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ABSTRACT

We explore the role of activity in the occurrence of the Mpemba effect within a system of an active colloid diffusing in a potential landscape devoid of metastable minimum. The Mpemba effect is characterized by a phenomenon where a hotter system reaches equilibrium quicker than a colder one when both are rapidly cooled to the same low temperature. While a minimal asymmetry in the potential landscape is crucial for observing this effect in passive colloidal systems, the introduction of activity can either amplify or reduce the threshold of this minimal asymmetry, resulting in the activity-induced and suppressed Mpemba effect. We attribute these variations in the Mpemba effect to the effective translational shift in the phase boundaries, which occurs as activity is changed.

I. INTRODUCTION

The Mpemba effect refers to the anomalous relaxation phenomenon where a system that is initially hotter equilibrates faster than an identical system that is initially cooler when both systems are quenched to the same low temperature.[1] The phenomenon was originally seen experimentally in the freezing of water when the quench was done across the freezing temperature.[1–9] However, the effect is now known to be much more general and applicable to the relaxation of any stochastic process and does not necessarily require a quench across a phase transition. This generality has stimulated the study of the Mpemba effect in a variety of systems, both theoretically and experimentally.

Physical systems other than water where the Mpemba effect has now been experimentally observed include magnetic alloys,[10] polylactides,[11] clathrate hydrates,[12] colloidal systems,[13–15] single trapped ion qubits,[16] etc. Numerous theoretical studies on model systems have demonstrated the presence of the Mpemba effect in spin systems,[17–22] Markovian systems with few states,[23,24] particles diffusing in a potential,[25–32] active systems,[33] spin glasses,[34] molecular gases in contact with a thermal reservoir,[35–38] quantum systems,[39–51] systems with phase transitions,[20,52–55] and granular systems.[56–63]

One approach to understanding the Mpemba effect is to study it within a minimal model that exhibits the effect yet is analytically tractable. In a recent experiment,[13] the Mpemba effect was demonstrated unambiguously in the relaxation of a Brownian particle trapped in an asymmetric double well potential, showing that complex inter-particle interactions are not a necessity for the effect. Motivated by this experiment, we studied the overdamped dynamics of a Brownian particle in an asymmetric piecewise linear double-well[30] as well as a single-well potential.[31] The linearity of the potential makes it analytically tractable. The existence of the Mpemba effect in the single-well potential with minimal asymmetry demonstrated that the presence of metastable states is not a necessary condition for the effect,[31] in contrast to earlier notions that trapping of the initially colder state in the metastable minima leads to a faster relaxation of the hotter system as suggested in the Refs. 17, 18, 23, 25, 30, and 33. This makes the single-well potential a simple minimal model, devoid of inter-particle interactions as well as multiple minima, for studying the Mpemba effect.

In this paper, we now ask how the relaxation dynamics of the Brownian particle trapped in the single-well potential is modified in the presence of activity. Active Brownian particles, or active colloids, are self-propelling particles that convert chemical energy to mechanical energy, thus constantly pumping energy into the system. The presence of activity makes these systems far from equilibrium. The introduction of activity leads to behaviors that are quite distinct from the passive Brownian particle, such as the existence of non-Boltzmannian steady states,[64,65] wall accumulation,[66,67] activity induced ratchet motion,[68] vortices,[69] and motility induced phase separation.[70]

There have been a couple of earlier studies of the Mpemba effect in relaxation dynamics in the presence of activity. Along the lines of the experiment of the Mpemba effect in the system of a colloidal particle in an asymmetric double well potential,[13] the role of activity in the same setup was explored numerically in Ref. 33. It was shown that activity can induce the Mpemba effect in parameter regimes where it is absent for quenches in the passive model and vice versa for the heating protocol (inverse Mpemba effect). A similar study[71] was done by introducing activity in a discrete three-state Markov process generalizing the relaxation dynamics studied in Ref. 23. Here, there are energy barriers between the states resulting in activated dominated dynamics. It was shown that activity leads to unique relaxation phenomena such as the activity induced and suppressed Mpemba effects as well as oscillations in the transients of the relaxation process that are distinct from the passive models.

Both these studies are based on the presence of multiple minima in the underlying energy landscape. Since these minima are not required for observing the Mpemba effect in the passive case, the role of activity is best explored in a single-well potential, thus decoupling the possible role played by multiple minima. With this motivation, we consider the dynamics of an active Brownian particle diffusing in a single well potential landscape. Although a minimal asymmetry in the potential barriers at its left and right edges leads to the Mpemba effect in the passive model of the Brownian particle, the presence of activity can effectively reduce or enhance the minimal asymmetry. As such, it leads to a unique relaxation behavior, such as the activity induced and suppressed Mpemba effect when compared to the passive model. The cause of such a phenomenon can be mapped to the effective translational shift of the existing phase boundaries of the passive system in the presence of activity, leading to the activity induced and suppressed Mpemba effect for a given choice of parameters.

The remainder of this paper is organized as follows: We define the model and describe its formalism of the dynamics in Sec. II. Next, we define the Mpemba effect and its necessary criteria in Sec. III. In Sec. IV, we explore the role of activity in the presence or absence of the Mpemba effect. In Sec. V, we study the phase diagram of the parameter space of the model to understand the underlying significance of activity in the dynamics of the Mpemba effect. Section VI contains the summary of results and a discussion of their implications.

II. MODEL AND FORMALISM

We consider an active Brownian particle placed in a one dimensional potential, $\tilde{U}(\tilde{x})$, in the presence of a thermal environment characterized by damping $\tilde{\gamma}$ and noise $\tilde{\eta}$ with the statistics $\langle\tilde{\eta}(t)\rangle = 0$ and $\langle\tilde{\eta}(\tilde{t})\tilde{\eta}(\tilde{t}')\rangle = 2\gamma k_B\tilde{T}_b\delta(\tilde{t}-\tilde{t}')$. Here, $\tilde{T}_b$ is the temperature of the thermal bath and k_B is the Boltzmann's constant. The equation of motion of the active particle in the overdamped approximation, where the damping γ is large compared to the mass of the particle, is given by the Langevin equation,[33]

$$\frac{d\tilde{x}}{d\tilde{t}} = \tilde{v}_0 n - \frac{1}{\gamma}\frac{d\tilde{U}}{d\tilde{x}} + \tilde{\eta}(\tilde{t}), \quad (1)$$

where $\tilde{v}_0$ is the self-propulsion speed and n denotes its direction of motion where $n = \pm 1$ for $\pm x$-direction. The value of n flips stochastically with a waiting time $\tilde{t}_p$ drawn from an exponential distribution $p(\tilde{t}_p) = \tilde{\tau}_p^{-1}e^{-\tilde{t}_p/\tilde{\tau}_p}$ with $\tilde{\tau}_p$ being the persistence time.

We denote the probability density of the particle propelling to the right by $P_r(\tilde{x},\tilde{t})$ and that for the left with $P_l(\tilde{x},\tilde{t})$. The equations of motion in terms of the probability densities are given by[33]

$$\frac{\partial P_r}{\partial\tilde{t}} = \frac{1}{\gamma}\frac{\partial}{\partial\tilde{x}}\left[\frac{d\tilde{U}}{d\tilde{x}}P_r\right] - \tilde{v}_0\frac{\partial P_r}{\partial\tilde{x}} + \frac{k_B\tilde{T}_b}{\gamma}\frac{\partial^2 P_r}{\partial\tilde{x}^2} - \frac{P_r}{\tilde{\tau}_p} + \frac{P_l}{\tilde{\tau}_p}, \quad (2)$$

$$\frac{\partial P_l}{\partial\tilde{t}} = \frac{1}{\gamma}\frac{\partial}{\partial\tilde{x}}\left[\frac{d\tilde{U}}{d\tilde{x}}P_l\right] + \tilde{v}_0\frac{\partial P_l}{\partial x} + \frac{k_B\tilde{T}_b}{\gamma}\frac{\partial^2 P_l}{\partial\tilde{x}^2} - \frac{P_l}{\tilde{\tau}_p} + \frac{P_r}{\tilde{\tau}_p}. \quad (3)$$

We now introduce the following dimensionless variables: $x = (2\pi/L)\tilde{x}$, $T = \tilde{T}/\tilde{T}_b$, $U = \tilde{U}/(k_B T_b)$, $t = \tilde{t}/\tau_p$, $v_0 = (2\pi\tilde{v}_0\tau_p)/L$, and $\tau_p = (4\pi^2 k_B T_b\tilde{\tau}_p)/(\gamma L^2)$, where $\tilde{T}$ refers to an initial temperature higher than the temperature $\tilde{T}_b$. In terms of the dimensionless variables, the time evolution of the probability densities in Eqs. (2) and (3) can be rewritten as follows:

$$\frac{\partial P_l}{\partial t} = v_0\frac{\partial P_l}{\partial x} + \tau_p\frac{\partial}{\partial x}\left[\frac{dU}{dx}P_l\right] + \tau_p\frac{\partial^2 P_l}{\partial x^2} - P_l + P_r, \quad (4)$$

$$\frac{\partial P_r}{\partial t} = -v_0\frac{\partial P_r}{\partial x} + \tau_p\frac{\partial}{\partial x}\left[\frac{dU}{dx}P_r\right] + \tau_p\frac{\partial^2 P_r}{\partial x^2} - P_r + P_l. \quad (5)$$

The densities for the total occupation probability, $P(x,t)$, and for the polarization, $Q(x,t)$, are defined as

$$P(x,t) = P_r(x,t) + P_l(x,t), \quad (6)$$

$$Q(x,t) = P_r(x,t) - P_l(x,t). \quad (7)$$

The quantity $P(x,t)$ denotes the probability density of finding the active particle at position x and time t irrespective of its bias to move in a positive or negative x direction. On the other hand, the polarization density $Q(x,t)$ denotes the preferential bias at position x and time t to move in the positive x-direction over the negative direction. The quantities $P(x,t)$ and $Q(x,t)$ evolve in time as

$$\frac{\partial P}{\partial t} = -v_0\frac{\partial Q}{\partial x} + \tau_p\frac{\partial}{\partial x}\left[\frac{dU}{dx}P\right] + \tau_p\frac{\partial^2 P}{\partial x^2}, \quad (8)$$

$$\frac{\partial Q}{\partial t} = -v_0\frac{\partial P}{\partial x} + \tau_p\frac{\partial}{\partial x}\left[\frac{dU}{dx}Q\right] + \tau_p\frac{\partial^2 Q}{\partial x^2} - 2Q. \quad (9)$$

We consider a single well potential that is piecewise linear. The choice of such a potential gives an analytically tractable model for

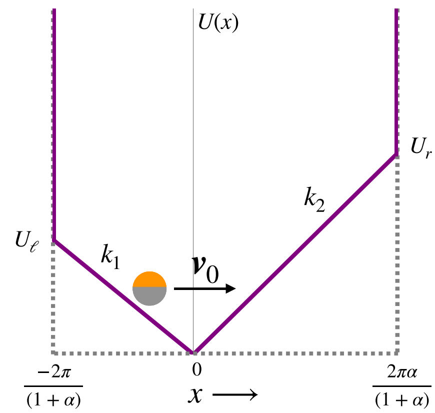

FIG. 1. Schematic diagram of the system of an active particle in a piecewise linear potential. The activity of the colloidal particle is characterized by its persistence velocity $\boldsymbol{v}_0$. The boundaries of the potential are situated at $\frac{-2\pi}{(1+\alpha)}$ and $\frac{2\pi\alpha}{(1+\alpha)}$ with the total length of the domain fixed at 2π. The parameters k_1 and k_2 refer to the various slopes, α denotes the asymmetry factor for the widths of left and right domains, and U_ℓ and U_r refer to the potential heights at the left and the right boundaries.

the system of passive Brownian particles and motivates us to consider the same in the case of the active system in order to make a direct comparison of the key results with the passive system for a given parameter space. The boundaries of the well are situated at (x_{min}, x_{max}) with $x_{min} = -(2\pi)/(1+\alpha)$ and $x_{max} = (2\pi\alpha)/(1+\alpha)$ such that the total length of the domains is fixed at 2π and the parameter α determines the asymmetry between the widths of the left and right domains. The construction of the potential in this way helps in reducing the number of variables in the model. As such, the parameters characterizing the configuration of the potential are U_ℓ, U_r, and α. The shape of the potential is shown in Fig. 1 and is described quantitatively as

$$U(x) = \begin{cases} U_\ell + k_1(x - x_{min}), & x_{min} < x < 0, \\ k_2 x, & 0 < x < x_{max}, \end{cases} \tag{10}$$

where $k_1 = U_\ell/x_{\min}$; $k_2 = U_r/x_{\max}$; and α, U_ℓ, and U_r are the constants.

Equations (8) and (9) can be written in a concise matrix form as

$$\frac{\partial \boldsymbol{P}}{\partial t} = \boldsymbol{\mathcal{L}}\boldsymbol{P}, \tag{11}$$

where the vector $\boldsymbol{P} = (P, Q)^{\mathrm{Tr}}$, where Tr represents transpose and matrix,

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \tau_p \frac{d^2U}{dx^2} + \tau_p \frac{dU}{dx}\frac{\partial}{\partial x} + \tau_p \frac{\partial^2}{\partial x^2} & -v_0 \frac{\partial}{\partial x} \\ -v_0 \frac{\partial}{\partial x} & \tau_p \frac{d^2U}{dx^2} + \tau_p \frac{dU}{dx}\frac{\partial}{\partial x} + \tau_p \frac{\partial^2}{\partial x^2} - 2 \end{bmatrix}. \tag{12}$$

Since the potential diverges at the boundaries, no flux boundary condition is implemented and is given by

$$\boldsymbol{j}(x_{min}) = \boldsymbol{j}(x_{max}) = 0, \tag{13}$$

where the probability current/flux is given by

$$\boldsymbol{j} = \begin{pmatrix} -\tau_p \frac{dU}{dx} P - \tau_p \frac{\partial P}{\partial x} + v_0 Q \\ -\tau_p \frac{dU}{dx} Q - \tau_p \frac{\partial Q}{\partial x} + v_0 P \end{pmatrix}. \tag{14}$$

The solution for Eq. (11) is obtained using the eigenspectrum decomposition method as

$$\boldsymbol{P}(x,t) = \boldsymbol{\pi}(x, T_b) + \sum_{i\ge 2} a_i(T, T_b)\boldsymbol{v}_i(x)e^{\lambda_i t}, \tag{15}$$

where $\boldsymbol{v}_i$ are the right eigenfunctions of $\boldsymbol{P}(x,t)$, λ_i are the eigenvalues that follow the order $\lambda_1 = 0 > \lambda_2 > \lambda_3 \ldots$, and $\boldsymbol{\pi}(x, T_b)$ is the final steady state distribution corresponding to $\lambda_1 = 0$. In order to compute the coefficient a_i, we consider the case of $t = 0$ such that

$$\boldsymbol{P}(x,0) = \boldsymbol{\pi}(x, T_b) + \sum_{i\ge 2} a_i(T, T_b)\boldsymbol{v}_i(x). \tag{16}$$

Note that $\boldsymbol{P}(x,0) \equiv \boldsymbol{\pi}(x,T)$ and is the initial steady state distribution at a temperature $T \ne T_b$ where the system is initially prepared before being quenched to the final bath temperature T_b. Then the coefficients $a_i(T, T_b)$ are given by the inner product of the left eigenfunctions $\boldsymbol{u}_i(x)$ (solved for bath temperature T_b) with the initial steady state distribution $\boldsymbol{\pi}(x,T)$ as

$$a_i(T, T_b) = \frac{\langle \boldsymbol{u}_i | \boldsymbol{\pi}(x,T)\rangle}{\langle \boldsymbol{u}_i | \boldsymbol{v}_i\rangle}. \tag{17}$$

We now discuss the computation of the left eigenfunctions $\boldsymbol{u}(x)$ and the initial steady state distribution $\boldsymbol{\pi}(x,T)$. To compute the left eigenfunctions $\boldsymbol{u}(x)$, one needs to work with the adjoint operator $\boldsymbol{\mathcal{L}}^\dagger$ of $\boldsymbol{\mathcal{L}}$. In order to compute $\boldsymbol{\mathcal{L}}^\dagger$, we consider two test functions $\boldsymbol{\psi}(x)$ and $\boldsymbol{\phi}(x)$ and do the following exercise:

$$\langle \boldsymbol{\psi}(x) | \boldsymbol{\mathcal{L}}\boldsymbol{\phi}(x)\rangle = \int_{x_{min}}^{x_{max}} dx \boldsymbol{\psi}^*(x)\boldsymbol{\mathcal{L}}\boldsymbol{\phi}(x). \tag{18}$$

After doing the integration by-parts for the components of the operator $\boldsymbol{\mathcal{L}}$ and rearranging, we obtain

$$\langle \boldsymbol{\psi}(x) | \boldsymbol{\mathcal{L}}\boldsymbol{\phi}(x)\rangle = \langle \boldsymbol{\psi}(x)\boldsymbol{\mathcal{L}}^\dagger | \boldsymbol{\phi}(x)\rangle - \left[\boldsymbol{\psi}.\boldsymbol{j} + \frac{k_B T_b}{\gamma}\frac{\partial \boldsymbol{\psi}}{\partial x}.\boldsymbol{\phi}\right]_{x_{min}}^{x_{max}}, \tag{19}$$

where the adjoint operator $\boldsymbol{\mathcal{L}}^\dagger$ is given by

$$\boldsymbol{\mathcal{L}}^\dagger = \begin{bmatrix} -\tau_p \frac{dU}{dx}\frac{\partial}{\partial x} + \tau_p \frac{\partial^2}{\partial x^2} & -v_0 \frac{\partial}{\partial x} \\ -v_0 \frac{\partial}{\partial x} & -\tau_p \frac{dU}{dx}\frac{\partial}{\partial x} + \tau_p \frac{\partial^2}{\partial x^2} - 2 \end{bmatrix}, \tag{20}$$

and it clearly obeys the Neumann boundary conditions given in terms of its eigenfunctions $\boldsymbol{\psi}(x)$ as

$$\frac{\partial \boldsymbol{\psi}}{\partial x} = 0 \text{ at } x = x_{min}, x_{max}. \tag{21}$$

The initial steady state distribution $\boldsymbol{\pi}(x,T)$ at temperature T is obtained by numerically solving the coupled differential equations (8) and (9) with the time derivative set to zero and with their appropriate boundary conditions. Although the coupled differential equations are not analytically solvable even with the piecewise linear, the solutions are numerically exact.

III. THE Mpemba EFFECT

With the above framework, we aim to study the consequence of the activity on the anomalous relaxation dynamics, namely, the Mpemba effect. It refers to the faster relaxation of an initially hotter system compared to an initially warmer system when both are quenched to a common final state characterized by an even colder temperature. For the current system under consideration, the temperature of the heat bath sets the temperature of the system. However, note that due to the presence of external active forces, the system has a non-equilibrium steady state instead of the equilibrium state, and it is characterized by the distribution $\boldsymbol{\pi}(x, T)$ corresponding to some choice of the active parameters v_0, τ_p, and temperature T of the heat bath. Keeping all the parameters the same, the quench from one steady state to another is done by changing the temperature of the bath.

Upon quenching, the time evolution of the probability distribution $\boldsymbol{P}(x, t)$ is given by Eq. (15). Since at large times, only the first non-zero largest eigenvalue dominates, the condition for the Mpemba effect can be obtained from the long time limit of the evolution equation (15), which is given by

$$\boldsymbol{P}(x,t) \simeq \boldsymbol{\pi}(x, T_b) + a_2(T, T_b)\boldsymbol{v}_2(x)e^{\lambda_2 t}, \tag{22}$$

where a_2 is given by

$$a_2 = \frac{\langle \boldsymbol{u}_2 | \boldsymbol{\pi}(x, T)\rangle}{\langle \boldsymbol{u}_2 | \boldsymbol{v}_2\rangle}. \tag{23}$$

Now, let us consider two identical systems that are prepared at two different initial steady states corresponding to the choice of hot and warm bath temperatures, T_h and T_c, respectively. As both the systems are quenched to the common steady state of the bath temperature T_b such that $T_h > T_c > T_b$, then the Mpemba effect is said to exist if

$$|a_2(T_h, T_b)| < |a_2(T_c, T_b)|. \tag{24}$$

It is so because, with the above condition, the distribution $\boldsymbol{P}(T_c, t)$ of the initially cold system lags behind the distribution $\boldsymbol{P}(T_h, t)$ of the initially hot system, leading to the Mpemba effect. Thus, the overall task reduces to computing the coefficient $a_2(T, T_b)$ in Eq. (23).

Equivalently, the relaxation process can also be described in terms of the distance from the steady state function, $D[\boldsymbol{P}(t), \boldsymbol{\pi}(T_b)]$, which measures the instantaneous distance of a distribution $\boldsymbol{P}(x, t)$ from the final steady state distribution, $\boldsymbol{\pi}(T_b)$. There exist numerous well defined distance measures in the literature, such as Kullback–Leibler (KL) divergence,[13,23,26] entropic distance, L_1-norm, and so on. Thus, in terms of the distance function, the initially hot and the cold system prepared at temperatures T_h and T_c, respectively, are denoted by the inequality $D[\boldsymbol{\pi}(T_h), \boldsymbol{\pi}(T_b)] > D[\boldsymbol{\pi}(T_c), \boldsymbol{\pi}(T_b)]$ for $T_h > T_c$. If it is followed by $D[\boldsymbol{P}_h(t), \boldsymbol{\pi}(T_b)] < D[\boldsymbol{P}_c(t), \boldsymbol{\pi}(T_b)]$ at a later time t, we state that the Mpemba effect exists.

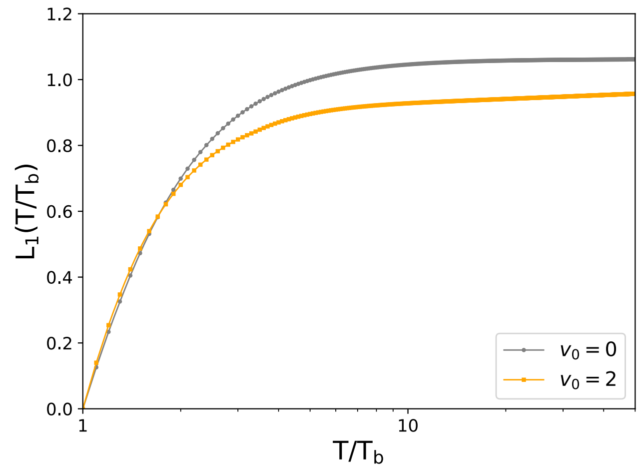

FIG. 2. Variation of the distance function L_1 as a function of the initial temperatures T. The final quenched temperature corresponds to $T_b = 1$. Monotonic rise of $L_1(T)$ with T implies that $L_1^h > L_1^c$ for the initially hot and cold systems with temperatures T_h and T_c, respectively, and the result is independent of the passive or active cases of the model as shown here for $v_0 = 0$ and $v_0 = 2$.

In this paper, we discuss the L_1-norm for its ease of convenience to compare with the discussion in terms of the a_2 measure. It is defined as

$$D[\boldsymbol{P}(t), \boldsymbol{\pi}(T_b)] \equiv L_1(t) = \int dx |\boldsymbol{P}(x,t) - \boldsymbol{\pi}(x, T_b)|. \tag{25}$$

We will now show that the use of measure $|a_2(T, T_b)|$ is equivalent to using any other distance measure, namely, the L_1-measure. For that purpose, we need to understand the behavior of L_1 at time $t = 0$ for different temperatures and also how it behaves at large times. First, we show that at $t = 0$, $L_1(T)$ is a monotonically increasing function of temperature T irrespective of the passive or active model as shown in Fig. 2. As a result, for two initially hot and cold systems with temperatures T_h and T_c with $T_h > T_c$, it means that $L_1^h > L_1^c$. At large times, using Eq. (22), we can write for $L_1(t)$ as

$$L_1(t) \simeq |a_2(T, T_b)| \int dx \; |\boldsymbol{v}_2(x)e^{\lambda_2 t}|. \tag{26}$$

Thus, if $|a_2(T_h, T_b)| < |a_2(T_c, T_b)|$, then $L_1^h < L_1^c$, i.e., the *hot* system is closer to the final steady state compared to the initial *cold* system. From the discussion, it is quite clear that the measure $a_2(T, T_b)$ is sufficient to describe the Mpemba effect for the present model.

IV. ROLE OF ACTIVITY IN THE Mpemba EFFECT

With the above formalism, we look at the effect of active parameters of the model v_0 and τ_p, in the existence of the Mpemba effect. Note that $v_0 = 0$ corresponds to the model of a passive Brownian particle, and $v_0 \neq 0$ corresponds to that of the active particle while keeping every other parameter of the potential and bath the same for both cases. Since the parameters v_0 and τ_p are coupled to each other, for the simplicity of the analysis, we shall set τ_p as unity unless otherwise mentioned, and an increase in activity will henceforth refer

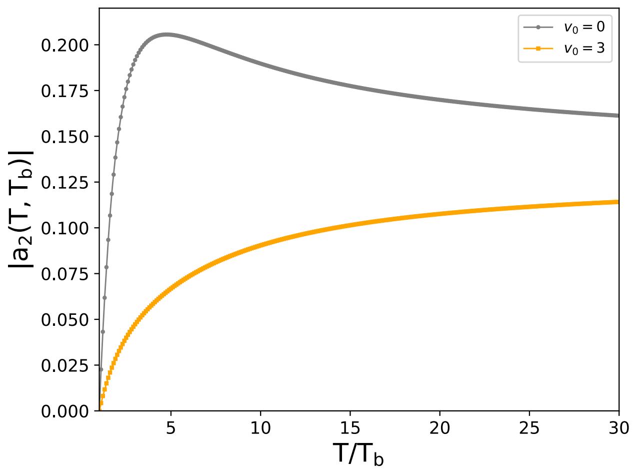

FIG. 3. Increase in activity suppressing the Mpemba effect. Variation of $|a_2(T, T_b)|$ with T for the chosen set of parameters: $U_\ell = 4$, $U_r = 10$, $\gamma = 1$, and $\tau_p = 1$; and for the choices of the persistence velocities: $v_0 = 0$ and $v_0 = 3$. The final quenched temperature corresponds to $T_b = 1$. With the increase in v_0, the Mpemba effect vanishes.

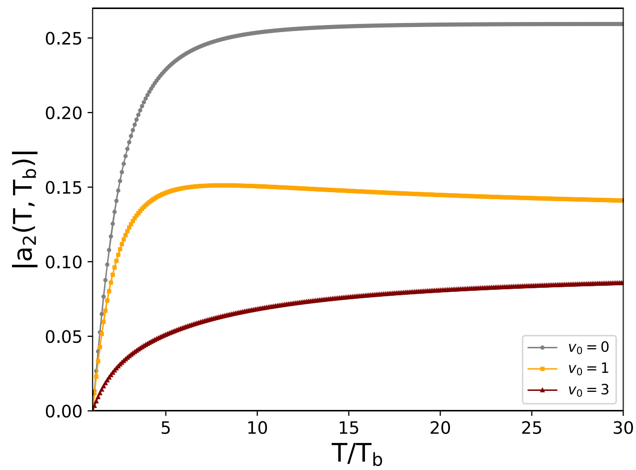

FIG. 4. Presence of activity induces the Mpemba effect, although the further increase in activity suppresses the effect. Variation of $|a_2(T, T_b)|$ with T for the chosen set of parameters: $U_\ell = 6$, $U_r = 10$, $\gamma = 1$, and $\tau_p = 1$; and for the choices of the persistence velocities: $v_0 = 0$, $v_0 = 1$, and $v_0 = 3$. The final quenched temperature corresponds to $T_b = 1$. Compared to the passive case $(v_0 = 0)$, where there is no Mpemba effect, it emerges as v_0 is increased, but it vanishes with further increases in the activity.

to an increase in v_0. To that end, we explore various configurations of the external potential and ask if the presence of activity $(v_0 \neq 0)$ helps or suppresses the existing phase space of the Mpemba effect that is already known for the passive scenario of the model.

A. Mpemba effect suppressed due to activity

We first consider a case where the Mpemba effect already exists in the passive scenario of the model. In our example, it corresponds to making a choice of the potential parameters $U_\ell = 4.0$, $U_r = 10.0$, and $\alpha = 1.0$, and the bath temperature corresponding to the final steady state is set at $T_b = 1.0$. The presence of the Mpemba effect in this parameter space is confirmed by the non-monotonic variation of the coefficient $|a_2(T, T_b)|$ with T for $v_0 = 0$ as shown in Fig. 3.

Now with the introduction of the activity parameter v_0 and keeping the same configuration of the potential, the Mpemba effect persists over a wide range of initial temperatures. However, as we further increase v_0, the Mpemba effect vanishes as characterized by the monotonic rise of $|a_2(T, T_b)|$ with T as shown in Fig. 3 for $v_0 = 3$. Thus, with the example that we considered, an increase in activity, or, in other words, increasing the persistence velocity of the particle, suppresses the Mpemba effect. A systematic analysis of the above discussion is illustrated in Fig. 6(a). The activity suppressed Mpemba effect can be qualitatively explained in terms of the fact that the presence of additional fluctuations due to activity can help overcome any slowness in the dynamics of the initially warmer system, leading to no Mpemba effect.

B. Intermediate activity can induce Mpemba effect

In order to verify if the role of activity in the Mpemba effect as appeared in the above example is universal, we next consider the opposite case where the Mpemba effect is originally absent for the passive particle ($v_0 = 0$). In our example, it corresponds to making a choice of the external potential $U_\ell = 6.0$, $U_r = 10.0$, and $\alpha = 1.0$ with $T_b = 1.0$. The absence of the Mpemba effect is evident from the monotonic rise in $|a_2(T, T_b)|$ with T for $v_0 = 0$ as shown in Fig. 4. However, in contrast to the previous case where activity above a certain threshold suppresses the Mpemba effect, on the introduction of the active degree of freedom of the particle characterized by non-zero v_0, the system shows the Mpemba effect as evident from the non-monotonic rise of $|a_2(T, T_b)|$ with T for $v_0 = 1$ (see Fig. 4). Thus, in this case, we observe an opposite behavior of the activity enhancing the Mpemba effect as compared to the previous case. However, on further increasing the activity of the system to $v_0 = 3$, the Mpemba effect vanishes, and the relaxation behavior turns out to be the same as in the passive system. A systematic analysis of the phenomena where intermediate activity induces the Mpemba effect is illustrated in Fig. 6(b). An intuitive explanation of such a behavior can be given in terms of the possibility that in the presence of intermediate activity, the initially warmer system can experience momentary traps in the energy landscape due to its persistence in a particular state leading to an activity induced Mpemba relaxation. On the other hand, the initially hot system does not experience such traps due to its high temperature. However, in the limit of a larger activity, it again goes back to its diffusive behavior[65] and exhibits similar results as its passive case.

V. PHASE DIAGRAM

We have seen that having an intermediate value of activity leads to the Mpemba effect, where it is absent in the passive case. On the contrary, for the case where the Mpemba effect is already present in the passive model, introducing activity above a certain threshold suppresses the Mpemba effect. However, in the presence of large activity, the Mpemba effect is always absent irrespective of whether it is present or absent in the passive model. In order to have a broader understanding of the role of activity in the Mpemba effect, it can be more informative to seek how the presence of activity affects, in general, the phase diagram of system parameters that leads to the Mpemba effect for the passive model.

To that end, we study the different configurations of the potential well to differentiate which configurations allow and do not allow the Mpemba like relaxations, as the activity is varied starting from the scenario where $v_0 = 0$ that corresponds to the phase diagram of the passive model. This allows us to understand the pattern in which the phase diagram of the passive model changes with the introduction of activity.

Note that the configuration of the potential well depends on the three parameters: U_ℓ, U_r, and α. From the study of the Mpemba effect for passive colloids in the presence of single well potential, it is known that the asymmetry in the potential $U(x)$ is the key, and it can be introduced through $U_\ell \neq U_r$ and/or $\alpha \neq 1$. In a similar vein, we explore the effect of all three parameters and also study the change in behavior of the phase diagram as the activity is varied starting from the passive model. For that purpose, we determine the phase diagram in the U_ℓ–U_r plane for different α and v_0 as shown in Fig. 5.

For the passive model ($v_0 = 0$), the symmetric case $U_\ell = U_r$ does not show the Mpemba effect even if $\alpha \neq 1$ [see Ref. 31 and Figs. 5(d) and 5(g)]. The same is true even in the presence of activity ($v_0 \neq 0$), and thus a minimal asymmetry in U_ℓ and U_r is the key to observe the Mpemba effect. Next, having known that asymmetry in U_ℓ and U_r is important, we now probe the behavior of the parameter space region as the parameters α and v_0 are varied.

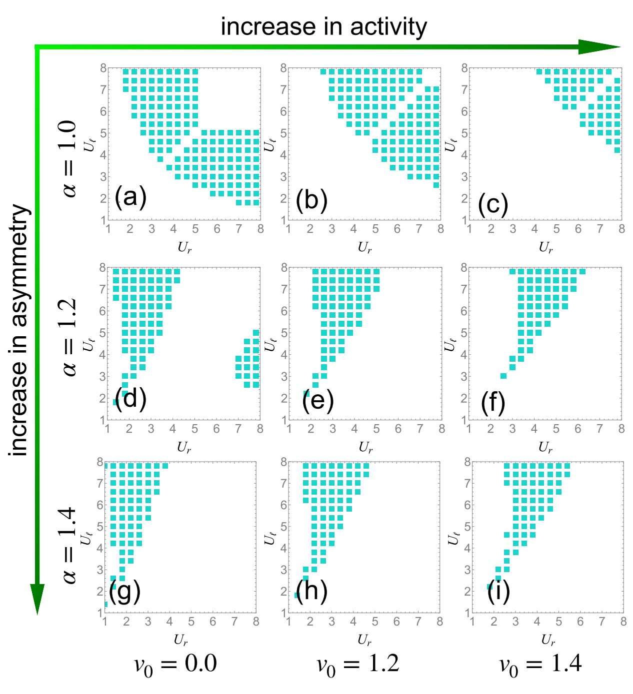

FIG. 5. Change in the parameter space region of initial conditions that show the Mpemba effect (shown as a colored region) as a function of the change in activity characterized by persistence velocity, v_0, and change in asymmetry parameter, α. The final quenched temperature corresponds to $T_b = 1$. The effect of the variation in asymmetry and activity on the parameter space region is illustrated in panels (a)–(i).

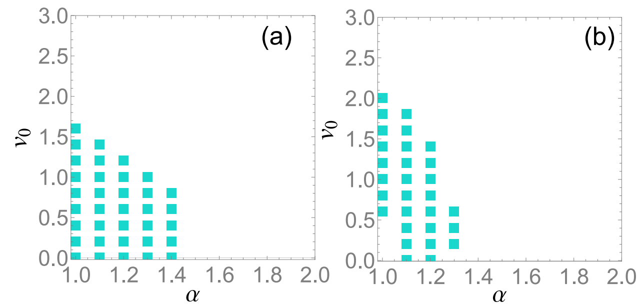

FIG. 6. Phase diagram illustrating the activity suppressed and induced Mpemba effect in the v_0–α plane, keeping U_ℓ and U_r fixed. The colored regions in the parameter space show the Mpemba effect. Panel (a) shows that while the Mpemba effect is present in the passive case ($v_0 = 0$) with symmetric domain widths ($\alpha = 1$), an increase in v_0 and α suppresses the Mpemba effect. Here, $U_\ell = 4$ and $U_r = 10$ are kept fixed. Panel (b) shows that the presence of either the intermediate activity or asymmetry in domain widths can induce the Mpemba effect while it is absent in the passive-symmetric case ($v_0 = 0$ and $\alpha = 1$). Further increases in v_0 and α, however, suppress the Mpemba effect. Here, $U_\ell = 6$ and $U_r = 10$ are kept fixed. The final quenched temperature corresponds to $T_b = 1$.

We know the effect of the asymmetry parameter α for the passive model, that it shrinks the phase boundaries of the Mpemba effect as α is increased. Now in the same study, we seek to understand how the phase boundaries behave as the activity is varied. We find that with the simultaneous increase in activity along with α, the phase boundaries are affected in two ways: first, unlike the passive case, the increase in α can either decrease or increase the region of phase boundaries depending on the value of the activity. As shown in Fig. 5, the region of phase boundaries decreases with the increase in α for $v_0 = 1.2$, while it increases for $v_0 = 1.4$. Thus, in the presence of activity, the role of asymmetry in the increase or decrease of the phase boundaries seems ambiguous. However, for a given α, the increase in activity effectively shifts the phase boundaries of the passive model. As a result, it leads to the emergence (and depletion) of the parameter space region that shows the Mpemba effect in the presence of activity where it was initially absent (or present) for the passive case. Thus, it leads to the activity induced and suppressed Mpemba effect.

With the above knowledge of the effect of activity and asymmetry on the parameter space of initial conditions leading to the Mpemba effect, now we highlight phase diagrams in the v_0–α plane depicting the phenomena of activity suppressed and induced Mpemba effects more comprehensively, as illustrated in Figs. 6(a) and 6(b), respectively. Figure 6(a) shows that while the Mpemba effect is present in the passive case ($v_0 = 0$) with symmetric domain widths ($\alpha = 1$), an increase in v_0 and α suppresses the Mpemba effect. On the contrary, Fig. 6(b) shows that the presence of either the intermediate activity or asymmetry in domain widths can induce the Mpemba effect while it is absent in the passive-symmetric case ($v_0 = 0$, $\alpha = 1$). Further increases in v_0 and α, however, suppress the Mpemba effect.

VI. CONCLUSION

In summary, we studied the Mpemba effect for an active Brownian particle in a single well potential that is piecewise linear. While the presence of the Mpemba effect in the setup of a single well potential with a passive Brownian particle had been confirmed earlier where the relaxation starts from an equilibrium Boltzmann distribution, the presence of activity presents a non-equilibrium steady state, and the consequences of the presence of the activity in the relaxation process were the main motive of the current study. To that end, we explored how the phase space pertaining to different configurations of the external potential that leads to the Mpemba effect is affected by the change in activity parameters.

We first show that the presence of activity can show the Mpemba effect for a given parameter space where it was absent for the passive Brownian particle and vice versa, leading to the activity induced and suppressed Mpemba effect compared to the system of passive colloid. The change in the pattern of the parameter space regions that show the Mpemba effect, with the increase in activity, sheds light on the activity induced and suppressed Mpemba effect.

While the parameter space that shows the Mpemba effect changes in the presence of the activity, the necessity of a minimal asymmetry in U_ℓ and U_r of the external potential for the Mpemba effect remains unchanged as in the passive case. However, unlike the parameter space of the Mpemba effect for the passive case that decreases with the increase in additional asymmetry in terms of parameter α of the potential well, the region of parameter space showing the Mpemba effect can increase or decrease with α depending on the value of the activity. Moreover, for a given value of the asymmetry parameter α, the increase in activity leads to an effective translational shift in the region of parameter space that shows the Mpemba effect when compared to the passive case. As a result, addition and deletion in the existing parameter space of the passive setup take place, and hence it clarifies the fact of why the activity induced Mpemba effect takes place for a set of system parameters where it is absent in the passive case and vice versa.

Several groups have developed experimental setups that are suitable for exploring the questions raised in this paper.[72–74] Another direction of future research using the current setup would be to explore the time delayed cooling protocols[75] to investigate the emergence of the Mpemba and Kovacs effect.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Apurba Biswas: Conceptualization (equal); Investigation (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal). **R. Rajesh**: Conceptualization (equal); Investigation (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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