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End corrections for rarefied gas flows through capillaries of finite length

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ABSTRACT

The influence of end effects on rarefied gas flows through moderately long capillaries is investigated by implementing the concept of effective capillary length, representing a sum of the real length of capillary and its increment. To calculate the length increment, a flow field near the inlet cross section of capillary is analyzed on the basis of the linearized kinetic equation. Far from the inlet inside of capillary, the solution is matched at the level of the distribution function with the one-dimensional solution corresponding to the flow in an infinite capillary. Far from the inlet outside of capillary, the gas is assumed to be in equilibrium at a specific pressure and temperature. The capillary length increment has been calculated as a function of the gas rarefaction. Using these results it is possible to estimate a flow rate through a moderately long capillary without hard calculations for the complete geometry.

Rarefied gas flows through long capillaries of various cross section shapes are very important in many technologies such as vacuum systems, gaseous microfluidics, etc. Such flows have been deeply studied on the basis of linearized kinetic equations in the whole range of gas rarefaction, see e.g. the review [1] and some papers [2–6] published recently. All these works assume the flow to be fully developed based on the fact that the capillary is sufficiently long so that the end effects can be neglected. This assumption allows us to simplify the kinetic equation and to obtain accurate results with modest computational effort.

However, the assumption of fully developed flow is not always fulfilled in practice, leading to a significant error in flow rate estimation because of the capillary inlet/outlet influence. For short capillaries, the end effects can be taken into account by including into consideration additional regions before and after the capillary [7,8]. Such an approach becomes computationally costly in case of long capillaries and therefore in most applications only empirical formulas are applied [9,10].

To circumvent this difficulty, it is convenient to introduce the concept of effective length, i.e. the real capillary length L is corrected by an additional length ΔL related to the end effect. Once the end correction ΔL is known, the flow rate through a finite capillary can be calculated by multiplying the flow rate for an infinitely long capillary by the factor $L/(L+\Delta L)$. The end correction ΔL was initially calculated for slow viscous flows [11]. Then, the same idea was used for rarefied gas flows through moderately long channels [12,13]. The corresponding data on ΔL for channel can be also found in Ref. [1]. To extract the correction ΔL, the flow field in the middle section of the channel is analyzed. If it is one-dimensional and fully developed, the correction ΔL is calculated from the pressure gradient in the middle section. The same idea was also explored in Refs. [14,15].

The objective of the present work is to calculate the correction ΔL directly, i.e. only the inlet/outlet region is considered using the fully developed flow as the boundary condition. The knowledge of this quantity allows us to avoid a solution of the complete problem and to reduce significantly the computational effort without loss of accuracy. The idea is similar to that applied to the velocity slip and temperature jump problems [16], where only the Knudsen layer is considered using the asymptotic behavior in the gas bulk as the boundary condition.

To demonstrate the idea, a planar channel of height a is considered here. Hereinafter, the two-dimensional position vector $\boldsymbol{r}=(x,y)$, channel length L and end correction ΔL will be expressed in units of the height a.

The two-dimensional flow scheme under consideration is displayed in Fig. 1, where the region $x>0$ represents the container, while the region $x\le 0$ corresponds to the channel. The gas flows from (or into) the long channel in the x direction into (or from) the

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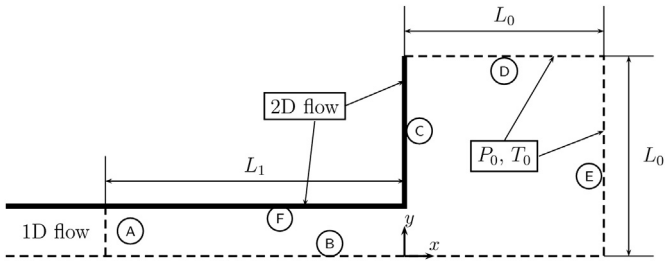

Fig. 1. Scheme of flow and coordinates.

infinitely large container, where it is maintained at pressure P_0 and temperature T_0 far from the channel inlet/outlet. Inside the channel ($x < 0$), a constant pressure gradient ξ is maintained far from the channel entrance, i.e.

$$\xi = (1/P)(\partial P/\partial x) = \text{const at } x \to -\infty. \tag{1}$$

As it is shown in Sec. 3 of Ref. [1], this assumption is well fulfilled for long capillaries. The sign of ξ determines the flow direction. We assume the pressure gradient to be sufficiently small, i.e. $|\xi| \ll 1$, to linearize the kinetic equation.

The expected pressure variation along the symmetry axis ($y = 0$) is depicted in Fig. 2, i.e. it smoothly varies from the linear dependence on the longitudinal coordinate x to its constant value P_0. However, at $x \to -\infty$ the pressure does not tend to the linear function $x\xi$, but it has a jump ΔP, i.e. its asymptotic behavior is as follows

$$P(x) \to P_0(1 + x\xi) + \Delta P, \text{ at } x \to -\infty. \tag{2}$$

From Fig. 2, it can be seen that the pressure jump ΔP is related to the end correction ΔL as

$$\Delta L = -\Delta P/(\xi P_0). \tag{3}$$

Thus, if the pressure jump ΔP is known, the end correction ΔL is calculated immediately.

Since we are going to consider a wide range of the gas rarefaction, the problem is solved on the basis of the velocity distribution function. Due to the smallness of the pressure gradient ξ, the distribution function can be presented as

$$f(\boldsymbol{r}, \boldsymbol{c}) = n_R(\boldsymbol{r}) \frac{e^{-c^2}}{\pi^{3/2} {v_0}^3} [1 + h(\boldsymbol{r}, \boldsymbol{c})\xi], \tag{4}$$

where $h(\boldsymbol{r},\boldsymbol{c})$ is the perturbation function, $v_0 = \sqrt{2k_BT_0/m}$ is the most probable molecular speed, k_B is the Boltzmann constant, m is

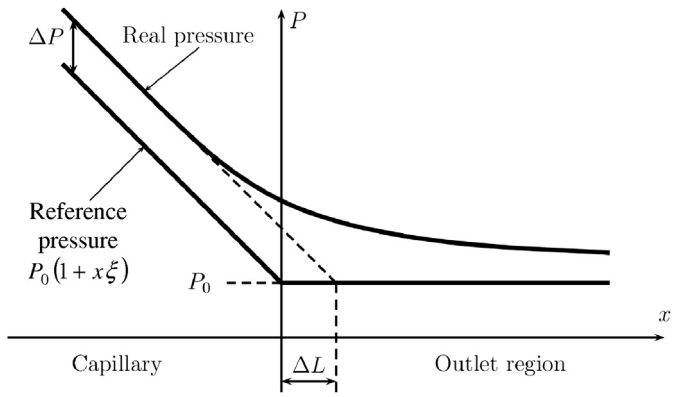

Fig. 2. Pressure distribution along the symmetry axis at the outlet part.

molecular mass of the gas, $\boldsymbol{c}$ is the molecular velocity expressed in units of v_0, and $n_R(\boldsymbol{r})$ is the reference number density. Then, the perturbations of the number density $n(\boldsymbol{r})$ and temperature $T(\boldsymbol{r})$ are defined as

$$n(\boldsymbol{r}) = n_R(\boldsymbol{r})[1 + \rho(\boldsymbol{r})\xi], \tag{5}$$

$$T(\boldsymbol{r}) = T_0[1 + \tau(\boldsymbol{r})\xi]. \tag{6}$$

The dimensionless bulk velocity $\boldsymbol{u}$ is related to the dimensional one $\widehat{\boldsymbol{u}}$ as

$$\widehat{\boldsymbol{u}}(\boldsymbol{r}) = \boldsymbol{u}(\boldsymbol{r}) v_0 \xi, \tag{7}$$

Thus, the moment perturbations are related to the distribution function perturbation as

$$\begin{bmatrix} \rho \\ \boldsymbol{u} \\ \tau \end{bmatrix} = \frac{1}{\pi^{3/2}} \int \begin{bmatrix} 1 \\ \boldsymbol{c} \\ \frac{2}{3}c^2 - 1 \end{bmatrix} e^{-c^2} h(\boldsymbol{r}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c}. \tag{8}$$

In order to approach the linear trend (2) of the pressure at large distances from the channel exit, the reference number density is defined as

$$n_R = \begin{cases} n_0 & \text{at } x \geq 0, \\ n_0(1 + x\xi) & \text{at } x < 0, \end{cases} \tag{9}$$

where $n_0 = P_0/(k_BT_0)$ is the equilibrium number density.

In our notations, the linearized Bhatnagar–Gross–Krook (BGK) model [17] reads

$$\boldsymbol{c} \cdot \frac{\partial h}{\partial \boldsymbol{r}} = \delta \left[\rho + 2\boldsymbol{c} \cdot \boldsymbol{u} + \tau \left(c^2 - \frac{3}{2} \right) - h(\boldsymbol{r}, \boldsymbol{c}) \right] + g(\boldsymbol{r}, \boldsymbol{c}) \tag{10}$$

where δ is the rarefaction parameter defined as

$$\delta = aP_0/(\mu v_0), \tag{11}$$

μ is the gas viscosity and g is the source function given as

$$g(\boldsymbol{r}, \boldsymbol{c}) = -\frac{c_x}{\xi n_0} \frac{\partial n_R}{\partial x} = \begin{cases} 0 & \text{at } x \geq 0, \\ -c_x & \text{at } x < 0. \end{cases} \tag{12}$$

The discontinuous source function is the main difference of the present approach from all previously published works.

On the solid surfaces. i.e. Ⓕ and Ⓒ according to Fig. 1, diffuse scattering is assumed. The particles entering through the soft surfaces Ⓓ and Ⓔ, see Fig. 1, have the equilibrium distribution function, i.e. their perturbation h is zero. The particles entering through the soft surface Ⓐ have the perturbation $h_1(y,\boldsymbol{c})$ corresponding to the one dimensional flow obeying the one-dimensional BGK equation

$$c_y \frac{\partial h_1}{\partial y} = \delta(2u_xc_x - h_1) - c_x. \tag{13}$$

The soft surface Ⓑ represents the symmetry axis and therefore specular reflection is assumed at $y = 0$.

Once the kinetic Eq. (10) is solved, the moments ρ, $\boldsymbol{u}$ and τ defined by Eq. (8) are known. Then, the length increment ΔL is calculated from the density asymptotic value far from the entrance inside the channel, i.e.

$$\Delta L = \lim_{x \to -\infty} \rho(x,y), \tag{14}$$

which follows from Eqs. (2), (3) and (5), and the state equation $P = nk_BT$.

The kinetic Eq. (10) has been solved by the discrete velocity method described in details in our previous papers [18–20]. Here, just the parameters of the numerical scheme are given.

The computational domain was covered by a regular grid with constant increments $\Delta x = \Delta y = 1/N_x$, where N_x is integers. The calculations were carried out for $N_x = 200$. Polar coordinates were used in the velocity space, i.e. $c_x = c_p\cos\theta$ and $c_y = c_p\sin\theta$. The quantity c_p was discretized to M values according to the roots of the Mth order Legendre polynomial mapped in $[0,c_{p,\max}]$. The angle θ varying from 0 to 2π was divided in N_θ regular intervals. The values $M = 20$, $c_{p,\max} = 5$, and $N_\theta = 400$ were used in our calculations. The iteration scheme was completed when all macroscopic quantities ρ, $\boldsymbol{u}$ and τ have converged, i.e. the sum of the absolute differences between two consecutive iterations reaches the value 10^{-9}. The computational domain size parameters L_0 and L_1 used in the calculations are given in the fourth and fifth columns of Table 1. These scheme parameters provide the numerical error of the increment ΔL with an accuracy of at least two significant figures.

The values of ΔL for the rarefaction parameter δ varying in the range from 0.2 to 10 are given in Table 1. It is observed that the quantity ΔL decreases by increasing the rarefaction parameter δ showing that the end effect becomes more significant at $\delta \to 0$. A comparison to corresponding values previously reported in the review [1], based on Refs. [12,13] is also provided. In the papers [12,13], the quantity ΔL was extracted from a numerical solution for the complete channel flow and contains a significant numerical error.

The density perturbation ρ along the symmetry axis ($y = 0$) is shown in Fig. 3 for $\delta = 0.2$, 1 and 10. It is seen that as $x \to -L_1$ the density perturbation reaches a constant value. The perturbation ρ reaches its asymptotic value faster for the larger values of rarefaction. It can be said that when $x < -1/\delta$, the quantity ρ does not vary any more, i.e. like the velocity slip problem, here we also deal with the Knudsen layer having the size of the molecular free path.

The mass flow rate through an infinite planar channel can be expressed as [1]

$$\dot{M} = -G_P a P_\xi^\xi / v_0, \tag{15}$$

where $G_P(\delta)$ is the local Poiseuille coefficient determined only by the local rarefaction parameter δ. In the same manner, the global Poiseuille coefficient G is defined by

$$\dot{M} = -Ga(P_{in} - P_{out})/(v_0 L) \tag{16}$$

determined by the inlet δ_{in} and outlet δ_{out} rarefaction parameters, and by the length L, i.e. $G = G(\delta_{in},\delta_{out},L)$. Combining Eqs. (1), (15) and (16) and the local rarefaction parameter $\delta(x) = aP(x)/(\mu v_0)$ we obtain

$$G(\delta_{in} - \delta_{out})/L = G_P(\mathrm{d}\delta/\mathrm{d}x). \tag{17}$$

Then it is transformed in

$$G\mathrm{d}x = [L/(\delta_{in} - \delta_{out})]G_P(\delta)\mathrm{d}\delta. \tag{18}$$

According to Fig. 2, the limit values of the rarefaction δ_{in} and δ_{out} correspond to the limits of integration with respect to x from $-\Delta L_{in}$ to $L + \Delta L_{out}$. Here, ΔL_{in} and ΔL_{out} correspond to δ_{in} and δ_{out}, respectively. Thus, by integrating Eq. (18) we obtain the relation of $G(\delta_{in},\delta_{out},L)$ in terms of $G_P(\delta)$

$$G(\delta_{in},\delta_{out},L) = \frac{L}{(L + \Delta L_{in} + \Delta L_{out})(\delta_{in} - \delta_{out})} \int_{\delta_{out}}^{\delta_{in}} G_P(\delta)\mathrm{d}\delta. \tag{19}$$

In the particular case, when the pressure difference is small, i.e. $(P_{in} - P_{out}) \ll P_{in}$, Eq. (19) is simplified as

$$G(\delta,L) = \frac{L}{(L + 2\Delta L)} G_P(\delta). \tag{20}$$

Here, we assume, $\delta = \delta_{in} = \delta_{out}$ and $\Delta L = \Delta L_{in} = \Delta L_{out}$. In the case of long channel, where the end effects are negligible, i.e. $\Delta L \ll L$, Eq. (19) takes the form obtained previously in Ref. [1].

The dimensionless flow rate G calculated by Eq. (20) and that reported in Refs. [1,21,22] based on calculations of the complete flow field are compared in Table 2 for five values of δ and two values of the length $L = 5$ and 10. It can be seen that the results obtained by different authors are in agreement within 2% between them, i.e. within the accuracy declared in the cited works. The relative difference $\Delta G/G$ between the value calculated by Eq. (20) and that reported in Ref. [21] is given in the sixth column of Table 2. In all cases, except $L = 5$ and $\delta = 0.2$, the simple Eq. (20) provides the Poiseuille coefficient G with a discrepancy not larger than 1%. In the last column of Table 2, the quantity $2\Delta L/L$, which corresponds to the relative influence of the end effect into the flow rate G, is given. Even if this influence is significant, e.g. it is 0.62 at $L = 5$ and $\delta = 0.4$, the analytical expression (20) provides reliable results.

Thus, the effective length concept has been introduced to investigate the end correction for rarefied gas flows through capillaries of finite length by considering the gas flow only in the

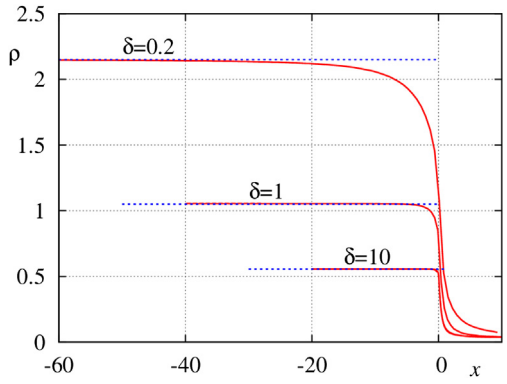

Fig. 3. Density deviation ρ distribution along the symmetry axis $y = 0$: solid lines – numerical solution, pointed lines – limit value.

Table 1
Length increment ΔL and computational domain sizes L_0 and L_1 vs rarefaction parameter δ.

δ	ΔL		L_0	L_1
	Present	Ref. [1]		
0.2	2.15	1.76	10	60
0.4	1.55	1.34	10	50
1	1.05	1.01	12	40
2	0.827	0.820	12	30
4	0.654	0.715	15	30
8	0.574	0.720	15	20
10	0.556	0.665	15	20

Table 2
Dimensionless flow rate G calculated by Eq. (20) compared with the one calculated considering the whole flow domain in Refs. [21,1,22], relative difference $\Delta G/G$ between Eq. (20) and results reported in Ref. [21], and relative end correction $2\Delta L/L$ vs rarefaction parameter δ and length L.

L	δ	G				$\Delta G/G$	$2\Delta L/L$
		Eq. (20)	Ref. [1]	Ref. [21]	Ref. [22]		
5	0.2	0.972	1.02	1.010	1.02	0.04	0.86
	0.4	1.013	1.03	1.023	–	0.01	0.62
	1	1.084	1.10	1.080	1.09	0.004	0.42
	2	1.198	1.19	1.196	1.21	0.002	0.33
	10	2.265	2.18	2.234	2.28	0.01	0.22
10	0.2	1.264	1.28	1.279	1.29	0.01	0.43
	0.4	1.253	1.26	1.256	–	0.002	0.31
	1	1.272	1.27	1.269	1.29	0.002	0.21
	2	1.368	1.36	1.365	1.39	0.002	0.16
	10	2.491	2.44	2.471	2.50	0.008	0.11

capillary inlet/outlet. The length increment has been calculated from the linearized kinetic equation for several values of the gas rarefaction. A simple formula for the flow rate through a channel of finite length has been obtained. A comparison with results obtained by time-consuming calculations considering the complete domain of the gas flow showed that the simple formula can be used in a wide range of the channel length, even if the end corrections are not small.

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