}_1-\bar{k}_1\,\bar{p}_2)\rangle\\
&- \frac{i}{4m^2}\left(\frac{-Z\alpha}{2\pi^2}\right)\int \frac{d\bar{k}_2}{\bar{k}_2^{\,2}}\langle\varphi_0(p_1p_2)\mid \bar{\sigma}_2\cdot[\bar{k}_2\times(\bar{p}_2+\bar{k}_2)]\mid \varphi_0(p_1p_2+k_2)\rangle\\
&\equiv \sum_{i=1}^{8}\Delta W_c^{(4)}(i)\equiv \Delta W_c^{(4)}.
\end{aligned}
\tag{4.33}
$$

Including the $[1 \rightleftarrows 2]$ exchange terms by doubling the direct terms we find

$$
\begin{aligned}
&\langle\varphi_0\mid \Delta H_c^{(6)}\mid \varphi_0\rangle\\
&= \frac{3i}{16m^4}\left(\frac{-Z\alpha}{2\pi^2}\right)\int \frac{d\bar{k}_1}{\bar{k}_1^{\,2}}\\
&\quad\times\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_1\cdot(\bar{p}_1\times\bar{k}_1)[\bar{p}_1^{\,2}+(\bar{p}_1-\bar{k}_1)^2]\mid \varphi_0(\bar{p}_1-\bar{k}_1\bar{p}_2)\rangle\\
&\quad+\frac{3i}{16m^4}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2}\\
&\quad\times\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_1\cdot(\bar{p}_1\times\bar{k})[\bar{p}_1^{\,2}+(\bar{p}_1-\bar{k})^2]\mid \varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle\\
&\quad+\frac{i}{16m^4}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2}\\
&\quad\times\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_1\cdot(\bar{p}_1\times\bar{k})[\bar{p}_2^{\,2}+(\bar{p}_2+\bar{k})^2]\mid \varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle\\
&\quad+\frac{1}{16m^4}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2}\\
&\quad\times\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \bar{\sigma}_1\cdot\bar{p}_1\,\bar{\sigma}_2\cdot\bar{p}_2\,\sigma_1\cdot(\bar{p}_1-\bar{k})\,\bar{\sigma}_2\cdot(\bar{p}_2+\bar{k})\mid \varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle\\
&\equiv \sum_{i=2}^{5}\Delta W_c^{(6)}(i).
\end{aligned}
\tag{4.34}
$$

Defining

$$
\Delta W_c^{(6)}(1)=\sum_n{}'\frac{\langle\varphi_0\mid \Delta H_c^{(4)}\mid \varphi_n\rangle\langle\varphi_n\mid \Delta H_c^{(4)}\mid \varphi_0\rangle}{W_0-W_n}
\tag{4.35}
$$

we may write the total $\alpha^6 m$ contribution from the Coulomb ladder equation as

$$\Delta W_c^{(6)} = \sum_{i=1}^{5} \Delta W_c^{(6)}(i). \tag{4.36}$$

5. Single Transverse Photon Exchange

In this section we investigate the effect of the exchange of a single transverse photon; i.e., of the operator $\mathcal{G}_T$. This operator is responsible for the well known Breit interaction, plus numerous correction terms. Also it is natural at this point to deal with the diagrams in which the transverse photon is crossed by an arbitrary number of uncrossed Coulomb interactions, Fig. 7. Thus, for this section the interaction operator considered is

$$J \to J_\tau \equiv \mathcal{G}_T + \sum_{n=1}^{\infty} \mathcal{G}_{T\times C^n}. \tag{5.1}$$

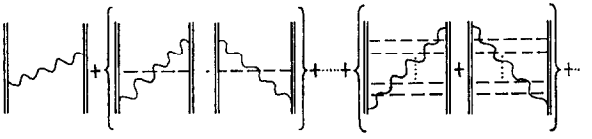

FIG. 7. Crossed Coulomb diagrams.

In a certain approximation these diagrams can be summed analytically. This is necessary for the calculation of $\alpha^5 m$ effects coming from very low momentum of the transverse photon. It will turn out that for the ^3P splitting calculation terms only through $n = 2$ contribute. It is however convenient to work with the summed expression.

Before proceeding we wish to discuss the structure of the general interaction term. The first order energy shift expression is, Eq. (3.52),

$$\Delta E_{(1)} = \int (d\epsilon/-2\pi i) \langle \Phi_c \mid I_c \mathcal{F}^{-1} J \mathcal{F}^{-1} I_c \mid \Phi_c \rangle. \tag{5.2}$$

The exterior $\mathcal{F}^{-1}$ operators may be decomposed as, Eqs. (3.2) and (3.21),

$$\mathcal{F}^{-1} = \mathcal{L}_{++} S_{1+} S_{2+} + \mathcal{L}_{+-} S_{1+} S_{2-} + \mathcal{L}_{-+} S_{1-} S_{2+} + \mathcal{L}_{--} S_{1-} S_{2-}, \tag{5.3}$$

giving the corresponding decomposition of $\Delta E_{(1)}$

$$\begin{aligned}\Delta E_{(1)} &= \sum_{\alpha\beta\gamma\delta=\pm\pm\pm\pm} \Delta E_{(1)\alpha\beta\gamma\delta} \\ &= \sum_{\alpha\beta\gamma\delta} \int (d\epsilon/-2\pi i)\, \langle \Phi_c \mid I_c \mathscr{L}_{\alpha\beta} S_{1\alpha} S_{2\beta} J S_{1\gamma} S_{2\delta} \mathscr{L}_{\gamma\delta} I_c \mid \Phi_c \rangle.\end{aligned} \tag{5.4}$$

In calculating, all internal propagators of J are broken in positive and negative energy projections also. Use of Eq. (3.48b) allows one to write the no pair term in a form with the flanking I_c's eliminated

$$\Delta E_{(1)++++} = \int (d\epsilon/-2\pi i)\, \langle \Phi_c \mid (S_{1+} + S_{2+})\, J (S_{1+} + S_{2+}) \mid \Phi_c \rangle. \tag{5.5}$$

In the remaining terms at least one of the I_c's is not removable, corresponding to reducible diagrams in which a Coulomb photon is exchanged (in conjunction with the production of a pair) before and/or after the interaction (see Fig. 8). The "blob" corresponds to the general interaction operator J.

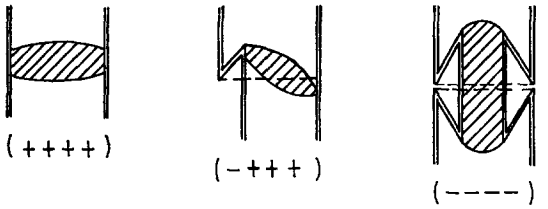

FIG. 8. Decomposition of general diagram.

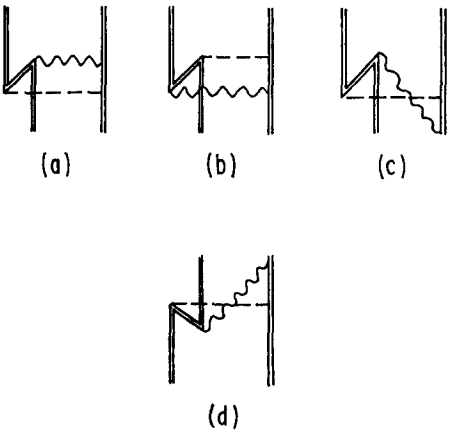

FIG. 9. Coulomb transverse diagrams with one intermediate or one external pair.

The simplest single photon exchange effects to calculate are the diagrams in which a pair is present. It has been found that only eight diagrams, those with a single Coulomb photon and single pair, contribute in order $\alpha^6 m$. The four involving a negative energy intermediate state on the path of particle one are shown in Fig. 9; (a) and (b) are irreducible, coming from the operator $\mathcal{G}_{T\times c}$ with no external I_c's, and (c) and (d) are the reducible diagrams generated by $\mathcal{G}_T$ with one external I_c.

Examples of the energy shift formulas, corresponding to Figs. 9(c) and 9(a), are

$$\begin{aligned}\Delta E_{9c} &= \int \frac{d\epsilon}{-2\pi i}\langle\Phi_c \mid I_c \mathscr{L}_{-+} S_{1-} S_{2+} \mathscr{G}_T (S_{1+} + S_{2+}) \mid \Phi_c\rangle \\ &= \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta}\frac{d\bar{k}'}{\bar{k}'^2}\langle\Phi_c(\bar{p}_1\bar{p}_2) \mid \\ &\quad \times \frac{\mathscr{L}_{1-}(\bar{p}_1 - \bar{k}')\,\mathscr{L}_{2+}(\bar{p}_2 + \bar{k}')}{E_c + \mathscr{E}(\bar{p}_1 - \bar{k}') - \mathscr{E}(\bar{p}_2 + \bar{k}')}(S_{1-}(\bar{p}_1 - \bar{k}'\epsilon) + S_{2+}(\bar{p}_2 + \bar{k}'\epsilon)) \\ &\quad \times \alpha_1{}^i\alpha_2{}^i(S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega) + S_{2+}(\bar{p}_2 + \bar{k} + \bar{k}'\epsilon - \omega)) \\ &\quad \times \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle \qquad (5.6)\end{aligned}$$

and

$$\begin{aligned}\Delta E_{9a} &= \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta}\frac{d\bar{k}'}{\bar{k}'^2}\langle\Phi_c(\bar{p}_1\bar{p}_2) \mid \\ &\quad \times (S_{1+}(\bar{p}_1, \epsilon) + S_{2+}(\bar{p}_2\epsilon))\,\mathscr{L}_{1-}(\bar{p}_1 - \bar{k}')\,S_{1-}(\bar{p}_1 - \bar{k}'\epsilon - \omega') \\ &\quad \times \alpha_1{}^i\alpha_2{}^i\mathscr{L}_{2+}(\bar{p}_2 + \bar{k})\,S_{2+}(\bar{p}_2 + \bar{k}\epsilon - \omega)(S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega - \omega') \\ &\quad + S_{2+}(\bar{p}_2 + \bar{k} + \bar{k}'\epsilon - \omega - \omega')) \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle, \qquad (5.7)\end{aligned}$$

with k and k' the momenta transferred by the transverse and Coulomb interactions. The ϵ integration dives zero immediately for the $S_{1-}S_{2+}$, $S_{2+}S_{2+}$ terms of ΔE_{9c} using Eq. (3.22b). Resolving the transverse photon propagator into simple poles with

$$\frac{1}{\omega^2 - \bar{k}^2 + i\Delta} = \frac{1}{2k}\left[\frac{1}{\omega - k + i\Delta} - \frac{1}{\omega + k - i\Delta}\right], \qquad k \equiv \mid \bar{k} \mid \qquad (5.8)$$

the $S_{2+}(\bar{p}_2 + \bar{k}'\epsilon)\,S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega)$ term can be integrated exactly with the result

$$\begin{aligned}\Delta E'_{9c} &= \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{2k}\frac{d\bar{k}'}{\bar{k}'^2}\langle\Phi_c(\bar{p}_1\bar{p}_2) \mid \frac{\mathscr{L}_{1-}(\bar{p}_1 - \bar{k}')\,\mathscr{L}_{2+}(\bar{p}_2 + \bar{k}')}{E_c + \mathscr{E}(\bar{p}_1 - \bar{k}') - \mathscr{E}(\bar{p}_2 + \bar{k}')} \\ &\quad \times \alpha_1{}^i\frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k} - \bar{k}') - \mathscr{E}(\bar{p}_2 + \bar{k}') - k} \\ &\quad \times \alpha_2{}^i \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \qquad (5.9)\end{aligned}$$

The nonrelativistic approximation is obtained by letting $E_c \to 2m$, $\mathscr{E} \to m$, $V \to 0$ and using the wave function approximation

$$\Phi_c = U_1^{-1}U_2^{-1}\varphi^{++} \cong \left(1 + \frac{\bar{\alpha}_1 \cdot \bar{p}_1}{2m}\right)\left(1 + \frac{\bar{\alpha}_2 \cdot \bar{p}_2}{2m}\right)\varphi_0\,; \tag{5.10}$$

$$\begin{aligned}\Delta E'_{9c} = &-\frac{1}{16m^3}\left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{k^2}\frac{d\bar{k}'}{k'^2}\,\langle\varphi_0(\bar{p}_1\bar{p}_2) \mid \bar{\sigma}_1 \cdot \bar{k}'\sigma_1{}^i \\ &\times [\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k}')\,\sigma_2{}^i + \sigma_2{}^i\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k} + \bar{k}')] \\ &\times \mid \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle,\end{aligned} \tag{5.11}$$

of nominal order $\alpha^6 m$. Again it is found that the integral is divergent for singlets but convergent for triplets. The remaining term of ΔE_{9c}, $S_{1-}S_{1+}$, cannot be integrated exactly due to noncommutativity of operators, so we make the approximation of dropping external potentials, finding for the denominator product replacing that in Eq. (5.9)

$$\frac{1}{E_c + E_{p_1-k'} - E_{p_2+k'}} \cdot \frac{1}{E_{p_1-k'} + E_{p_1-k-k'} + k},$$

which approaches $1/4m^2$ for small momenta. This term is thus of nominal order $\alpha^7 m$ and is neglected. For basically the same reason the diagrams with one Coulomb photon and two pairs are negligible.

The sum of all the single pair Coulomb-transverse diagrams can be written compactly in sixteen component notation as

$$\begin{aligned}\Delta E^{TC}_{(-+)} = &\left(\frac{\alpha}{2\pi^2}\right)^2\left(\frac{-1}{4m}\right)\int \frac{d\bar{k}}{k^2}\frac{d\bar{k}'}{k'^2}\,\langle\Phi_c(\bar{p}_1\bar{p}_2)\mid [\alpha_1{}^i\Lambda_{1-}(\bar{p}_1 - \bar{k}) + \Lambda_{1-}(\bar{p}_1 - \bar{k}')\,\alpha_1{}^i] \\ &\times [\alpha_2{}^i\Lambda_{2+}(\bar{p}_2 + \bar{k}) + \Lambda_{2+}(\bar{p}_2 + \bar{k}')\,\alpha_2{}^i] \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle,\end{aligned} \tag{5.12}$$

$$\begin{aligned}\Delta E^{TC}_{(+-)} = &\left(\frac{\alpha}{2\pi^2}\right)^2\left(\frac{-1}{4m}\right)\int \frac{d\bar{k}}{k^2}\frac{d\bar{k}'}{k'^2}\,\langle\Phi_c(\bar{p}_1\bar{p}_2)\mid [\alpha_1{}^i\Lambda_{1+}(\bar{p}_1 - \bar{k}) + \Lambda_{1+}(\bar{p}_1 - \bar{k}')\,\alpha_1{}^i] \\ &\times [\alpha_2{}^i\Lambda_{2-}(\bar{p}_2 + \bar{k}) + \Lambda_{2-}(\bar{p}_2 + \bar{k}')\,\alpha_2{}^i] \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle,\end{aligned} \tag{5.13}$$

where $\Delta E^{TC}_{(-+)}$ refers to the diagrams of Fig. 9 and $\Delta E^{TC}_{(+-)}$ to the corresponding set

with the negative energy state on the path of particle two. These are equal, and so we present the Pauli form of $\Delta E^{TC}_{(-+)}$ and double;

$$\Delta E^{TC}_{(-+)} + \Delta E^{TC}_{(+-)}$$
$$= \left(\frac{\alpha}{2\pi^2}\right)^2 \left(\frac{-1}{8m^3}\right) \int \frac{d\bar{k}}{\bar{k}^2} \frac{d\bar{k}'}{\bar{k}'^2} \langle \varphi_0(\bar{p}_1 \bar{p}_2) \mid [\bar{\sigma}_1 \cdot \bar{k}' \sigma_1{}^i - \sigma_1{}^i \bar{\sigma}_1 \cdot \bar{k}']$$
$$\times [2(\bar{\sigma}_2 \cdot \bar{p}_2 \sigma_2{}^i + \sigma_2{}^i \bar{\sigma}_2 \cdot \bar{p}_2) + 2\sigma_2{}^i \bar{\sigma}_2 \cdot \bar{k} + (\bar{\sigma}_2 \cdot \bar{k}' \sigma_2{}^i + \sigma_2{}^i \bar{\sigma}_2 \cdot \bar{k}')]$$
$$\times \mid \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}' \; \bar{p}_2 + \bar{k} + \bar{k}') \rangle. \tag{5.14}$$

Using Eq. (4.32) to eliminate surplus spin matrices and the formula

$$\sum_{i=1}^{2} A_i B_i = \bar{A} \cdot \bar{B} - \bar{A} \cdot \hat{k} \; \bar{B} \cdot \hat{k}$$

to eliminate transverse operators yields

$$\Delta E^{TC}_{(-+)} + \Delta E^{TC}_{(+-)}$$
$$= \left(\frac{\alpha}{2\pi^2}\right)^2 \left(\frac{-1}{8m^3}\right) \int \frac{d\bar{k}}{\bar{k}^2} \frac{d\bar{k}'}{\bar{k}'^2} \langle \varphi_0(\bar{p}_1 \bar{p}_2) \mid$$
$$\times [8i\{\bar{p}_2 \cdot (\bar{\sigma}_1 \times \bar{k}') - \bar{p}_2 \cdot \hat{k}(\bar{\sigma}_1 \times \bar{k}') \cdot \hat{k}\} - 4i(\bar{\sigma}_1 \times \bar{k}') \cdot \hat{k}\bar{k}' \cdot \hat{k}$$
$$+ 4(\bar{\sigma}_1 \times \bar{k}') \cdot (\bar{\sigma}_2 \times \bar{k}) \mid \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}' \; \bar{p}_2 + \bar{k} + \bar{k}') \rangle. \tag{5.15}$$

The no pair diagrams of Fig. 10 also contribute to the splitting. The simplest

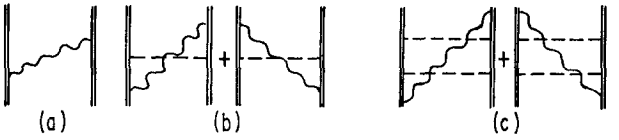

FIG. 10. No pair Coulomb transverse diagrams.

diagram to calculate is the single photon exchange term $\Delta E^{T}_{(++++)}$, Fig. 10(a), given by

$$\Delta E^{T}_{(++++)}$$
$$= \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta} \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid [S_{1+}(\bar{p}, \epsilon) + S_{2+}(\bar{p}_2 \epsilon)]$$
$$\times \alpha_1{}^i \alpha_2{}^i [S_{1+}(\bar{p}_1 - \bar{k}\epsilon - \omega) + S_{2+}(\bar{p}_2 + \bar{k}\epsilon - \omega)] \mid \Phi_c(\bar{p}_1 - \bar{k} \; \bar{p}_2 + \bar{k}) \rangle. \tag{5.16}$$

Using the methods described for the TC diagrams we find exactly

$$\Delta E^{\tau}_{(++++)} = \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{2k} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \alpha_1{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k} \alpha_2{}^i$$
$$+ \alpha_2{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k} \alpha_1{}^i \mid \Phi_c(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle$$
$$\equiv \langle \Phi_c \mid B \mid \Phi_c \rangle, \tag{5.17}$$

with B the "Breit operator." The order of the $\alpha_1{}^i$, $\alpha_2{}^i$ takes account of the fact that $[\alpha^i, \mathscr{E}] \neq 0$ and we could just as well have $\mathscr{E} \to H$ due to the manner in which the operators are ordered with respect to the wave functions.

Even though it is claimed that it is necessary to include only the diagrams of Fig. 10, it is convenient to make use of the result of Sucher [10] for the sum of diagrams with an arbitrary number of uncrossed Coulomb photons exchanged and no pairs in intermediate states

$$\Delta E^{\tau}_{(++++)}$$
$$= \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{2k} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid$$
$$\times \alpha_1{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - \mathscr{L}_{++}(\bar{p}_1 - \bar{k}\bar{p}_2)\, I_c \mathscr{L}_{++}(\bar{p}_1 - \bar{k}\bar{p}_2) - k} \alpha_2{}^i$$
$$+ \alpha_2{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - \mathscr{L}_{++}(\bar{p}_1\bar{p}_2 + \bar{k})\, I_c \mathscr{L}_{++}(\bar{p}_1\bar{p}_2 + \bar{k}) - k} \alpha_1{}^i$$
$$\times \mid \Phi_c(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle. \tag{5.18}$$

Reexpansion of this result in I_c yields exactly each diagram going into the sum.

The relation

$$\Phi_c = U_1^{-1} U_2^{-1} \varphi^{++} \tag{5.19}$$

may be used to rewrite this in the exact reduced form

$$\Delta E^{\tau}_{(++++)} = \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{2k} \langle \varphi^{++}(\bar{p}_1\bar{p}_2) \mid \Big\{ [U_1(\bar{p}_1)\, \alpha_1{}^i U_1^{-1}(\bar{p}_1 - \bar{k})]^{++}$$
$$\times \frac{1}{E_c - H_{\mathrm{red}}(\bar{p}_1 - \bar{k}\bar{p}_2) - k} [U_2(\bar{p}_2)\, \alpha_2{}^i U_2^{-1}(\bar{p}_2 + \bar{k})]^{++}$$
$$+ [U_2(\bar{p}_2)\, \alpha_2{}^i U_2^{-1}(\bar{p}_2 + \bar{k})]^{++} \frac{1}{E_c - H_{\mathrm{red}}(\bar{p}_1\bar{p}_2 + \bar{k}) - k}$$
$$\times [U_1(\bar{p}_1)\, \alpha_1{}^i U_1^{-1}(\bar{p}_1 - \bar{k})]^{++} \Big\} \mid \varphi^{++}(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle, \tag{5.20}$$

with H_{red} given by Eq. (4.22). Defining the order V term of the transformed α^i matrix as

$$
\begin{aligned}
A_1{}^i = {} & [U_1(\bar{p}_1)\,\alpha_1{}^i U_1^{-1}(\bar{p}_1 - \bar{k})]_{V_1}^{++} \\
& \times \Bigg[\frac{\bar{\sigma}_1 \cdot \bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\,\frac{V_1'}{2m}\left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\sigma_1{}^i\left(\frac{E_{p_1-k}+m}{2E_{p_1-k}}\right)^{1/2} \\
& + \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\sigma_1{}^i\left(\frac{E_{p_1-k}+m}{2E_{p_1-k}}\right)^{1/2}\frac{V_1'}{2m}\,\frac{\bar{\sigma}_1\cdot(\bar{p}_1-\bar{k})}{(2E_{p_1-k}(E_{p_1-k}+m))^{1/2}}\Bigg] \\
& - \Bigg[\left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\frac{V_1'}{2m}\,\frac{\bar{\sigma}_1\cdot\bar{p}_1}{(2E_{p_1}(E_{p_1}+m))^{1/2}}\,\sigma_1{}^i\left(\frac{E_{p_1-k}+m}{2E_{p_1-k}}\right)^{1/2} \\
& + \left(\frac{E_{p_1}+m}{2E_{p_1}}\right)^{1/2}\sigma_1{}^i\,\frac{\bar{\sigma}_1\cdot(\bar{p}_1-\bar{k})}{(2E_{p_1-k}(E_{p_1-k}+m))^{1/2}}\,\frac{V_1'}{2m}\left(\frac{E_{p_1-k}+m}{2E_{p_1-k}}\right)^{1/2}\Bigg]
\end{aligned}
\tag{5.21}
$$

with a corresponding expression for

$$A_2{}^i = [U_2(\bar{p}_2)\,\alpha_2{}^i U_2^{-1}(\bar{p}_2 + \bar{k})]_{V_2}^{++}$$

(terms higher order in V are negligible) the energy shift may be written

$$
\begin{aligned}
\Delta E^{\tau}_{(++++)} = {} & \left(\frac{\alpha}{2\pi^2}\right)\int\frac{d\bar{k}}{2k}\,\langle\varphi^{++}(\bar{p}_1\bar{p}_2)\mid R_1{}^i\,\frac{1}{E_c - H_{\text{red}}(\bar{p}_1-\bar{k},\,\bar{p}_2)}\,R_2{}^i \\
& + R_2{}^i\,\frac{1}{E_c - H(p_1,\,p_2+k) - k}\,R_1{}^i \mid \varphi^{++}(\bar{p}_1-\bar{k},\,\bar{p}_2+\bar{k})\rangle,
\end{aligned}
\tag{5.22a}
$$

where

$$
\begin{aligned}
R_1{}^i &= \frac{\bar{\sigma}_1\cdot\bar{p}_1\sigma_1{}^i(E_{p_1-k}+m) + (E_{p_1}+m)\,\sigma_1{}^i\bar{\sigma}_1\cdot(\bar{p}_1-\bar{k})}{(2E_{p_1}(E_{p_1}+m))^{1/2}\,(2E_{p_1-k}(E_{p_1-k}+m))^{1/2}} + A_1{}^i, \\
R_2{}^i &= \frac{\bar{\sigma}_2\cdot\bar{p}_2\sigma_2{}^i(E_{p_2+k}+m) + (E_{p_2}+m)\,\sigma_2{}^i\bar{\sigma}_2\cdot(\bar{p}_2+k)}{(2E_{p_1}(E_{p_2}+m))^{1/2}\,(2E_{p_2+k}(E_{p_2+k}+m))^{1/2}} + A_2{}^i.
\end{aligned}
\tag{5.22b}
$$

Terms of the form V_1V_2 may also be dropped. The roughest approximation to this is the order $\alpha^4 m$ Breit energy, obtained by dropping the A^i's, letting Ep's $\to m$, neglecting $E_c - H_{\text{red}}$ compared to k and using φ_0 for φ^{++}:

$$
\begin{aligned}
\Delta E_B^{(4)} = {} & -\left(\frac{\alpha}{2\pi^2}\right)\int\frac{d\bar{k}}{k^2}\,\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid\frac{1}{4m^2}\,[\bar{\sigma}_1\cdot\bar{p}_1\bar{\sigma}_1{}^i + \sigma_1{}^i\bar{\sigma}_1\cdot(\bar{p}_1-\bar{k})] \\
& \times[\bar{\sigma}_2\cdot\bar{p}_2\sigma_2{}^i + \sigma_2{}^i\sigma_2\cdot(\bar{p}_2+\bar{k})]\mid\varphi_0(\bar{p}_1-\bar{k},\,\bar{p}_2+\bar{k})\rangle \\
= {} & -\frac{1}{4m^2}\left(\frac{\alpha}{2\pi^2}\right)\int\frac{d\bar{k}}{k^2}\,\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid[4(\bar{p}_1\cdot\bar{p}_2 - \bar{p}_1\cdot\hat{k}\,\bar{p}_2\cdot\hat{k}) \\
& - 2i\bar{p}_1\cdot(\bar{\sigma}_2\times\bar{k}) + 2i\bar{p}_2\cdot(\bar{\sigma}_1\times\bar{k}) + (\bar{\sigma}_1\times\bar{k})\cdot(\bar{\sigma}_2\times\bar{k})] \\
& \times\mid\varphi_0(\bar{p}_1-\bar{k},\,\bar{p}_2+\bar{k})\rangle \\
= {} & \langle\varphi_0\mid H_{\text{retardation}}\mid\varphi_0\rangle \equiv \langle\varphi_0\mid B_{\text{red}}\mid\varphi_0\rangle \propto \alpha^4 m,
\end{aligned}
\tag{5.23}
$$

which defines the momentum space form of the well known "reduced Breit operator," which is precisely $H_{\text{retardation}}$, Eq. (1.4). The refinement of all four of the above mentioned approximations yields α^2 corrections to $\Delta E_B^{(4)}$; however, each correction can be made independently. First, we notice that the fine structure terms contained in $(E_c - H_{\text{red}})$ are of order $\alpha^4 m$, i.e., of order α^3 compared to $k \sim \alpha m$, and thus to order $\alpha^6 m$ in the shift we can replace $(E_c - H_{\text{red}})$ by $(W_0 - H_0)$.

To proceed, the propagator is decomposed as

$$\frac{1}{W_0 - H_0 - k} = -\frac{1}{k} - \frac{W_0 - H_0}{k^2} - \frac{(W_0 - H_0)^2}{k^2[k - (W_0 - H_0)]}$$
$$\equiv P_1 + P_2 + P_3 . \tag{5.24}$$

P_1 yields $\Delta E_B^{(4)}$, and the order V and order $\bar{p}^2$ corrections to the reduced α^i matrices as well as the α^2 correction to φ_0 given by Eq. (4.31) must be included. These are not necessary for P_2 and P_3. The $\bar{p}^2$ and V corrections to P_1 are found to be (calculating for one particle and doubling)

$$\Delta E_B(\bar{p}^2) = \frac{1}{16m^4}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2} \langle \varphi_0(\bar{p}_1\bar{p}_2) \mid [\bar{\sigma}_2 \cdot \bar{p}_2\sigma_2{}^i + \sigma_2{}^i\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k})]$$
$$\times [\bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^i((\bar{p}_1 - \bar{k})^2 + 3\bar{p}_1{}^2) + \sigma_1{}^i\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k})(\bar{p}_1{}^2 + 3(\bar{p}_1 - \bar{k})^2)]$$
$$\times \mid \varphi_0(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle \tag{5.25}$$

$$\Delta E_B(V) = \frac{-1}{4m^3}\left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{\bar{k}^2} \langle \varphi_0(\bar{p}_1\bar{p}_2) \mid [\bar{\sigma}_2 \cdot \bar{p}_2\sigma_2{}^i + \sigma_2{}^i\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k})]$$
$$\times (\sigma_1{}^i[V_1, \bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k})] - [V_1, \bar{\sigma}_1 \cdot \bar{p}_1]\,\sigma_1{}^i) \mid \varphi_0(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle. \tag{5.26}$$

The corrections to φ_0 yield an expression corresponding to the second order perturbation theory cross term between the Coulomb fine structure of Section 4, and the retardation fine structure

$$\Delta E_B(\Delta\varphi) = \sum_n{}' \frac{\langle \varphi_0 \mid \Delta H_c^{(4)} \mid \varphi_n\rangle\langle \varphi_n \mid B_{\text{red}} \mid \varphi_0\rangle + \langle \varphi_0 \mid B_{\text{red}} \mid \varphi_n\rangle\langle \varphi_n \mid \Delta H_c^{(4)} \mid \varphi_0\rangle}{W_0 - W_n} . \tag{5.27}$$

P_2 gives an integral of nominal order $\alpha^5 m$. The spin dependent and spin independent parts may be separated with the use of

$$\bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^i + \sigma_1{}^i\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k}) = 2p_1{}^i + i(\bar{\sigma}_1 \times \bar{k})^i,$$
$$\bar{\sigma}_2 \cdot \bar{p}_2\sigma_2{}^i + \sigma_2{}^i\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k}) = 2p_2{}^i - i(\bar{\sigma}_2 \times \bar{k})^i. \tag{5.28}$$

It is seen that the spin independent $p_1{}^i p_2{}^i$ terms of both P_2 and P_3 diverge as $k \to 0$, however their sum does not. Since the spin dependent terms contain at least one extra power of k the separation into P_2 and P_3 is useful. One finds, by using the equation for φ_0, that the P_2 "direct" and "exchange" (see Eq. (5.22a)) spin dependent terms combine to give zero. P_3 is of nominal order $\alpha^6 m$. (The $I_c{}^2$ term in $(W_0 - H_0)^2$ comes originally from the diagrams with two Coulomb photons—these are the only three photon terms that must be included in the calculation.) For spin dependent terms $W_0 - H_0$ may be neglected compared to k in the denominator without incurring an infrared divergence (this eliminates higher order crossed Coulomb diagrams), yielding the "pure" $\alpha^6 m$ "recoil" energy shift

$$\begin{aligned}
\Delta E_R = &- \left(\frac{\alpha}{2\pi^2}\right) \frac{1}{2m^2} \int \frac{d\bar{k}}{\bar{k}^4} \langle \varphi_0(p_1 p_2) | \\
&\times \{-(i/2)(\bar{\sigma}_2 \times \bar{k})[\bar{p}_1(W_0 - H_0(\bar{p}_1 - \bar{k}\,\bar{p}_2))^2 + (W_0 - H_0(\bar{p}_1 \bar{p}_2 + \bar{k}))^2\, \bar{p}_1] \\
&+ (i/2)(\bar{\sigma}_1 \times \bar{k})[\bar{p}_2(W_0 - H_0(\bar{p}_1 \bar{p}_2 + \bar{k}))^2 + (W_0 - H_0(\bar{p}_1 - \bar{k}\,\bar{p}_2))^2\, \bar{p}_2] \\
&+ \tfrac{1}{4}(\bar{\sigma}_1 \times \bar{k}) \cdot (\bar{\sigma}_2 \times \bar{k})[(W_0 - H_0(\bar{p}_1 - \bar{k}\,\bar{p}_2))^2 + (W_0 - H_0(\bar{p}_1 \bar{p}_2 + \bar{k}))^2] \\
&\times | \varphi_0(\bar{p}_1 - \bar{k}\,\bar{p}_2 + \bar{k}) \rangle \\
\equiv\ & \Delta E_{R,\bar{\sigma}_2} + \Delta E_{R,\bar{\sigma}_1} + \Delta E_{R,\bar{\sigma}_1 \bar{\sigma}_2}\,. \qquad (5.29)
\end{aligned}$$

It has been found convenient to use the equation for φ_0 to manipulate ΔE_R into a form in which the potential operators in H_0 appear only linearly, with the result

$$\begin{aligned}
\Delta E_{R,\bar{\sigma}_1} = &\left(\frac{\alpha}{2\pi^2}\right) \frac{-1}{2m^2} \int \frac{d\bar{k}}{\bar{k}^4} \langle \varphi_0(\bar{p}_1 \bar{p}_2) | \\
&\times \frac{i}{2}(\bar{\sigma}_1 \times \bar{k}) \cdot \left(-\left\{[\bar{p}_2\,, F], \frac{(\bar{p}_1 - \bar{k})^2}{2m}\right\} + \left\{[\bar{p}_2\,, F], \frac{\bar{p}_1{}^2}{2m}\right\}\right. \\
&+ \bar{p}_2 \left(-2\,\frac{\bar{p}_1{}^2}{2m}\,\frac{\bar{p}_2{}^2}{2m} - 2\,\frac{(\bar{p}_1 - \bar{k})^2}{2m}\,\frac{(\bar{p}_2 + \bar{k})^2}{2m} + 2\,\frac{(\bar{p}_1 - \bar{k})^2}{2m}\,\frac{\bar{p}_2{}^2}{2m}\right. \\
&\left.\left. + 2\,\frac{\bar{p}_1{}^2}{2m}\,\frac{(\bar{p}_2 + \bar{k})^2}{2m}\right)\right) | \varphi_0(\bar{p}_1 - \bar{k}\,\bar{p}_2 + \bar{k}) \rangle, \qquad (5.30)
\end{aligned}$$

where

$$F = V_1 + V_2 + I_c\,, \qquad (5.31)$$

and an equivalent form for $\Delta E_{R,\bar{\sigma}_2}$. The potential terms drop out of $\Delta E_{R,\bar{\sigma}_1\bar{\sigma}_2}$ and the result is

$$\Delta E_{R,\bar{\sigma}_1\bar{\sigma}_2} = \left(\frac{\alpha}{2\pi^2}\right)\frac{-1}{2m^2}\int\frac{d\bar{k}}{\bar{k}^4}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid \tfrac{1}{4}(\bar{\sigma}_1\times\bar{k})\cdot(\bar{\sigma}_2\times\bar{k})$$
$$\times\left(-2\frac{\bar{p}_1^{\,2}}{2m}\frac{\bar{p}_2^{\,2}}{2m} - 2\frac{(\bar{p}_1-\bar{k})^2}{2m}\frac{(\bar{p}_2+\bar{k})^2}{2m} + 2\frac{\bar{p}_1^{\,2}}{2m}\frac{(\bar{p}_2+\bar{k})^2}{2m}\right.$$
$$\left. + 2\frac{(\bar{p}_1-\bar{k})^2}{2m}\frac{\bar{p}_2^{\,2}}{2m}\right)\mid\varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle. \tag{5.32}$$

To summarize the results of this section we write the total single transverse photon exchange contribution to the fine structure splitting as

$$\Delta W_T = \Delta W_T^{(4)} + \Delta W_T^{(6)}, \tag{5.33}$$

where $\Delta W_T^{(4)}$ is the familiar $\alpha^4 m$ term and $\Delta W_T^{(6)}$ is the desired $\alpha^6 m$ correction. Repeating Eq. (5.23) we have

$$\Delta W_T^{(4)} \equiv \Delta E_B^{(4)} = \langle\varphi_0\mid B_{\text{red}}\mid\varphi_0\rangle$$
$$= \left(\frac{\alpha}{2\pi^2}\right)\frac{-1}{4m^2}\int\frac{d\bar{k}}{\bar{k}^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid 4(\bar{p}_1\cdot\bar{p}_2 - \bar{p}_1\cdot\hat{k}\,\bar{p}_2\cdot\hat{k})$$
$$+ 2i\bar{p}_2\cdot(\bar{\sigma}_1\times\bar{k}) - 2i\bar{p}_1\cdot(\bar{\sigma}_2\times\bar{k}) + (\bar{\sigma}_1\times\bar{k})\cdot(\bar{\sigma}_2\times\bar{k})$$
$$\times\mid\varphi_0(\bar{p}_1-\bar{k}\,\bar{p}_2+\bar{k})\rangle$$
$$\equiv \sum_{i=1}^{4}\Delta W_T^{(4)}(i) \tag{5.34}$$

(the spin independent term has been retained for completeness). The numerous contributions to

$$\Delta W_T^{(6)} = \sum_{i=1}^{8}\Delta W_T^{(6)}(i) \tag{5.35}$$

are given as follows:

(1) $\Delta W_T^{(6)}(1)$ is defined as the second order perturbation theory cross term between $\Delta H_c^{(4)}$ and B_{red} and is given by Eq. (5.27)

$$\Delta W_T^{(6)}(1) \equiv \Delta E_B(\Delta\varphi). \tag{5.36}$$

(2) $\Delta W_T^{(6)}(2)$ comes from the $\bar{p}^2$ corrections to reduced α^i matrices and is given by Eq. (5.25), and with transverse operators removed is

$$\begin{aligned}\Delta W_T^{(6)}(2) &\equiv \Delta E_B(\bar{p}^2)\\ &= \left(\frac{\alpha}{2\pi^2}\right)\frac{1}{16m^4}\int\frac{d\bar{k}}{\bar{k}^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\,|\,-i\bar{p}_1\cdot(\bar{\sigma}_2\times\bar{k})(4\bar{p}_1^2+4(\bar{p}_1-\bar{k})^2)\\ &\quad+2i\bar{p}_2\cdot(\bar{\sigma}_1\times\bar{k})(\bar{p}_1^2+3(\bar{p}_1-\bar{k})^2)\\ &\quad+2i\{\bar{p}_2\cdot(\bar{\sigma}_1\times\bar{p}_1)-\bar{p}_2\cdot\hat{k}(\bar{\sigma}_1\times\bar{p})\cdot\hat{k}\}(2\bar{p}_1^2-2(\bar{p}_1-\bar{k})^2)\\ &\quad+(\bar{\sigma}_1\times\bar{p}_1)\cdot(\bar{\sigma}_2\times\bar{k})(2\bar{p}_1^2-2(\bar{p}_1-\bar{k})^2)\\ &\quad+(\bar{\sigma}_1\times\bar{k})\cdot(\bar{\sigma}_2\times\bar{k})(\bar{p}_1^2+3(\bar{p}_1-\bar{k})^2)\,|\,\varphi_0(\bar{p}_1-\bar{k}\;\bar{p}_2+\bar{k})\rangle.\end{aligned} \tag{5.37}$$

(3) $\Delta W_T^{(6)}(3)$ is generated by external potential corrections to the reduced α^i matrices and is given by Eq. (5.26), and with the external potential expressed in integral form and transverse operators removed is

$$\begin{aligned}\Delta W_T^{(6)}(3) \equiv \Delta E_B(V) = \left(\frac{-Z\alpha}{2\pi^2}\right)\left(\frac{\alpha}{2\pi^2}\right)\frac{-1}{4m^3}\int\frac{d\bar{k}}{\bar{k}^2}\frac{d\bar{k}_1}{\bar{k}_1^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\,|\\ \times[4i\{\bar{p}_2\cdot(\bar{\sigma}_1\times\bar{k}_1)-\bar{p}_2\cdot\hat{k}(\bar{\sigma}_1\times\bar{k}_1)\cdot\hat{k}\}\\ +2(\bar{\sigma}_2\times\bar{k})\cdot(\bar{\sigma}_1\times\bar{k}_1)]\,|\,\varphi_0(\bar{p}_1-\bar{k}-\bar{k}_1\;\bar{p}_2+\bar{k})\rangle.\end{aligned} \tag{5.38}$$

(4) $\Delta W_T^{(6)}(4)$ comes from the Coulomb-pair exchange diagrams and is found in Eq. (5.15)

$$\Delta W_T^{(6)}(4) \equiv \Delta E_{(-+)}^{TC} + \Delta E_{(+-)}^{TC}\,. \tag{5.39}$$

(5) $\Delta W_T^{(6)}(5)$ is given by the spin–spin term of the recoil correction, Eq. (5.32),

$$\Delta W_T^{(6)}(5) \equiv \Delta E_{R,\bar{\sigma}_1\bar{\sigma}_2}\,. \tag{5.40}$$

(6) $\Delta W_T^{(6)}(6)$ denotes the no potential part of the spin orbit piece of the recoil correction, Eq. (5.30). Doubling to include $\bar{\sigma}_2$.

$$\begin{aligned}\Delta W_T^{(6)}(6) = \left(\frac{\alpha}{2\pi^2}\right)\frac{-1}{m^2}\int\frac{d\bar{k}}{\bar{k}^4}\langle\varphi_0(\bar{p}_1\bar{p}_2)\,|\,\frac{i}{2}(\bar{\sigma}_1\times\bar{k})\cdot\bar{p}_2\\ \times\left(-2\frac{\bar{p}_1^2}{2m}\frac{\bar{p}_2^2}{2m}-2\frac{(\bar{p}_1-\bar{k})^2}{2m}\frac{(\bar{p}_2+\bar{k})^2}{2m}+2\frac{\bar{p}_1^2}{2m}\frac{(\bar{p}_2+\bar{k})^2}{2m}\right.\\ \left.+2\frac{(\bar{p}_1-\bar{k})^2}{2m}\frac{\bar{p}_2^2}{2m}\right)\,|\,\varphi_0(\bar{p}_1-\bar{k}\;\bar{p}_2+\bar{k})\rangle.\end{aligned} \tag{5.41}$$

(7) $\Delta W_T^{(6)}(7)$ is defined as the external potential part of the recoil correction, Eq. (5.30), and written out in integral form is, likewise doubling,

$$\Delta W_T^{(6)}(7) = \left(\frac{-Z\alpha}{2\pi^2}\right)\left(\frac{\alpha}{2\pi^2}\right)\frac{-i}{2m^3}\int \frac{d\bar{k}}{\bar{k}^4}\frac{d\bar{k}_2}{\bar{k}_2^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid(\bar{\sigma}_1\times\bar{k})$$

$$\cdot\,\bar{k}_2(-2\bar{p}_1\cdot\bar{k}+\bar{k}^2)\mid\varphi_0(\bar{p}_1-\bar{k}\;\bar{p}_2+\bar{k}+\bar{k}_2)\rangle. \tag{5.42}$$

(8) $\Delta W_T^{(6)}(8)$ denotes the I_c term in the recoil correction, Eq. (5.30), and written in integral form is, again doubling,

$$\Delta W_T^{(6)}(8) = \left(\frac{\alpha}{2\pi^2}\right)\frac{-i}{2m^3}\int \frac{d\bar{k}}{\bar{k}^4}\frac{d\bar{k}'}{\bar{k}'^2}\langle\varphi_0(\bar{p}_1\bar{p}_2)\mid(\bar{\sigma}_1\times\bar{k})$$

$$\cdot\,\bar{k}'(-2\bar{p}_1\cdot\bar{k}+\bar{k}'\cdot\bar{k}+\bar{k}^2)\mid\varphi_0(\bar{p}_1-\bar{k}-\bar{k}'\;\bar{p}_2+\bar{k}+\bar{k}')\rangle. \tag{5.43}$$

Further rearrangement and Fourier transformation of these results is carried out in Section 8.

6. Double Transverse Photon Effects

In this section we calculate the effect of the double transverse photon exchange diagrams, i.e., $J\to \mathcal{G}_{T\times T}+\mathcal{G}_T\mathcal{F}^{-1}\mathcal{G}_T$, plus the last two terms of Eq. (3.54). Thus, we define

$$\Delta E^{TT} = \int\frac{d\epsilon}{-2\pi i}\langle\Phi_c\mid I_c[\mathcal{F}^{-1}\mathcal{G}_{T\times T}\mathcal{F}^{-1}+\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}]\,I_c\mid\Phi_c\rangle$$

$$+\langle\Phi_c\mid\left[D_c\int\frac{d\epsilon}{-2\pi i}\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}I_c\right]\Gamma\left[D_c\int\frac{d\epsilon'}{-2\pi i}\mathcal{F}^{-1}\mathcal{G}_T\mathcal{F}^{-1}I_c\right]\mid\Phi_c\rangle$$

$$-\Delta E\int\frac{d\epsilon}{-2\pi i}\langle\Phi_c\mid(S_{1+}+S_{2+})\,\mathcal{G}_T(S_1+S_2)\frac{1}{D_c^2}\,I_c\mid\Phi_c\rangle$$

$$\equiv \Delta E_a^{TT}+\Delta E_b^{TT}+\Delta E_c^{TT}. \tag{6.1}$$

The terms of ΔE_a^{TT} involving one or more external pairs, and, thus, one or two extra factors of I_c are negligible, the terms without external I_c being already of

order $\alpha^6 m$ insofar as they contribute to the ^3P splitting. Thus we find, referring to Eqs. (5.2), and (5.5)

$$\begin{aligned}\Delta E_a^{TT} &= \int \frac{d\epsilon}{-2\pi i} \langle \Phi_c \mid (S_{1+} + S_{2+})\, \mathcal{G}_{T\times T}(S_{1+} + S_{2+}) \mid \Phi_c \rangle \\ &\quad + \int \frac{d\epsilon}{-2\pi i} \langle \Phi_c \mid (S_{1+} + S_{2+})\, \mathcal{G}_T \mathcal{F}^{-1} \mathcal{G}_T (S_{1+} + S_{2+}) \mid \Phi_c \rangle \\ &\equiv \Delta E^{T\times T} + \Delta E^{T\cdot T}. \end{aligned} \tag{6.2}$$

Inserting the expressions for the operators $\mathcal{G}_T$, $\mathcal{G}_{T\times T}$ we find the explicit formulas

$$\begin{aligned}\Delta E^{T\times T} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int & \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\omega'}{-2\pi i} \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta} \frac{dk'}{\omega'^2 - \bar{k}'^2 + i\Delta'} \\ & \times \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid [S_{1+}(\bar{p}_1 \epsilon) + S_{2+}(\bar{p}_2 \epsilon)] \\ & \times \alpha_1{}^j S_1(\bar{p}_1 - \bar{k}'\epsilon - \omega')\, \alpha_1{}^i \alpha_2{}^i S_2(\bar{p}_2 + \bar{k}\epsilon - \omega)\, \alpha_2{}^j \\ & \times [S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega - \omega') + S_{2+}(\bar{p}_2 + \bar{k} + \bar{k}'\epsilon - \omega - \omega')] \\ & \times \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}') \rangle \\ \equiv \Delta E^{T\times T}_{(++)} + \Delta E^{T\times T}_{(+-)} + \Delta E^{T\times T}_{(-+)} + \Delta E^{T\times T}_{(--)} \end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\Delta E^{T\cdot T} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int & \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\omega'}{-2\pi i} \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta} \frac{d\bar{k}'}{\omega'^2 - \bar{k}'^2 + i\Delta'} \\ & \times \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid [S_{1+}(\bar{p}_1 \epsilon) + S_{2+}(\bar{p}_2 \epsilon)] \\ & \times \alpha_1{}^i S_1(\bar{p}_1 - \bar{k}\epsilon - \omega)\, \alpha_1{}^j \alpha_2{}^i S_2(\bar{p}_2 + \bar{k}\epsilon - \omega)\, \alpha_2{}^j \\ & \times [S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega - \omega') + S_{2+}(\bar{p}_2 + \bar{k} + \bar{k}'\epsilon - \omega - \omega')] \\ & \times \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}') \rangle \\ \equiv \Delta E^{T\cdot T}_{(++)} + \Delta E^{T\cdot T}_{(+-)} + \Delta E^{T\cdot T}_{(-+)} + \Delta E^{T\cdot T}_{(--)}, \end{aligned} \tag{6.4}$$

where the $(\pm\pm)$ subscripts refer to the positive, negative energy decomposition of the interior propagators; the superscripts i refer to orthogonal directions perpendicular to $\bar{k}$ and j to $\bar{k}'$.

The simplest of these $T \times T$, $T \cdot T$ terms are the $(+-)$, $(-+)$ projections, and their evaluation is similar to that of the TC terms of Section 5. All external potentials may be neglected. This allows denominators to be combined straightforwardly

using Eqs. (3.22a,b). We can also let $E_p \to m$ in the resultant energy denominators, finding that as in the TC terms they reduce to the simple result

$$1/\bar{k}^2 \cdot 1/\bar{k}'^2. \tag{6.5}$$

Also the wave function approximation Eq. (5.10) is adequate. The result of these approximations is a convergent $\alpha^6 m$ integral (again convergent only for triplet states). These calculations can be summarized in a form analogous to Eq. (5.13), doubling $(+-)$ to include the equal effect of $(-+)$,

$$\begin{aligned}\Delta E^{TT}_{(+-)} + \Delta E^{TT}_{(-+)} &= (\Delta E^{T\times T}_{(+-)} + \Delta E^{T\cdot T}_{(+-)}) + (\Delta E^{T\times T}_{(-+)} + \Delta E^{T\cdot T}_{(-+)}) \\ &= \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{\bar{k}^2}\frac{d\bar{k}'}{\bar{k}'^2} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \alpha_2{}^i \Lambda_{2-}(\bar{p}_2 + \bar{k})\, \alpha_2{}^j \\ &\quad \times [\alpha_1{}^j \Lambda_{1+}(\bar{p}_1 - \bar{k}')\, \alpha_1{}^i + \alpha_1{}^i \Lambda_{1+}(\bar{p}_1 - \bar{k})\, \alpha_1{}^j] \\ &\quad \times \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \end{aligned} \tag{6.6}$$

Transforming this to F–W form using Eq. (5.10) gives the intermediate result

$$\begin{aligned}\Delta E^{TT}_{(+-)} + \Delta E^{TT}_{(-+)} &= \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{16m^3} \int \frac{d\bar{k}}{\bar{k}^2}\frac{d\bar{k}'}{\bar{k}'^2} \langle \varphi_0(\bar{p}_1\bar{p}_2) \mid (\sigma_2{}^i\sigma_2{}^j + \sigma_2{}^j\sigma_2{}^i) \\ &\quad \times [(\bar{p}_1 - \bar{k})^2\, \sigma_1{}^i\sigma_1{}^j + \bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^i\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k})\, \sigma_1{}^j \\ &\quad + \sigma_1{}^i\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k})\, \sigma_1{}^j\sigma_1 \cdot (\bar{p}_1 - \bar{k} - \bar{k}') \\ &\quad + \bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^i\sigma_1{}^j\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k} - \bar{k}')] \\ &\quad \times \mid \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \end{aligned} \tag{6.7}$$

Further manipulation yields the final result

$$\begin{aligned}&\Delta E^{TT}_{(+-)} + \Delta E^{TT}_{(-+)} \\ &= \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{8m^3} \int \frac{d\bar{k}}{\bar{k}^2}\frac{d\bar{k}'}{\bar{k}'^2} \langle \varphi_0(\bar{p}_1\bar{p}_2) \mid [8i\{\bar{p}_1 \cdot (\bar{\sigma}_1 \times \bar{k}) - p_1 \cdot \hat{k}'(\bar{\sigma}_1 \times \bar{k}) \cdot \hat{k}'\} \\ &\quad + 4i(\bar{\sigma}_1 \times \bar{k}) \cdot \hat{k}'\bar{k} \cdot \hat{k}'] \mid \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \end{aligned} \tag{6.8}$$

The term $\Delta E^{T\times T}_{(++)}$ presents an extra complication but will be found negligible.

Proceeding according to the discussion following Eq. (6.4) we find the nonrelativistic approximation analogous to Eq. (6.7)

$$\begin{aligned}\Delta E_{(++)}^{T\times T} = \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{-1}{8m^2} \int d\bar{k}\, d\bar{k}' \left(\frac{1}{\bar{k}^3\bar{k}'^2} + \frac{1}{\bar{k}^2\bar{k}'^3}\right) \langle \varphi_0(\bar{p}_1\bar{p}_2) \,|\\ \times [(\bar{p}_1 - \bar{k}')^2\, \sigma_1{}^j\sigma_1{}^i + \sigma_1{}^j\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k}')\, \sigma_1{}^i\sigma_1 \cdot (\bar{p}_1 - \bar{k} - \bar{k}')\\ + \bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^j\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k}')\, \sigma_1{}^i + \bar{\sigma}_1 \cdot \bar{p}_1\sigma_1{}^j\sigma_1{}^i\bar{\sigma}_1 \cdot (\bar{p}_1 - \bar{k} - \bar{k}')]\\ \times [(\bar{p}_2 + \bar{k})^2\, \sigma_2{}^i\sigma_2{}^j + \sigma_2{}^i\sigma_2 \cdot (\bar{p}_2 + \bar{k})\, \sigma_2{}^j\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k} + \bar{k}')\\ + \bar{\sigma}_2 \cdot \bar{p}_2\sigma_2{}^i\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k})\, \sigma_2{}^j + \bar{\sigma}_2 \cdot \bar{p}_2\sigma_2{}^i\sigma_2{}^j\bar{\sigma}_2 \cdot (\bar{p}_2 + \bar{k} + \bar{k}')]\\ \times |\, \varphi_0(\bar{p}_1 - \bar{k} - \bar{k}'\ \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \qquad (6.9)\end{aligned}$$

This expression is convergent for large values of the momenta. One finds by counting powers of momenta the nominal order of this term is $\alpha^7 m$. However, the number of inverse powers of k or k' in the factor $(1/k^3k'^2 + 1/k^2k'^3)$ causes an infrared divergence for either k or $k' \to 0$. The neglect of $E_p - m$ factors in denominators has led to these divergences, and these act as a natural cutoff for the integrals. We are led to the conclusion that ΔE_{++}^{TT} is actually of order $\alpha^7 \log \alpha m + \alpha^7 m$, but still negligible.

The nonrelativistic order of the sum $\Delta E_{--}^{TT} = \Delta E_{--}^{T\times T} + \Delta E_{--}^{T\cdot T}$ is $\alpha^5 m$ but it is spin independent, the spin matrix combination being simply $[\sigma_1{}^i\sigma_1{}^j + \sigma_1{}^j\sigma_1{}^i][\sigma_2{}^i\sigma_2{}^j + \sigma_2{}^j\sigma_2{}^i]$. Also it is found that this combination is infrared convergent. Spin dependent corrections must come from a wave function approximation more accurate than Eq. (5.10). As found in Section 5 these are of relative order α^2 and, thus, to order $\alpha^6 m$ ΔE_{--}^{TT} may be neglected.

The remaining terms ΔE_b^{TT}, ΔE_c^{TT}, and $\Delta E_{++}^{T\cdot T}$ are found to combine into a simple result. Performing the ϵ and ω integrations in the positive energy piece of $\Delta E_c^{T\cdot T}$ (pair terms involve an extra I_c and are negligible) we find

$$\begin{aligned}\Delta E_c^{TT} = -\Delta E \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{2k} \langle \Phi_c(\bar{p}_1\bar{p}_2) \,| \Big[\alpha_1{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k}\, \alpha_2{}^i\\ + \alpha_2{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k}\, \alpha_1{}^i\Big]\\ \times \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})} \,|\, \Phi_c(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})\rangle, \qquad (6.10)\end{aligned}$$

clearly of order $\alpha^6 m$.

In ΔE_b^{TT} $\mathscr{L}_{++}$'s are inserted to the left of the I_c's and the negative energy part of Γ is dropped (see Eq. (3.41)), corrections to both of these approximations being

negligible three photon operators. We are then left with a sum over the positive energy solutions to CLE

$$\Delta E_b^{TT} = \sum_{E_n>0}{}' \frac{\left\{\begin{array}{l}\langle\Phi_c \mid \int (d\epsilon/-2\pi i)\, D_c\mathscr{F}^{-1}\mathscr{G}_T\mathscr{F}^{-1}\mathscr{L}_{++}I_c \mid \Phi_n\rangle \\ \quad \times \langle\Phi_n \mid \int (d\epsilon'/-2\pi i)\, D_c\mathscr{F}^{-1}\mathscr{G}_T\mathscr{F}^{-1}\mathscr{L}_{++}I_c \mid \Phi_c\rangle\end{array}\right\}}{E_c - E_n}. \quad (6.11)$$

Using the equation for Φ_c, and $(D_n - \mathscr{L}_{++}I_c)\,\Phi_n = 0$, and also making the separation

$$D_n = D_c - (E_c - E_n) \quad (6.12)$$

we find on performing the relative energy integrations, and defining

$$\mathscr{M}(\bar{k}) = \alpha_1{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k}\, \alpha_2{}^i + \alpha_2{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k}\, \alpha_1{}^i, \quad (6.13a)$$

$$\begin{aligned}\Delta E_b^{TT} &= \sum_{E_n>0}{}' \frac{\langle\Phi_c \mid (\alpha/2\pi^2)\int (d\bar{k}/2k)\,\mathscr{M}(\bar{k}) \mid \Phi_n\rangle\langle\Phi_n \mid (\alpha/2\pi^2)\int (d\bar{k}'/2k')\,\mathscr{M}(\bar{k}') \mid \Phi_c\rangle}{E_c - E_n} \\ &\quad - \sum_{E_n>0}{}' \langle\Phi_c \mid \left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}}{2k} \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})} \mid \Phi_n\rangle \\ &\quad \times \langle\Phi_n \mid \left(\frac{\alpha}{2\pi^2}\right)\int \frac{d\bar{k}'}{2k'}\,\mathscr{M}(\bar{k}') \mid \Phi_c\rangle \\ &\equiv \Delta E_B^{(2)} + \Delta E_e^{TT}. \end{aligned} \quad (6.13b)$$

The notation for the first term is chosen to denote the fact that it is precisely the "full Breit operator" B, Eq. (5.17), in second order. ΔE_e^{TT} is dealt with as follows: in the left side matrix element we make the shift $\bar{p}_1 \to \bar{p}_1 + \bar{k}$, $\bar{p}_2 \to \bar{p}_2 - \bar{k}$ and then use completeness of the positive energy spectrum in the form

$$\sum_{E_n>0}{}' \mid \Phi_n(\bar{p}_1\bar{p}_2)\rangle\langle\Phi_n(\bar{p}_1\bar{p}_2) \mid = \mathscr{L}_{++}(\bar{p}_1\bar{p}_2) - \mid \Phi_c(\bar{p}_1\bar{p}_2)\rangle\langle\Phi_c(\bar{p}_1\bar{p}_2) \mid; \quad (6.14)$$

p_1 and p_2 are then shifted back yielding

$$\Delta E_e^{TT} = -\left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{2k}\frac{d\bar{k}'}{2k'} \langle\Phi_c(\bar{p}_1\bar{p}_2) \mid \left\{\alpha_1{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k}\, \alpha_2{}^i + \alpha_2{}^i \frac{1}{E_c - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k}\, \alpha_1{}^i\right\} \frac{\mathscr{L}_{1+}(\bar{p}_1 - \bar{k})\,\mathscr{L}_{2+}(\bar{p}_2 + \bar{k})}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})}$$

$$\times \left\{ \alpha_1{}^j \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k} - \bar{k}') - \mathscr{E}(\bar{p}_2 + \bar{k}) - k'} \alpha_2{}^j \right.$$
$$\left. + \alpha_2{}^j \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k} + \bar{k}') - k'} \alpha_1{}^j \right\}$$
$$\times | \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}', \bar{p}_2 + \bar{k} + \bar{k}') \rangle + \langle \Phi_c | B | \Phi_c \rangle \left(\frac{\alpha}{2\pi^2}\right) \int \frac{d\bar{k}}{2k}$$
$$\times \langle \Phi_c(\bar{p}_1 \bar{p}_2) | \mathscr{M}(\bar{k}) \frac{1}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})} | \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}) \rangle. \tag{6.15}$$

Since for Φ_c we know already that

$$\Delta E = \langle \Phi_c | B | \Phi_c \rangle + \begin{cases} \mathcal{O}(\alpha^5 m) \text{ singlet}, \\ \mathcal{O}(\alpha^6 m) \text{ triplet}, \end{cases}$$

it is evident that the second term of ΔE_e^{TT} cancels ΔE_c^{TT}.

Finally, we examine the remaining piece of ΔE^{TT}, $\Delta E_{++}^{T\cdot T}$. From Eq. (6.4) we have

$$\Delta E_{(++)}^{T\cdot T} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\omega'}{-2\pi i} \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta} \frac{d\bar{k}'}{\omega'^2 - \bar{k}'^2 + i\Delta'}$$
$$\times \langle \Phi_c(\bar{p}_1 \bar{p}_2) | [S_{1+}(\bar{p}_1 \epsilon) + S_{2+}(\bar{p}_2 \epsilon)] \alpha_1{}^i S_{1+}(\bar{p}_1 - \bar{k}\epsilon - \omega)$$
$$\times \mathscr{L}_{1+}(\bar{p}_1 - \bar{k}) \alpha_1{}^j \alpha_2{}^i S_{2+}(\bar{p}_2 + \bar{k}\epsilon - \omega) \mathscr{L}_{2+}(\bar{p}_2 + \bar{k}) \alpha_2{}^j$$
$$\times [S_{1+}(\bar{p}_1 - \bar{k} - \bar{k}'\epsilon - \omega - \omega') + S_{2+}(\bar{p}_2 + \bar{k} + \bar{k}'\epsilon - \omega - \omega')]$$
$$\times | \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}', \bar{p}_2 + \bar{k} + \bar{k}') \rangle, \tag{6.16}$$

which we write out explicitly as

$$\Delta E_{(++)}^{T\cdot T} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\omega'}{-2\pi i} \frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta} \frac{dk'}{\omega'^2 - \bar{k}'^2 + i\Delta'}$$
$$\times \langle \Phi_c(\bar{p}_1 \bar{p}_2) | \left[\frac{1}{E_c/2 + \epsilon - \mathscr{E}(\bar{p}_1) + i\delta} + \frac{1}{E_c/2 - \epsilon - \mathscr{E}(\bar{p}_2) + i\delta} \right]$$
$$\times \alpha_1{}^i \alpha_2{}^i \frac{\mathscr{L}_{1+}(\bar{p}_1 - \bar{k}) \mathscr{L}_{2+}(\bar{p}_2 + \bar{k})}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})}$$
$$\times \left[\frac{1}{E_c/2 + \epsilon - \omega - \mathscr{E}(\bar{p}_1 - \bar{k}) + i\delta} \right.$$
$$\left. + \frac{1}{E_c/2 - \epsilon + \omega - \mathscr{E}(\bar{p}_2 + \bar{k}) + i\delta} \right] \alpha_1{}^j \alpha_2{}^j$$
$$\times \left[\frac{1}{E_c/2 + \epsilon - \omega - \omega' - \mathscr{E}(\bar{p}_1 - \bar{k} - \bar{k}') + i\delta} \right.$$
$$\left. + \frac{1}{E_c/2 - \epsilon + \omega + \omega' - \mathscr{E}(\bar{p}_2 + \bar{k} + \bar{k}') + i\delta} \right]$$
$$\times | \Phi_c(p_1 - k - k', p_2 + k + k') \rangle. \tag{6.17}$$

The relation Eq. (3.48b) has been used to rewrite the interior propagator expression. The ϵ, ω, and ω' integrations are facilitated by setting $\mathscr{E} = m$ in the exterior $[S_{1+} + S_{2+}]$ factors. Performing these integrations we obtain

$$\begin{aligned}\Delta E^{T\cdot T}_{(++)} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{2k}\frac{d\bar{k}'}{2\bar{k}'} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \alpha_1{}^i\alpha_2{}^i \frac{\mathscr{L}_{1+}(\bar{p}_1 - \bar{k})\,\mathscr{L}_{2+}(\bar{p}_2 + \bar{k})}{E_c - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2 + \bar{k})} \\ \times \Bigg[\frac{-1}{k + k'}\left(\frac{1}{m + \mathscr{E}(\bar{p}_1 - \bar{k}) - k'} + \frac{1}{m - \mathscr{E}(\bar{p}_2 + \bar{k}) - k'}\right) \\ - \frac{1}{k + k'}\left(\frac{1}{m + \mathscr{E}(\bar{p}_1 - \bar{k}) - k} + \frac{1}{m - \mathscr{E}(\bar{p}_2 + \bar{k})}\right) \\ + \frac{1}{m - \mathscr{E}(\bar{p}_2 + \bar{k}) - k'}\frac{1}{m - \mathscr{E}(p_2 + k) - k} \\ + \frac{1}{m - \mathscr{E}(p_1 - k) - k'}\frac{1}{m - \mathscr{E}(p_1 - k) - k}\Bigg] \\ \times \alpha_1{}^j\alpha_2{}^j \mid \Phi_c(\bar{p}_1 - \bar{k} - \bar{k}',\, \bar{p}_2 + \bar{k} + \bar{k}')\rangle. \end{aligned} \tag{6.18}$$

Evaluating this nonrelativistically (giving an $\alpha^6 m$ contribution to triplet states) one makes the approximation

$$\frac{1}{m - \mathscr{E} - k} \to -\frac{1}{k}, \qquad \frac{1}{m - \mathscr{E} - k'} \to -\frac{1}{k'}$$

and likewise in the first term of ΔE_e^{TT}, Eq. (6.15). These are then seen to cancel.

We have arrived at the pleasing result that for triplet states to order $\alpha^6 m$

$$\Delta E^{T\cdot T}_{(++)} + \Delta E_b^{TT} + \Delta E_c^{TT} = \Delta E_B^{(2)}, \tag{6.19}$$

where

$$\Delta E_B^{(2)} = \sum_{E_n>0}{}' \frac{\langle \Phi_c \mid B \mid \Phi_n\rangle\langle \Phi_n \mid B \mid \Phi_c\rangle}{E_c - E_n}. \tag{6.20}$$

This can be expressed in terms of the Pauli-type wave function $\Phi_c = U_1^{-1}U_2^{-1}\varphi^{++}$ by introduction of the usual F–W transformation, i.e.,

$$\Delta E_B^{(2)} = \sum_{E_n>0}{}' \frac{\langle \varphi^{++} \mid [U_1U_2BU_1^{-1}U_2^{-1}]^{++++} \mid \varphi_n^{++}\rangle\langle \varphi_n^{++} \mid [U_1U_2BU_1^{-1}U_2^{-1}]^{++++} \mid \varphi^{++}\rangle}{E_c - E_n} \tag{6.21}$$

The nonrelativistic approximation to $\Delta E_B^{(2)}$ is obtained by replacing φ_n^{++}'s by

φ_n's, neglecting $E - \mathscr{E}(1) - \mathscr{E}(2)$ compared to k in the denominator of B, letting $E_c - E_n \to W_0 - W_n$, and using

$$U_1^{-1}U_2^{-1} \cong (1 + \bar{\alpha}_1 \cdot \bar{p}_1/2m)(1 + \bar{\alpha}_2 \cdot \bar{p}_2/2m).$$

One then arrives at

$$\Delta E_B^{(2)} \cong \sum_n{}' \frac{\langle \varphi_0 \mid B_{\text{red}} \mid \varphi_n \rangle \langle \varphi_n \mid B_{\text{red}} \mid \varphi_0 \rangle}{W_0 - W_n}, \tag{6.22}$$

the reduced Breit operator in second order. Power counting arguments as well as the successful calculation of this sum in coordinate space by Hambro [25] indicate that this sum is convergent and thus also that corrections to the approximations listed above are of higher order than $\alpha^6 m$.

We have thus arrived at the simple result for the sum of all two transverse photon contributions to the triplet P splitting

$$\Delta W_{TT}^{(6)} = \Delta W_{TT}^{(6)}(1) + \Delta W_{TT}^{(6)}(2) \tag{6.23}$$

with $\Delta W_{TT}^{(6)}(1)$ defined as the second order perturbation sum of Eq. (6.22) (see Eq. (5.23) for the definition of B_{red}), and $\Delta W_{TT}^{(6)}(2)$ denoting the pair terms of Eq. (6.8),

$$\Delta W_{TT}^{(6)}(2) = \Delta E_{(+-)}^{TT} + \Delta E_{(-+)}^{TT}. \tag{6.24}$$

We note for completeness that the contribution of $\mathscr{G}_{c\times c}$ (double Coulomb photon exchange, Fig. 5) can be analyzed by methods identical in principle but much simpler in practice to those applied to $\mathscr{G}_{T\times T}$. While the spin dependent terms of the $(++, ++)$ (i.e., no external pair) part are nominally of order $\alpha^6 mc^2$ they take the form

$$\Delta E^{c\times c} = -\frac{i}{8m^3}\left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\bar{k}}{\bar{k}^2}\frac{d\bar{k}'}{\bar{k}'^2} \langle \varphi_0(\bar{p}_1\bar{p}_2) \mid (\bar{\sigma}_1 - \bar{\sigma}_2) \cdot (\bar{k} \times \bar{k}') \mid \varphi_0(p_1 - k - k', \bar{p}_2 + \bar{k} + \bar{k}') \rangle, \tag{6.25}$$

and hence (on account of $\bar{k}$, $\bar{k}'$ antisymmetry) vanish.

7. Radiative Corrections

The rigorous treatment of $\mathscr{G}^{(\text{rad})}$ has not been completed. We shall present instead a phenomenological treatment based on the assumption that radiative corrections to electron interaction may be taken into account by ascribing to the

electron a modified charge form factor and modified static magnetic moment. The expressions we shall write down will in total yield order $\alpha^5 m$ and order $\alpha^6 m$ corrections to the $\alpha^4 m$ spin dependent fine structure. These corrections may be symply interpreted as arising from replacement of the electron magnetic moment $\mu_0 = e/2m$ by $\mu_0(1 + \Delta)$, where according to Refs. [23, 32, 37] $\Delta = \alpha/2\pi - .328(\alpha/\pi)^2$ is the order α plus α^2 radiative correction factor.

In accordance with the above prescription the pointlike vertex operators are replaced by the modified operators [31, Section 6]

$$\beta = \gamma_0 \to \gamma_0 + \Lambda_0(q) = \gamma_0 + \Lambda_0^c + \Lambda_0^M, \tag{7.1a}$$

$$\beta\alpha^i = \gamma_i \to \gamma_i + \Lambda_i(q) = \gamma_i + \Lambda_i^c + \Lambda_i^M, \tag{7.1b}$$

where

$$\Lambda_0^c = \frac{\alpha}{3m^2} q^2 \gamma_0 \left(\ln \frac{m}{\lambda} - \frac{3}{8}\right), \qquad \Lambda_i^c = \frac{\alpha}{3m^2} q^2 \gamma_i \left(\ln \frac{m}{\lambda} - \frac{3}{8}\right) \tag{7.2a}$$

$$\Lambda_0^M = \frac{\Delta}{4m} [\not{q}, \gamma_0], \qquad \Lambda_i^M = \frac{\Delta}{4m} [\not{q}, \gamma_i]. \tag{7.2b}$$

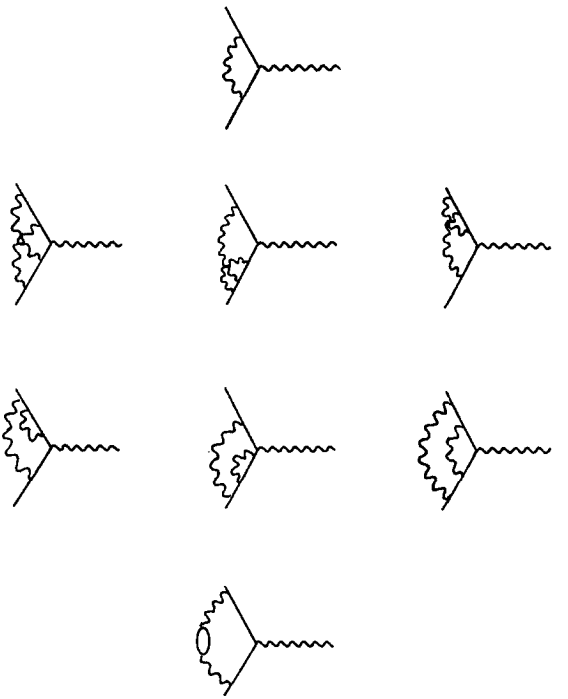

FIG. 11. Vertex correction diagrams.

Here q is the four momentum transfer to the electron in question and may be provided by either the other electron or the external potential. The $\Lambda^c_{0,i}$ terms represent the charge form factor of the electron to order α; λ is a photon mass introduced to avoid an infrared divergence and indicates the need for binding corrections. The $\Lambda^M_{0,i}$ terms represent the magnetic moment correction to order α plus α^2. Second and fourth order Feynman diagram contributions to $\Lambda^c + \Lambda^M$ are illustrated in Fig. 11. Since it will be found that the charge term does not contribute to $\alpha^5 m$ and $\alpha^6 m$ fine structure an order α correction is not included.

To investigate the effect of radiative corrections to photon exchange we shall substitute $\gamma_{0,i} \to \Lambda^c_{0,i} + \Lambda^M_{0,i}$ in the one and two photon exchange diagrams; see Fig. 12. Radiative correction to interaction with the external potential is taken into account by replacing the mass operators Σ_{1v}, Σ_{2v} of Eq. (2.19) by $\Lambda^c_{10} + \Lambda^M_{10}$, $\Lambda^c_{20} + \Lambda^M_{20}$. The expressions arrived at are then equivalent to the vertex correction to a single scattering in the external potential.

The interaction operators corresponding to Figs. 12(a)–12(j), call them $\mathcal{G}_{CR}$, $\mathcal{G}_{VR}$, $\mathcal{G}_{TR}$, $\mathcal{G}_{CR^2}$, $\mathcal{G}_{VR^2}$, $\mathcal{G}_{TR^2}$, $\mathcal{G}_{CR\times C}$, $\mathcal{G}_{TR\times C}$, $\mathcal{G}_{T\times CR}$, $\mathcal{G}_{T\times TR}$, are derived simply by employing the rules of Appendix II and substituting the vertex correction

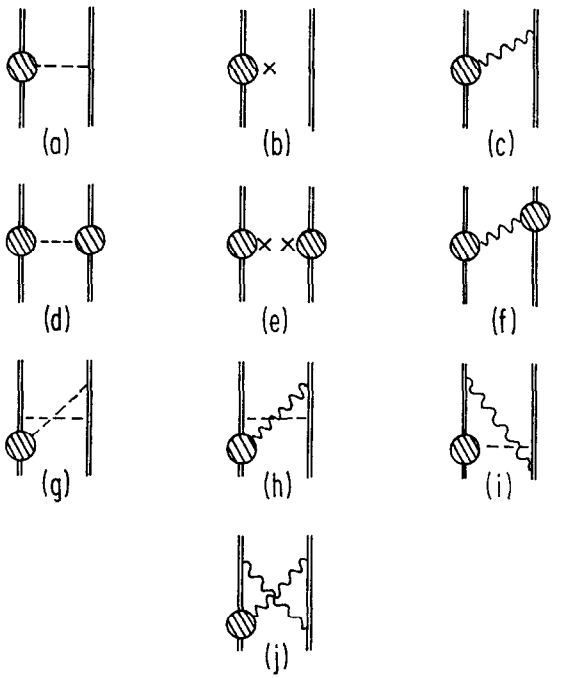

FIG. 12. Vertex modified exchange diagrams.

operator for the appropriate matrix. Then the energy shifts generated by these operators are calculated in the usual manner by means of Eqs. (5.2) and (5.4). It is found that in all diagrams $\Lambda^{c}_{0,i}$ yields energy splittings the largest of which (from $\mathcal{G}_{CR}$, $\mathcal{G}_{VR}$, $\mathcal{G}_{TR}$) are of nominal order $\alpha^7 m$ and thus negligible (the true order of such terms is $\alpha^7 \log \alpha$ as indicated by the factor $\log \lambda$ in the vertex operator; see for example Bethe, Barranger, and Feynman [38]). In calculating with $\Lambda^{M}_{0,i}$ it is found that the only operators that contribute in the order of interest are $\mathcal{G}_{CR}$, $\mathcal{G}_{VR}$, $\mathcal{G}_{TR}$, $\mathcal{G}_{TR\times C}$, and $\mathcal{G}_{TR^2}$. Furthermore we need retain only those terms in which no external pairs are present, and in $\mathcal{G}_{TR\times C}$ only positive energy intermediate states contribute to the order of interest.

The relations

$$\beta_{1,2}[\not{k}, \gamma_{1,20}] = 2\bar{\gamma}_{1,2} \cdot \bar{k}, \tag{7.3a}$$

$$\beta_{1,2}[\not{k}, \gamma_{1,2i}] = 2\omega\gamma_{1,2i} + 2\alpha^{i}_{1,2}\bar{\gamma}_{1,2} \cdot \bar{k} \tag{7.3b}$$

are used to rewrite the magnetic vertex operator. Performing the relative energy integrations and including factors of two to account for diagrams with the radiative correction on the path of particle 2 we find

$$\Delta E_{CR} = 2 \cdot \left(\frac{\alpha}{2\pi^2}\right) \frac{\Delta}{2m} \int \frac{d\bar{k}}{\bar{k}^2} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \bar{\gamma}_1 \cdot \bar{k} \mid \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}) \rangle, \tag{7.4a}$$

$$\Delta E_{VR} = 2 \cdot \left(\frac{-Z\alpha}{2\pi^2}\right) \frac{\Delta}{2m} \int \frac{d\bar{k}_1}{\bar{k}_1^2} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \bar{\gamma}_1 \cdot \bar{k}_1 \mid \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2) \rangle, \tag{7.4b}$$

$$\begin{aligned}
\Delta E_{TR} = 2 \cdot \left(\frac{\alpha}{2\pi^2}\right) \frac{\Delta}{2m} \int \frac{d\bar{k}}{2k} \langle \Phi_c(\bar{p}_1\bar{p}_2) \mid \\
\times \Big\{ \Big[\alpha_1{}^{i}\bar{\gamma}_1 \cdot \bar{k} \, \frac{1}{E - \mathcal{E}(\bar{p}_1 - \bar{k}) - \mathcal{E}(\bar{p}_2) - k} \, \alpha_2{}^{i} \\
+ \alpha_2{}^{i} \, \frac{1}{E - \mathcal{E}(\bar{p}_1) - \mathcal{E}(\bar{p}_2 + \bar{k}) - k} \, \alpha_1{}^{i}\bar{\gamma}_1 \cdot \bar{k} \Big] \\
\times \Big[\gamma_{1i} \, \frac{E - \mathcal{E}(\bar{p}_1 - \bar{k}) - \mathcal{E}(\bar{p}_2)}{E - \mathcal{E}(\bar{p}_1 - \bar{k}) - \mathcal{E}(\bar{p}_2) - k} \, \alpha_2{}^{i} \\
+ \alpha_2{}^{i} \, \frac{E - \mathcal{E}(\bar{p}_1) - \mathcal{E}(\bar{p}_2 + \bar{k})}{E - \mathcal{E}(\bar{p}_1) - \mathcal{E}(\bar{p}_2 + \bar{k}) - k} \, \gamma_{1i} \Big] \Big\} \mid \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}) \rangle,
\end{aligned} \tag{7.4c}$$

$$\Delta E_{TR^2} = -\left(\frac{\alpha}{2\pi^2}\right)\left(\frac{\Delta}{2m}\right)^2 \int \frac{d\bar{k}}{\bar{k}^2} \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid$$
$$\times \Big\{ -\alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k} \alpha_2{}^i \bar{\gamma}_2 \cdot \bar{k} + \tfrac{1}{2}(-k\gamma_{2i} + \alpha_2{}^i \bar{\gamma}_2 \cdot \bar{k})$$
$$\times \frac{E - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k})}{E - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k} (-k\gamma_{1i} + \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k})$$
$$+ \tfrac{1}{2}(k\gamma_{1i} + \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k}) \frac{E - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2)}{E - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k}$$
$$\times (k\gamma_{2i} + \alpha_2{}^i \bar{\gamma}_2 \cdot \bar{k}) \Big\} \mid \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}) \rangle, \tag{7.4d}$$

$$\Delta E_{TR\times C} = 2 \cdot \left(\frac{\alpha}{2\pi^2}\right) \frac{\Delta}{2m} \int \frac{d\bar{k}}{2k} \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid$$
$$\times \Big\{ \alpha_2{}^i \frac{\mathscr{L}_{2+}(\bar{p}_2 + \bar{k})}{E - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k} I_c \frac{\mathscr{L}_{1+}(\bar{p}_1)}{E - \mathscr{E}(\bar{p}_1) - \mathscr{E}(\bar{p}_2 + \bar{k}) - k}$$
$$\times (-k\gamma_{1i} + \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k}) + (k\gamma_{1i} + \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k})$$
$$\times \frac{\mathscr{L}_{1+}(\bar{p}_1 - \bar{k})}{E - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k} I_c \frac{\mathscr{L}_{2+}(\bar{p}_2)}{E - \mathscr{E}(\bar{p}_1 - \bar{k}) - \mathscr{E}(\bar{p}_2) - k} \Big\}$$
$$\times \mid \Phi_c(\bar{p}_1 - \bar{k}, \bar{p}_2 + \bar{k}) \rangle. \tag{7.4e}$$

For the purpose of extracting the $\alpha^5 m$ and $\alpha^6 m$ contributions we may drop the second square bracket of ΔE_{TR}, the second and third terms of ΔE_{TR^2}; further, in $\Delta E_{TR\times C}$ we may drop the $k\gamma_{1i}$ terms and also neglect $E - \mathscr{E}(1) - \mathscr{E}(2)$ compared to k in the denominators (i.e., neglect "recoil"). ΔE_{TR} contributes in order $\alpha^5 m$ and we need to retain a recoil term obtained by writing

$$\frac{1}{E - \mathscr{E}(1) - \mathscr{E}(2) - k} \cong -\frac{1}{k} - \frac{E - \mathscr{E}(1) - \mathscr{E}(2)}{k^2}.$$

Using the equation for Φ_c we may eliminate $E - \mathscr{E}(1) - \mathscr{E}(2)$ in favor of I_c and find for the total of the recoil term of ΔE_{TR} plus $\Delta E_{TR\times C}$

$$\Delta E_{TR}(\alpha^6 m) + \Delta E_{TR\times C}$$
$$\cong -2 \cdot \left(\frac{\alpha}{2\pi^2}\right) \frac{\Delta}{2m} \int \frac{d\bar{k}}{2\bar{k}^2} \langle \Phi_c(\bar{p}_1 \bar{p}_2) \mid$$
$$\times \{\alpha_2{}^i \mathscr{L}_{2+}(\bar{p}_2 + \bar{k}) I_c \mathscr{L}_{1+}(\bar{p}_1) \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k} + \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k} \mathscr{L}_{1+}(\bar{p}_1 - \bar{k}) I_c \mathscr{L}_{2+}(\bar{p}_2) \alpha_2{}^i$$
$$+ I_c \mathscr{L}_{1+}(\bar{p}_1) \mathscr{L}_{2+}(\bar{p}_2) \alpha_2{}^i \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k} + \alpha_2{}^i \alpha_1{}^i \bar{\gamma}_1 \cdot \bar{k} \mathscr{L}_{1+}(\bar{p}_1 - \bar{k}) \mathscr{L}_{2+}(\bar{p}_2 + \bar{k}) I_c\}$$
$$\times \mid \Phi_c(p_1 - k, p_2 + k) \rangle. \tag{7.5}$$

Making the usual F–W transformation and nonrelativistic approximations it is found that this expression is spin independent to order $\alpha^6 m$, analogous to the spin independence of P_1, Eq. (5.24).

TABLE I

Coordinate Space Form of $\alpha^4 m$ Operators

Contribution	Formula
$\Delta W_c^{(4)}(1)$	$-\frac{1}{8}\alpha^2 \langle \bar{\nabla}_1^2 \varphi_0 \mid \bar{\nabla}_1^2 \mid \varphi_0 \rangle - \frac{1}{8}\alpha^2 \langle \bar{\nabla}_2^2 \varphi_0 \mid \bar{\nabla}_2^2 \mid \varphi_0 \rangle$
$\sum_{i=2}^{4} \Delta W_c^{(4)}(i)$	$+\pi\alpha^2 \langle \varphi_0 \mid -2\delta^3(\bar{r}) + Z\delta^3(\bar{r}_1) + Z\delta^3(\bar{r}_2) \mid \varphi_0 \rangle$
$\Delta W_c^{(4)}(5)$	$-\frac{1}{4}\alpha^2 \langle \varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}/r^3 \times \bar{\nabla}_1/i) \mid \varphi_0 \rangle$
$\Delta W_c^{(4)}(6)$	$+\frac{1}{4}\alpha^2 \langle \varphi_0 \mid \bar{\sigma}_2 \cdot (\bar{r}/r^3 \times \bar{\nabla}_2/i) \mid \varphi_0 \rangle$
$\Delta W_c^{(4)}(7)$	$-\frac{1}{4}\alpha(-Z\alpha) \langle \varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}_1/r_1^3 \times \bar{\nabla}_1/i) \mid \varphi_0 \rangle$
$\Delta W_c^{(4)}(8)$	$-\frac{1}{4}\alpha(-Z\alpha) \langle \varphi_0 \mid \sigma_2 \cdot (\bar{r}_2/r_2^3 \times \bar{\nabla}_2/i) \mid \varphi_0 \rangle$
$\Delta W_T^{(4)}(1)$	$-\frac{1}{2}\alpha^2 \langle \varphi_0 \mid (1/r)[\bar{\nabla}_1/i \cdot \bar{\nabla}_2/i - \hat{r} \cdot (\hat{r} \cdot \bar{\nabla}_1/i)\, \bar{\nabla}_2/i] \mid \varphi_0 \rangle$
$\Delta W_T^{(4)}(2)$	$+\frac{1}{2}\alpha^2 \langle \varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}/r^3 \times \bar{\nabla}_2/i) \mid \varphi_0 \rangle$
$\Delta W_T^{(4)}(3)$	$-\frac{1}{2}\alpha^2 \langle \varphi_0 \mid \bar{\sigma}_2 \cdot (\bar{r}/r^3 \times \bar{\nabla}_1/i) \mid \varphi_0 \rangle$
$\Delta W_T^{(4)}(4)$	$+\frac{1}{4}\alpha^2 \langle \varphi_0 \mid -(8\pi/3)\bar{\sigma}_1 \cdot \bar{\sigma}_2 \delta^3(\bar{r}) + 1/r^3\,[\bar{\sigma}_1 \cdot \bar{\sigma}_2 - 3\bar{\sigma}_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}] \mid \varphi_0 \rangle$

The remaining expressions reduce simply to multiples of the $\alpha^4 m$ fine structure. That is, we find expressed in terms of the $\alpha^4 m$ fine structure operators listed in Eqs. (4.33) and (5.34) and Table I

$$\Delta E_{CR} = \Delta \cdot (2\Delta W_c^{(4)}(5) + 2\Delta W_c^{(4)}(6)), \tag{7.6a}$$

$$\Delta E_{VR} = \Delta \cdot (2\Delta W_c^{(4)}(7) + 2\Delta W_c^{(4)}(8)), \tag{7.6b}$$

$$\Delta E_{TR} = \Delta \cdot (\Delta W_T^{(4)}(2) + \Delta W_T^{(4)}(3)), \tag{7.6c}$$

$$\Delta E_{TR^2} = \Delta^2 \cdot \Delta W_T^{(4)}(4). \tag{7.6d}$$

Finally, we separate the results into order $\alpha^5 m$, $\alpha^6 m$, defining

$$\Delta W_R^{(5)} = \sum_{i=1}^{4} \Delta W_R^{(5)}(i), \tag{7.7}$$

with

$$\Delta W_R^{(5)}(1) = (\alpha/\pi)(\Delta W_C^{(4)}(5) + \Delta W_C^{(4)}(6)), \tag{7.8a}$$

$$\Delta W_R^{(5)}(2) = (\alpha/\pi)(\Delta W_C^{(4)}(7) + \Delta W_C^{(4)}(8)), \tag{7.8b}$$

$$\Delta W_R^{(5)}(3) = (\alpha/2\pi)(\Delta W_T^{(4)}(2) + \Delta W_T^{(4)}(3)), \tag{7.8c}$$

$$\Delta W_R^{(5)}(4) = (\alpha/\pi)\,\Delta W_T^{(4)}(4), \tag{7.8d}$$

and

$$\Delta W_R^{(6)} = \sum_{i=1}^{4} \Delta W_R^{(6)}(i), \tag{7.9}$$

with

$$\Delta W_R^{(6)}(1) = 2 \cdot [-.328(\alpha/\pi)^2](\Delta W_C^{(4)}(5) + \Delta W_C^{(4)}(6)), \tag{7.10a}$$

$$\Delta W_R^{(6)}(2) = 2 \cdot [-.328(\alpha/\pi)^2](\Delta W_C^{(4)}(7) + \Delta W_C^{(4)}(8)), \tag{7.10b}$$

$$\Delta W_R^{(6)}(3) = [-.328(\alpha/\pi)^2](\Delta W_T^{(4)}(2) + \Delta W_T^{(4)}(3)), \tag{7.10c}$$

$$\Delta W_R^{(6)}(4) = (2 \cdot [-.328(\alpha/\pi)^2] + (\alpha/2\pi)^2)\,\Delta W_T^{(4)}(4). \tag{7.10d}$$

In this phenomenological treatment we have rederived the well known anomalous moment fine structure contributions of order $\alpha^5 m$ [10, 16, 17, 39] plus similar terms of order $\alpha^6 m$ which may be described as arising from the next higher term in the expansion of the anomalous moment. We have also shown that $\alpha^6 m$ corrections which cannot be simply interpreted as arising from a modification of the electron moment do not occur in the phenomenological treatment. Such a term does occur in ΔE_{TR}, but it is cancelled by a similar term in $\Delta E_{TR\times C}$.

It is clear that a systematic exact reduction of $\mathcal{G}^{(\mathrm{rad})}$, Eqs. (2.19) and (2.37), is preferable to the above treatment. Such an analysis appears to be necessary in order to definitively determine whether or not there are order α corrections to the $\alpha^5 m$ terms coming from the binding of the electron in the atom. In the case of the analogous problem of P state hydrogen hyperfine structure [40] order α binding corrections have been shown to be absent. It would, therefore, not be surprising if they were absent for helium as well. On the other hand, in the calculation of Ref. [40] binding is provided by a fixed external potential and the complications provided by relative energy behavior are not present. A rigorous analysis in progress indicates that the above treatment in fact yields the correct result. It is based on use of the Yennie–Fried [41] form of the photon propagators for the photon lines beginning and ending on the same electron line. Systematic renormal-

ization and reduction of terms to forms which can be estimated proves to be possible but lengthy. Selected "dangerous" terms have been investigated and shown to yield results consistent with the phenomenological analysis. However, the detailed demonstration that all terms which might conceivably contribute to order $\alpha^6 mc^2$ fine structure in fact do not do so has not been completed.

8. Results in Coordinate Space

The purpose of this section is to reexpress our result in a form convenient for a computerized numerical evaluation procedure. Because numerical evaluation of helium wave functions has been found to be most conveniently carried out in coordinate space it is useful to express all of our results in terms of coordinate space operators.

In the field-theoretic calculations natural units have been employed, with the unit of charge chosen so that $e^2 = \alpha \cong 1/137$. Momenta can be rescaled in terms of the Bohr momentum $P_B = \alpha m$ and the Schroedinger equation becomes

$$\left(\frac{\bar{p}_1^{\,2}}{2} + \frac{\bar{p}_2^{\,2}}{2}\right) \varphi_0(\bar{p}_1 \bar{p}_2) + \frac{1}{2\pi^2} \int \frac{d\bar{k}}{\bar{k}^2} \varphi_0(\bar{p}_1 - \bar{k}\ \bar{p}_2 + \bar{k})$$

$$+ \frac{-Z}{2\pi^2} \int \frac{d\bar{k}_1}{\bar{k}_1^{\,2}} \varphi_0(\bar{p}_1 - \bar{k}_1 \bar{p}_2) + \frac{-Z}{2\pi^2} \int \frac{d\bar{k}_2}{\bar{k}_2^{\,2}} \varphi_0(\bar{p}_1 \bar{p}_2 + \bar{k}_2)$$

$$= \nu_0 \varphi_0(\bar{p}_1 \bar{p}_2), \tag{8.1}$$

with

$$\nu_0 = W_0/\alpha^2 m, \tag{8.2}$$

the energy measured in atomic units (a.u.).

Our convention for Fourier transformation is that

$$\varphi_0(\bar{p}_1 \bar{p}_2) = \frac{1}{(2\pi)^3} \int d\bar{r}_1\, d\bar{r}_2\, e^{-i\bar{p}_1 \cdot \bar{r}_1} e^{-i\bar{p}_2 \cdot \bar{r}_2} \varphi_0(\bar{r}_1 \bar{r}_2), \tag{8.3a}$$

which implies

$$\varphi_0(\bar{r}_1 \bar{r}_2) = \frac{1}{(2\pi)^3} \int d\bar{p}_1\, d\bar{p}_2\, e^{i\bar{p}_1 \cdot \bar{r}_1} e^{i\bar{p}_2 \cdot \bar{r}_2} \varphi_0(\bar{p}_1 \bar{p}_2) \tag{8.3b}$$

(the symbol φ_0 will be used to denote both the coordinate space and momentum

space wave function). In addition use will be made of the following transform formulas:

$$\frac{1}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^2}\, e^{i\bar{k}\cdot\bar{r}} = \frac{1}{r} \equiv \frac{1}{|\bar{r}|}, \tag{8.4a}$$

$$\frac{1}{2\pi^2}\int d\bar{k}\, e^{i\bar{k}\cdot\bar{r}} = 4\pi\delta^3(\bar{r}), \tag{8.4b}$$

$$\frac{1}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^2}\, k_i e^{i\bar{k}\cdot\bar{r}} = \frac{ir_i}{r^3}, \tag{8.4c}$$

$$\frac{1}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^4}\, k_i k_j e^{i\bar{k}\cdot\bar{r}} = \frac{1}{2r}\,[\delta_{ij} - \hat{r}_i\hat{r}_j], \tag{8.4d}$$

$$\frac{1}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^2}\, k_i k_j e^{i\bar{k}\cdot\bar{r}} = \frac{4\pi}{3}\,\delta^3(\bar{r})\,\delta_{ij} + \frac{1}{r^3}\,[\delta_{ij} - 3\hat{r}_i\hat{r}_j], \tag{8.4e}$$

$$\frac{1}{2\pi^2}\int \frac{d\bar{k}}{\bar{k}^4}\, k_i k_j k_l e^{i\bar{k}\cdot\bar{r}} = \frac{i}{2r^2}\,[\delta_{jl}\hat{r}_i + \delta_{il}\hat{r}_j + \delta_{ij}\hat{r}_l - 3\hat{r}_i\hat{r}_j\hat{r}_l]. \tag{8.4f}$$

All but Eq. (8.4f) may be found in [1, p. 180] in a form slightly different from that presented here; Eq. (8.8f) may be obtained by methods described there or by directly performing the $\bar{k}$ integral.

Using the above relations the Schroedinger equation may be transformed to coordinate space with the result

$$\left[-\frac{\bar{\nabla}_1^2}{2} - \frac{\bar{\nabla}_2^2}{2} + \frac{-Z}{r_1} + \frac{-Z}{r_2} + \frac{1}{r}\right]\varphi_0(\bar{r}_1\bar{r}_2) = \nu_0\varphi_0(\bar{r}_1\bar{r}_2),$$
$$\bar{r} = \bar{r}_1 - \bar{r}_2\,, \qquad r = |\bar{r}|, \tag{8.5}$$

the wave equation in atomic units; that is distances are measured in terms of the Bohr radius $a_0 = \hbar^2/me^2$ (numerical calculations are usually carried out in these units). Expectation values will be similarly transformed and energy shifts expressed as α^4(a.u.) $\times$ numerical factor $\times$ integral over dimensionless wave function, where 1 a.u. $= \alpha^2 mc^2 = 2$ Rydberg.

The total energy shift formula accurate to order $\alpha^6 mc^2$ and valid only for calculating the ^3P splitting may be summarized as

$$\Delta W = \Delta W^{(4)} + \Delta W^{(5)} + \Delta W^{(6)}, \tag{8.6a}$$

with

$$\Delta W^{(6)} = \Delta W_{\mathrm{I}}^{(6)} + \Delta W_{\mathrm{II}}^{(6)}. \tag{8.6b}$$

$\Delta W^{(4)}$ is the well known αm fine structure

$$\Delta W^{(4)} = \Delta W_c^{(4)} + \Delta W_T^{(4)}, \tag{8.7}$$

given by Eqs. (4.33), (5.34); $\Delta W^{(5)} = \Delta W_R^{(5)}$ is given by Eq. (7.7) and included in the result of Schwartz [22]; $\Delta W_{II}^{(6)}$ is defined as the sum of all second order perturbation sums and is found by adding Eqs. (4.35), (5.36), and (6.22)

$$\begin{aligned}\Delta W_{II}^{(6)} &= \Delta W_c^{(6)}(1) + \Delta W_T^{(6)}(1) + \Delta W_{TT}^{(6)}(1) \\ &= \sum_n{}' \frac{\langle \varphi_0 \mid \Delta H_c^{(4)} + B_{\text{red}} \mid \varphi_n \rangle \langle \varphi_n \mid \Delta H_c^{(4)} + B_{\text{red}} \mid \varphi_0 \rangle}{W_0 - W_n},\end{aligned} \tag{8.8}$$

i.e., the total $\alpha^4 m$ fine structure taken to second order. $\Delta W_I^{(6)}$ is the sum of expectation values of all the operators of intrinsic order $\alpha^6 m$, and which can be found in Eqs. (4.34), (5.37)–(5.43), (6.24), and (7.9); i.e.,

$$\Delta W_I^{(6)} = \sum_{i=2}^{5} \Delta W_c^{(6)}(i) + \sum_{i=2}^{8} \Delta W_T^{(6)}(i) + \Delta W_{TT}^{(6)}(2) + \sum_{i=1}^{4} \Delta W_R^{(6)}(i). \tag{8.9}$$

There is, naturally a wide latitude of forms available for our final expressions. The following guidelines have influenced our choice. Nonvanishing contact terms are eliminated in favor of gradients operating on one or both wave functions. This choice simplifies the systematics of the numerical work. In addition an effort is made to keep the convergence properties of the expressions as self-evident as possible.

It is appropriate at this point to discuss the behavior of $\varphi_0(\bar{r}_1, \bar{r}_2)$ as $\bar{r} \to 0$. Eq. (8.5) can be expressed in center of mass variables

$$\bar{R} = \tfrac{1}{2}(\bar{r}_1 + \bar{r}_2), \qquad \bar{r} = \bar{r}_1 - \bar{r}_2, \tag{8.10}$$

which imply

$$\bar{\nabla}_1 = \tfrac{1}{2}(\bar{\nabla}_R + \bar{\nabla}), \qquad \bar{\nabla}_2 = \tfrac{1}{2}(\bar{\nabla}_R - \bar{\nabla}). \tag{8.11}$$

To examine the limit $\bar{r} \to 0$ we drop all terms in Eq. (8.5) that are no more singular than φ_0 as $\bar{r} \to 0$, finding

$$\bar{\nabla}^2 \varphi_0(\bar{r}\bar{R}) \underset{\bar{r}\to 0}{\sim} \frac{1}{r}\, \varphi_0(\bar{r}\bar{R}). \tag{8.12}$$

The antisymmetry of φ_0 requires

$$\varphi_0(\bar{r}, \bar{R}) = -\varphi_0(-\bar{r}\bar{R}). \tag{8.13}$$

Eqs. (8.12) and (8.13) imply for the small $\bar{r}$ behavior of φ_0

$$\varphi_0(\bar{r}\bar{R}) \cong \bar{G}(\bar{R}) \cdot \bar{r}(1 + \tfrac{1}{4}r) + \mathcal{O}(r^3), \tag{8.14}$$

with G regular for $\bar{R} \neq 0$. It follows that

$$\varphi_0(\bar{r}\bar{R}) \sim r, \tag{8.15a}$$

$$\bar{\nabla}\varphi_0\,,\ \bar{\nabla}_1\varphi_0\,,\ \bar{\nabla}_2\varphi_0 \sim \bar{G}(\bar{R}), \tag{8.15b}$$

$$\bar{\nabla}^2\varphi_0\,,\ \bar{\nabla}_1{}^2\varphi_0\,,\ \bar{\nabla}_2{}^2\varphi_0\,,\ -\bar{\nabla}_1 \cdot \bar{\nabla}_2\varphi_0 \sim \bar{G}(\bar{R}) \cdot \bar{r} + \lambda\bar{\nabla}_R \cdot \bar{G}, \tag{8.15c}$$

where the coefficient λ depends on which of the operators is being considered. Also we find for arbitrary components

$$\bar{\nabla}_{1i}\bar{\nabla}_{2j}\varphi_0 \sim \frac{-1}{4r}(G_j r_i + G_i r_j) - \frac{1}{4}\bar{G} \cdot \bar{r}\left(\frac{\delta_{ij}}{r} - \frac{\bar{r}_i\bar{r}_j}{r^3}\right) - (\bar{\nabla}_{Ri}\bar{G}_j - \bar{\nabla}_{Rj}\bar{G}_i) + \mathcal{O}(r). \tag{8.16}$$

The answers will be arranged so that only a $\bar{\nabla}_1\,,\ \bar{\nabla}_2\,,\ \bar{\nabla}_1{}^2,\ \bar{\nabla}_2{}^2,\ \bar{\nabla}_{1i}\bar{\nabla}_{2j}$ ever appears next to a wave function, and using the above estimates convergence can be easily verified.

We begin the Fourier transformation with the $\alpha^4 m$ terms finding the result listed in Table I. It is noticed that many terms in the $\alpha^6 m$ result, $\Delta W_1^{(6)}$, are of the same form and will combine. We feel however that it is useful to present the transformed form of each term in Eq. (8.9) separately and then combine like terms and group together those of similar structure. The Fourier transformation is straightforward, and we shall simply present the result. A few conventions have been adopted: (a) Many terms occur in complex conjugate pairs and we shall save writing by using one form, taking the real part and doubling, (b) In operators with a $\bar{\nabla}_2{}^2$ we shall arrange to use a form with $\bar{\nabla}_1{}^2$ (the transformation is effected by letting $\bar{r}_1 \rightleftarrows \bar{r}_2$ everywhere in an expression) and also use a form with the $\bar{\nabla}_1{}^2$ operating on the left, (c) In single spin operators we shall always let $\sigma_2 \to \sigma_1$ which is valid since the spin part of φ_0 is symmetric, (d) Terms whose spin dependence is $\sigma_1 \cdot \sigma_2$ shall be dropped because they contribute equally to all three levels of the triplet. The results are listed in Table II.

To simplify the answer terms of like structure are combined and a new labeling is introduced. Also the expectation values are real and we may drop "Re" on the answers. Terms involving the external potential combine to $E_1\,,\ E_2\,,\ E_3$. Terms coming from two photon exchange and the I_c term in the recoil energy combine to $E_4\,,\ E_5$. Spin orbit terms that originally contained one explicit power of α combine to $E_6\,,\ E_7\,,\ E_8\,,\ E_9$. The remaining spin-spin terms combine to $E_{10}\,,\ E_{11}\,,\ E_{12}\,,\ E_{13}\,,\ E_{14}$. The radiative diagrams are given by $E_{15}\,,\ E_{16}\,,\ E_{17}\,,\ E_{18}$. These results are listed in Table III.

TABLE II

Coordinate Space Form of $\alpha^6 m$ Operators

Contribution	Formula
$\Delta W_c^{(6)}(2)$	$-(3/8)\,\alpha^3(-Z\alpha)\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}_1/r_1^3\times\bar{\nabla}_1/i)\mid\varphi_0\rangle$
$\Delta W_c^{(6)}(3)$	$-(3/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_1/i)\mid\varphi_0\rangle$
$\Delta W_c^{(6)}(4)$	$+(1/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_2/i)\mid\varphi_0\rangle$
$\Delta W_c^{(6)}(5)$	$-(1/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_2/i)\mid\varphi_0\rangle$ $-(1/16)\,\alpha^4\,\mathrm{Re}\,\langle\varphi_0 \mid (1/r^3)\,\bar{\sigma}_1\cdot(\bar{\nabla}_2/i)\,\bar{\sigma}_2\cdot(\bar{\nabla}_1/i)\mid\varphi_0\rangle$ $+(3/8)\,\alpha^4\,\mathrm{Re}\,\langle\varphi_0 \mid (i/r^5)\,\bar{\sigma}_1\cdot[\bar{r}\times(\bar{r}\cdot\bar{\nabla}_2/i)\,\bar{\nabla}_1/i]\mid\varphi_0\rangle$ $-(3/16)\,\alpha^4\,\mathrm{Re}\,\langle\varphi_0 \mid (1/r^5)\,\bar{\sigma}_2\cdot[\bar{r}\times[\sigma_1\cdot(\bar{r}\times\bar{\nabla}_1/i)]\,\bar{\nabla}_2/i]\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(2)$	$-\frac{1}{2}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_1/i)\mid\varphi_0\rangle$ $-\frac{1}{4}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r)[(\bar{\sigma}_1\times\bar{\nabla}_1/i)\cdot\bar{\nabla}_2/i+\hat{r}\cdot[\hat{r}\cdot(\bar{\sigma}_1\times\bar{\nabla}_1/i)]\,\bar{\nabla}_2/i]\mid\varphi_0\rangle$ $+\frac{3}{4}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_2/i)\mid\varphi_0\rangle$ $-(9/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot\hat{r}\bar{\sigma}_2\cdot\hat{r}/r^3\mid\varphi_0\rangle$ $+\frac{1}{4}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid i(\bar{\sigma}_1\cdot\bar{r}/r^3)\,\bar{\sigma}_2\cdot\bar{\nabla}_1/i\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(3)$	$+\frac{1}{2}\alpha^3(-Z\alpha)\,\langle\varphi_0 \mid (1/r)\,[\bar{\sigma}_1\cdot(\bar{r}_1/r_1^3\times\bar{\nabla}_2/i)+\bar{\sigma}_1\cdot(\bar{r}_1/r_1^3\times\hat{r})\,\hat{r}\cdot\bar{\nabla}_2/i]\mid\varphi_0\rangle$ $-\frac{1}{2}\alpha^3(-Z\alpha)\,\langle\varphi_0 \mid \sigma_1\cdot\hat{r}\,\sigma_2\cdot\hat{r}_1/r^2r_1^2\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(4)$	$+\frac{1}{2}\alpha^4\,\langle\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^4\times\bar{\nabla}_2/i)\mid\varphi_0\rangle$ $-\frac{1}{2}\alpha^4\,\langle\varphi_0 \mid \bar{\sigma}_1\cdot\hat{r}\bar{\sigma}_2\cdot\hat{r}/r^4\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(5)$	$-(3/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot\hat{r}\bar{\sigma}_2\cdot\hat{r}/r^3\mid\varphi_0\rangle$ $-(1/8)\,\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r^2)[(\hat{r}\cdot\bar{\sigma}_1)\bar{\sigma}_2+(\hat{r}\cdot\bar{\sigma}_2)\bar{\sigma}_1-3(\hat{r}\cdot\bar{\sigma}_1)(\hat{r}\cdot\bar{\sigma}_2)\hat{r}]\cdot\bar{\nabla}_2/i\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(6)$	$-\frac{1}{2}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^3\times\bar{\nabla}_1/i)\mid\varphi_0\rangle$ $+\frac{1}{2}\alpha^4\,\mathrm{Re}\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r)[\sigma_1\cdot(\bar{\nabla}_1/i\times\bar{\nabla}_2/i)-\hat{r}\cdot[\hat{r}\cdot(\bar{\sigma}_1\times\bar{\nabla}_1/i)]\,\bar{\nabla}_2/i]\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(7)$	$-\frac{1}{2}\alpha^3(-Z\alpha)\,\mathrm{Re}\,\langle\varphi_0 \mid (1/r)[\bar{\sigma}_1\cdot(\bar{r}_1/r_1^3\times\bar{\nabla}_2/i)-\bar{\sigma}_1\cdot(\bar{r}_1/r_1^3\times\hat{r})\hat{r}\cdot\bar{\nabla}_2/i]\mid\varphi_0\rangle$
$\Delta W_T^{(6)}(8)$	$+\frac{1}{2}\alpha^4\,\langle\varphi_0 \mid \bar{\sigma}_1\cdot(\bar{r}/r^4\times\bar{\nabla}_1/i)\mid\varphi_0\rangle$
$\Delta W_{TT}^{(6)}(2)$	$-\Delta W_T^{(6)}(8)$

TABLE III

Summary of $\alpha^6 m$ Operators

Contribution	Formula
E_1	$-(3/8)\,\alpha^3(-Z\alpha)\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}_1/r_1^3 \times \bar{\nabla}_1/i) \mid \varphi_0\rangle$
E_2	$+\alpha^3(-Z\alpha)\,\langle \varphi_0 \mid (1/r)\bar{\sigma}_1 \cdot (\bar{r}_1/r_1^3 \times \bar{r})\hat{r} \cdot \bar{\nabla}_2/i \mid \varphi_0\rangle$
E_3	$-\frac{1}{2}\alpha^3(-Z\alpha)\langle \varphi_0 \mid \sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}_1/r^2 r_1^2 \mid \varphi_0\rangle$
E_4	$+\frac{1}{2}\alpha^4\,\langle \varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}/r^4 \times \bar{\nabla}_2/i) \mid \varphi_0\rangle$
E_5	$-\frac{1}{2}\alpha^4\,\langle \varphi_0 \mid \bar{\sigma}_1 \cdot \hat{r}\bar{\sigma}_2 \cdot \hat{r}/r^4 \mid \varphi_0\rangle$
E_6	$+\frac{1}{4}\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r)\,\bar{\sigma}_1 \cdot (\bar{\nabla}_1/i \times \bar{\nabla}_2/i) \mid \varphi_0\rangle$
E_7	$+\frac{3}{4}\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r^3)\bar{r} \cdot (\bar{\nabla}_2/i)\bar{\sigma}_1 \cdot (\bar{r} \times \bar{\nabla}_1/i) \mid \varphi_0\rangle$
E_8	$+\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid \bar{\sigma}_1 \cdot [\bar{r}/r^3 \times (\frac{3}{4}\bar{\nabla}_2/i - \frac{5}{8}\bar{\nabla}_1/i)] \mid \varphi_0\rangle$
E_9	$+(3/8)\,\alpha^4\,\langle \varphi_0 \mid (i/r^5)\,\bar{\sigma}_1 \cdot [\bar{r} \times (\bar{r} \cdot \bar{\nabla}_2/i)(\bar{\nabla}_1/i)] \mid \varphi_0\rangle$
E_{10}	$-(3/2)\,\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid \sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r}/r^3 \mid \varphi_0\rangle$
E_{11}	$+\frac{1}{4}\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid i(\bar{\sigma}_1 \cdot \bar{r}/r^3)\,\bar{\sigma}_2 \cdot \bar{\nabla}_1/i \mid \varphi_0\rangle$
E_{12}	$-\frac{1}{8}\alpha^4\,\langle \bar{\nabla}_1^2\varphi_0 \mid (i/r^2)[(\hat{r} \cdot \bar{\sigma}_1)\bar{\sigma}_2 + (\hat{r} \cdot \bar{\sigma}_2)\bar{\sigma}_1 - 3(\hat{r} \cdot \bar{\sigma}_1)(\hat{r} \cdot \bar{\sigma}_2)\hat{r}] \cdot \bar{\nabla}_2/i \mid \varphi_0\rangle$
E_{13}	$-(1/16)\,\alpha^4\,\langle \varphi_0 \mid (1/r^3)\,\bar{\sigma}_1 \cdot (\bar{\nabla}_2/i)\,\bar{\sigma}_2 \cdot (\bar{\nabla}_1/i) \mid \varphi_0\rangle$
E_{14}	$-(3/16)\,\alpha^4\,\langle \varphi_0 \mid (1/r^5)\,\bar{\sigma}_2 \cdot [\bar{r} \times [\bar{\sigma}_1 \cdot (\bar{r} \times (\bar{\nabla}_1/i))]\,\bar{\nabla}_2/i] \mid \varphi_0\rangle$
E_{15}	$-\frac{1}{2}\alpha^2\,\langle \varphi_0 \mid \sigma_1 \cdot (\bar{r}/r_3 \times \bar{\nabla}_1/i) \mid \varphi_0\rangle \cdot (2) \cdot [-.328(\alpha/\pi)^2]$
E_{16}	$-\frac{1}{2}\alpha(-Z\alpha)\,\langle \varphi_0 \mid \sigma_1 \cdot (\bar{r}_1/r_1^3 \times \bar{\nabla}_1/i) \mid \varphi_0\rangle \cdot (2) \cdot [-.328(\alpha/\pi)^2]$
E_{17}	$+\frac{1}{4}\alpha^2\,\langle \varphi_0 \mid (1/r^3)[-3\bar{\sigma}_1 \cdot \hat{r}\bar{\sigma}_2 \cdot \hat{r}] \mid \varphi_0\rangle\,[(\alpha/2\pi)^2 + 2(-.328(\alpha/\pi)^2)]$
E_{18}	$+\alpha^2\,\langle \varphi_0 \mid \bar{\sigma}_1 \cdot (\bar{r}/r^3 \times \bar{\nabla}_2/i) \mid \varphi_0\rangle\,[-.328\,(\alpha/\pi)^2]$

In conclusion we have found the simple result for the total $\alpha^6 mc^2$ contribution to the triplet P fine structure *splitting*:

$$\Delta W^{(6)} = \sum_{i=1}^{18} E_i + \Delta W_{\mathrm{II}}^{(6)} = \Delta W_{\mathrm{I}}^{(6)} + \Delta W_{\mathrm{II}}^{(6)}, \tag{8.17}$$

where $\Delta W_{\mathrm{II}}^{(6)}$, Eq. (8.8), is obtained by calculating the usual $\alpha^4 mc^2$ fine structure operator in second order with Schroedinger–Pauli wave functions [25].

The contributions to the splittings of $\Delta W_I^{(6)}$ have been calculated numerically [26] employing techniques similar to those of Ref. [22] with the result

$$\begin{aligned} \Delta\nu_{01} &= -3.331 \text{ MHz}, \\ \Delta\nu_{12} &= +1.542 \text{ MHz}. \end{aligned} \tag{8.18}$$

A compilation of all effects of order $\alpha^6 m$ as well as nuclear motion corrections may be found in Ref. [5]. The totals agree with experiment to within estimated numerical errors. It is clear that the $\alpha^6 m$ terms calculated here are required.

APPENDIX I: Units and Notation

Natural units $\hbar = c = 1$ are used. m is the electronic mass, e the positive quantum of charge in nonrationalized units; $e^2 = \alpha \cong 1/137$. $V = -ze^2/|\bar{r}|$ is the Coulomb potential energy. $\bar{A}$ is a three vector with components $A_i (i = 1, 2, 3)$ and length $|\bar{A}| = (A_i A_i)^{1/2}$. B is a four vector with components $B_\mu (\mu = 0, 1, 2, 3)$. The summation convention for four vectors is $A_\mu B_\mu = A_0 B_0 - A_1 B_1 - A_3 B_3 = A_0 B_0 - \bar{A} \cdot \bar{B}$. The convention for Dirac matrices is $\gamma_i = \beta\alpha_i$, $\gamma_0 = \beta$, with

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} I & \\ & -I \end{bmatrix};$$

the σ_i are the usual 2×2 Pauli matrices. Also we define

$$\not{A} = A_\mu \gamma_\mu = A_0 \gamma_0 - \bar{A} \cdot \bar{\gamma}.$$

APPENDIX II. Rules for Construction of the Interaction Operator $\mathscr{G}_I$

Following Sucher [10, Section I], we present the rules for the construction of the interaction operator

$$\mathscr{G}_I = \sum_j \mathscr{G}_I^{(j)}. \tag{AII.1}$$

The most general interaction diagram, due to fermion conservation, contains two unbroken electron lines entering and leaving, connected by a combination of Coulomb, and transverse photon lines. One finds that the Feynman rules [7, 31] forming $\bar{G}$ together with the variable change Eqs. (2.21) and (2.24) and the transformation Eq. (2.25) lead to the following rules for the direct construction of $\mathscr{G}_I^{(j)}$ in momentum space: In the diagram j label the photon lines corresponding to instantaneous or transverse interactions with the four vectors k, k, k^1,... (we use a

dashed line for Coulomb interaction and a wavy line for transverse), assigning each line a definite sense; for example, let all momenta k^i be carried from particle 2 to particle 1. Label the final sections of the lines of electrons 1 and 2 with the four vectors $p_1 = (E/2 + \epsilon, \bar{p}_1)$, $p_2 = (E/2 - \epsilon, \bar{p}_2)$. To label the remaining electron lines conserve four momentum at each vertex. With the diagram thus labeled write $\alpha_1{}^i$ or $\alpha_2{}^i$ for a transverse interaction vertex on the path of 1 or 2; and a factor $S_1(p_1)$ for a line segment with momentum p_1, and similarly label the segments of the particle 2 line. These factors are written in the order in which they occur in the diagram, with the final factors to the far left. For example, the diagram of Fig. (13) yields the expression

$$M^{(j)} = \alpha_1{}^i S_1(p_1 - k')\, S_1(p_1 - k' - k'')\, \alpha_1{}^i \alpha_2{}^i S_2(p_2 + k)\, S_2(p_2 + k + k'')\, \alpha_2{}^j.$$

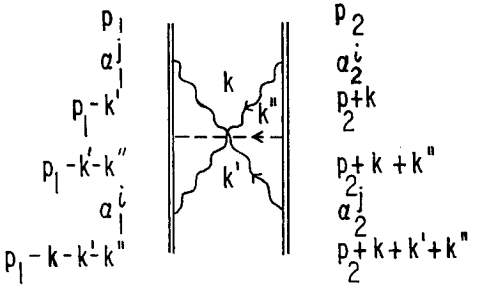

FIG. 13. Example of contribution to interaction operator.

The calculation of $\mathscr{G}_I^{(j)}\psi(\bar{p}, \bar{p}_2\epsilon)$ is completed by supplying a factor

$$g_c(\bar{k}) = \frac{e^2}{2\pi^2}\frac{1}{\bar{k}^2}\frac{1}{-2\pi i} \tag{AII.2}$$

for each Coulomb line,

$$g_T(\bar{k}) = \frac{e^2}{2\pi^2}\frac{1}{k\mu^2 + i\Delta}\frac{1}{-2\pi i} \tag{AII.3}$$

for each transverse line, displacing the wave function arguments by the sum of the transferred four momenta, and integrating over these. Thus, we have for our example

$$\mathscr{G}_I^{(j)}\psi = \left(\frac{e^2}{2\pi^2}\right)^3 \int \frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\omega''}{-2\pi i}\frac{d\bar{k}}{\omega^2 - \bar{k}^2 + i\Delta}\frac{d\bar{k}'}{\omega'^2 - \bar{k}'^2 + i\Delta}\frac{d\bar{k}''}{\bar{k}''^2}$$
$$\times M^{(j)} \times \psi(\bar{p}_1 - \bar{k} - \bar{k}' - \bar{k}'', \bar{p}_2 + \bar{k} + \bar{k}' + \bar{k}'', \epsilon - \omega - \omega' - \omega'').$$

Covariantly gauged photons may be handled by means of the substitution indicated in Eq. (2.36).

A word about our notation is perhaps in order. The meaning of the general single particle operator expression $O(P)$ is that the function obtained by operating with $O(P)$ is

$$O(P)f = Of(P) = \int O(PP')f(P')\,dP' \tag{AII.4}$$

with $O(PP')$ the kernel corresponding to the integral operator O, clearly a c number matrix function of two variables. The meaning of the displaced operator $O(P + k)$ is then

$$O(P + k)f = \int O(P + k, P')f(P')\,dP'.$$

This notation is useful for the following reason. A typical contribution to

$$\bar{G}\psi = \int \bar{G}(x_1x_2x_3x_4)\,\psi(x_3x_4)\,dx_3\,dx_4$$

contains a factor (suppressing particle 2 variables) roughly of the form

$$\bar{G}\psi(x_1\cdots) = \int dx_5\,dx_7 \ldots dx_n\,dx_3\,[K_{1V}(x_1x_5)\,\beta_1][K_{1V}(x\,x\,)\,\beta_1]\cdots[K_{1V}(x_nx_3)]$$
$$\times\, e^{-i[k_1x_1+k_5x_5+\cdots+k_nx_n+k_3x_3]}\psi(x_3\cdots) \tag{AII.5}$$

generated by the Feynman diagram rules for construction of $\bar{G}$. The exponentials come from photons exchanged in the diagram (see Fig. 14). Fourier transformation leads to an expression of the form

$$\bar{G}\psi(p_1\cdots) = \int dp_5\,dp_7\cdots dp_n\,dp_3\,S_1(p_1 - k_1\,,p_5)\,S_1(p_5 - k_5\,,p_7)$$
$$\cdots S_1(p_n - k_n\,,p_3)\,\psi(p_3 - k_3), \tag{AII.6}$$

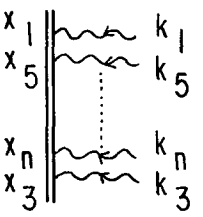

FIG. 14. Illustration of notation convention.

where $S_1(p_1, p')$ is the kernel of the operator $S_1(p_1)$. It is clear from the above discussion that we may simply write Eq. (AII.6) in the operator form

$$(\bar{G}\psi)(p_1 \cdots) = S_1(p_1 - k_1)\, S_1(p_1 - k_1 - k_5) \cdots S_1(p_1 - k_1 - k_5 - \cdots - k_n)\, \psi(p_1 - k_1 - \cdots - k_n - k_3, \ldots), \tag{AII.7}$$

i.e., formally assigning to each electron line segment a given momentum even though the external potential transfers momenta to the electron between interactions. This allows one to write down momentum space diagrams in the external potential case in a manner completely analogous to the usual free propagation case as specified above. One must, of course, always keep in mind the fact that p_1 is an operator which fails to commute with S_1, V_1, and H_1.

APPENDIX III: Generalization of F–W Transformation to the External Potential Dirac Equation

Transforming $H = \alpha \cdot \bar{p} + \beta m + V$ with U_0 we find

$$U_0 H U_0^{-1} = \beta E_p + \epsilon_1 + O_1, \tag{AIII.1}$$

where ϵ_1 and 0_1 are even and odd operators of order V; specifically,

$$\epsilon_1 = \left(\frac{E_p + m}{2E_p}\right)^{1/2} V \left(\frac{E_p + m}{2E_p}\right)^{1/2} + \frac{\bar{\alpha} \cdot \bar{p}}{(2E_p(E_p + m))^{1/2}} V \frac{\alpha \cdot \bar{p}}{(2E_p(E_p + m))^{1/2}} \tag{AIII.2}$$

$$O_1 = \frac{-1}{(2E_p(E_p + m))^{1/2}} [\bar{\alpha} \cdot \bar{p}\beta(E_p + m) - (E_p + m)\, V\bar{\alpha} \cdot \bar{p}\beta] \frac{1}{(2E_p(E_p + m))^{1/2}}. \tag{AIII.3}$$

An additional transformation that removes O_1 and leaves a remaining odd part of order V^2 is desired. Note that if an operator W_1 is anti-Hermitean then

$$U_1 = (1 + W_1^2)^{1/2} + W_1 \tag{AIII.4}$$

is unitary. Letting W_1 be an operator of order V we find

$$U_1[\beta E_p + \epsilon_1 + O_1]\, U_1^{-1} = \beta E_p - [\beta E_p, W_1] + O_1 + \epsilon_1 + O_2 + \epsilon_2 + \cdots, \tag{AIII.5}$$

where O_2 and ϵ_2 are odd and even operators of order V^2. If W_1 is chosen to satisfy

$$[\beta E_p, W_1] = O_1, \tag{AIII.6}$$

then O_1 is removed. To satisfy Eq. (4.21) W_1 must be odd and thus anticommutes with β, yielding the condition

$$W_1E_p + E_pW_1 = \beta O_1 . \tag{AIII.7}$$

Since the general momentum space operator is of the form

$$gf(p) = \int g(pp')\, f(p')\, dp', \tag{AIII.8}$$

the kernel of W_1 is given by

$$W_1(\bar{p}\bar{p}') = (\beta/(E_p + E_{p'}))\, O_1(\bar{p}\bar{p}'). \tag{AIII.9}$$

This procedure may be repeated indefinitely, transforming H as

$$H' = U_nU_{n-1} \cdots U_1U_0HU_0^{-1}U_1^{-1} \cdots U_{n-1}^{-1}U_n^{-1} \tag{AIII.10}$$

with

$$U_n = (1 + W_n^2)^{1/2} + W_n \tag{AIII.11}$$

and

$$W_n(\bar{p}\bar{p}') = (\beta/(E_p + E_{p'}))\, O_n(\bar{p}\bar{p}'), \tag{AIII.12}$$

where O_n is the odd operator of order V^n generated by the previous n transformations.

Referring to Eqs. (AIII.2) and (AIII.3) we find

$$W_1(\bar{p}\bar{p}') = -\left[-\frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}}\left(\frac{E_{p'}+m}{2E_{p'}}\right)^{1/2} + \frac{\bar{\alpha}\cdot\bar{p}'}{(2E_{p'}(E_{p'}+m))^{1/2}}\left(\frac{E_p+m}{2E_p}\right)^{1/2}\right]\frac{V(\bar{p}\bar{p}')}{E_p+E_{p'}}, \tag{AIII.13}$$

or in operator form

$$W_1 = \frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}}\frac{V'}{2m}\left(\frac{E_p+m}{2E_p}\right)^{1/2} - \left(\frac{E_p+m}{2E_p}\right)^{1/2}\frac{V'}{2m}\frac{\bar{\alpha}\cdot\bar{p}}{(2E_p(E_p+m))^{1/2}} \tag{AIII.14}$$

with

$$V'(\bar{p}\bar{p}') = \frac{2m}{E_p+E_{p'}}\, V(\bar{p}\bar{p}') = \frac{2m}{E_p+E_{p'}}\left(\frac{-Ze^2}{2\pi^2}\right)\frac{1}{|\,\bar{p}-\bar{p}'\,|^2}. \tag{AIII.15}$$

Performing the transformation through U_1 we find

$$H' = \beta E_p + \left(\frac{E_p + m}{2E_p}\right)^{1/2} V \left(\frac{E_p + m}{2E_p}\right)^{1/2} + \frac{\bar{\alpha} \cdot \bar{p}}{(2E_p(E_p + m))^{1/2}} V \frac{\bar{\alpha} \cdot \bar{p}}{(2E_p(E_p + m))^{1/2}} + \beta[W_1 E_p W_1 + \tfrac{1}{2} W_1^2 E_p + \tfrac{1}{2} E_p W_1^2] + O_2 + \epsilon_3 + \cdots . \quad \text{(AIII.16)}$$

Additional transformations to remove O_2 etc. will not further change the even part of H' through order V^2.

To complete the generalization of Eqs. (4.11) it must be demonstrated that $U\mathscr{L}_+U^{-1} \sim 1$, and $[U_1U_2I_cU_1^{-1}U_2^{-1}]^{(++++)}$ must be calculated. For the first task we use the definition Eq. (3.13), and calculate in the transformed representation. The transformed H can be written

$$H' = \beta H_e + H_0 , \quad \text{(AIII.17)}$$

where H_e is an even operator given by

$$H_e = E_p + \mathcal{O}(V) + \mathcal{O}(V^2) + \cdots + \mathcal{O}(V^n) + \cdots, \quad \text{(AIII.18a)}$$

$$H_0 = \mathcal{O}(V^{n+1}) + \cdots \quad \text{(AIII.18b)}$$

(to our accuracy $n = 2$). $\mathscr{E}$ is calculated using

$$\mathscr{E} = +(H^2)^{1/2} = (H_e^2 + [\beta H_e , H_0] + H_0^2)^{1/2}. \quad \text{(AIII.19)}$$

Since the eigenvalue of $\mathscr{E}$ is positive by definition, and in the limit $V \to 0$ $\mathscr{E} \to E_p$ we write

$$\mathscr{E} = H_e + B, \quad \text{(AIII.20)}$$

where B is determined by setting

$$\mathscr{E}^2 = H_e^2 + \{B, H_e\} + B^2 = H_e^2 + \beta[H_e , H_0] + H_0^2. \quad \text{(AIII.21)}$$

From this it is seen that B is odd and of order H_0. This $\mathscr{E}^{-1}$ can be written

$$\mathscr{E}^{-1} = H_e^{-1} + H_e^{-1} Q H_e^{-1}, \quad \text{(AIII.22)}$$

where Q is odd and of order H_0 and the higher order even terms are of order H_0^2. Using Eq. (3.13) we have

$$\mathscr{L}_\pm = \tfrac{1}{2}(1 \pm (\beta H_e + H_0)(H_e^{-1} + H_e^{-1} Q H_e^{-1} + \cdots) = \tfrac{1}{2}(1 \pm \beta) \pm (H_0 H_e^{-1} + \beta Q H_e^{-1}) + \mathcal{O}(H_0)^2. \quad \text{(AIII.23)}$$

Thus, to the order in which we are working in the transformed representation, it is legitimate to put

$$\mathscr{L}_{\pm} = \tfrac{1}{2}(1 \pm \beta). \qquad \text{(AIII.24)}$$

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