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Monte Carlo Modeling of the Thermal Conductivity of Porous Cometary Ice

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The basic structure of comet nuclei is an aggregation of grains, with a size distribution that extends over several orders of magnitude and a similar distribution of pores. Although attempts have been made to assess the effect of porosity on the thermal conductivity, the effect of pore size distribution has been ignored. Modeling a porous structure with a wide size distribution would require a very fine 3-D grid, so as to accommodate the smallest and largest voids. In order to circumvent this difficulty, we adopt a hierarchical procedure. Thus we assume a random and fractal porous structure and use a 3-D Monte Carlo model. The basic configuration is a cube made of unit cells of two types, (ice) filled and void, randomly distributed. Their relative number corresponds to a prescribed porosity. We solve the heat transport equation for this cube until a steady state is obtained, and from this solution the effective thermal conductivity is derived. The calculations are repeated for a range of porosities and temperatures, since the ice conductivity is temperature dependent. The basic cube serves as a unit filled cell in a larger cube, and in this way the hierarchical structure of the medium is built up.

We find that the thermal conductivity is lowered by several orders of magnitude at high porosities. The correction factor, obtained as a fit to the results of our calculations, is expressed as a smooth function of the porosity, which tends to zero as the porosity approaches the percolation threshold of the solid. If only the porosity of the medium is known, this correction is not uniquely determined, but rather a range of values is possible. Only if the size distribution of the pores is known does the correction become uniquely determined.

Key Words: **comets; comets, composition; ices.**

1. INTRODUCTION

The thermal conductivity of low-temperature ice, as found in comet nuclei, has been determined both experimentally and from theoretical considerations. The formulae provided by Klinger (1980, 1981) have gained widespread use. But cometary material is highly porous, and although it is certain that porosity lowers the thermal conductivity, it is unclear to what extent, and how the correction depends on porosity and on the pore size distribution. So far, answers to these questions have been either vague or widely discrepant.

We should stress that pores will be assumed to be *empty,* whereas in reality, when the solid matrix is made predominantly of ice and temperatures are not too low, vapor may fill the pores and flow through them. Such is the case in the outer layers of cometary nuclei, for example. Often, heat transport by vapor can be more efficient than heat conduction by the solid (e.g., Smoluchowski 1982, Steiner and Kömle 1991). Rigorously, such situations should be described by simultaneous solutions of the mass and heat transport equations and then heat transfer by vapor arises independently of heat conduction by the solid (e.g., Espinasse *et al.* 1991, Prialnik 1992, Benkhoff and Huebner 1995). Only if mass transport is omitted, in order to simplify the calculations, does it become necessary to parametrize heat transport by vapor and add it to the thermal conductivity coefficient, which results in an *effective* thermal conductivity (e.g., Kossacki *et al.* 1994). Indeed, laboratory determinations of the thermal conductivity of porous water ice are complicated by the fact that a significant amount of heat is transported by the vapor that fills the pores, in addition to the heat conducted by the solid matrix (Seiferlin *et al.* 1996 and references therein). In the present paper, *effective thermal conductivity* is taken to represent the conductivity of a multicomponent *solid,* to be distinguished from the conductivity of a single component.

Generally, a porous medium may be regarded as a homogeneous two-phase material—one occupying a volume fraction p and the other $1 - p$—with two characteristic conductivities, K_p and K_s, respectively. Although heat may be transferred through the pores by radiation (or by a fluid flow), usually $K_p \ll K_s$. Simple analytical approximations for the effective conductivity of such a medium, expressed as ϕK_s in terms of the ratio of the conductivities of the two phases, $r_{ps} = K_p/K_s$, include: an arithmetic mean, $\phi = 1 + r_{ps}$, which is obviously inappropriate at high porosities if $r_{ps} \ll 1$ (such as in the case when pores are one of the phases); a weighted geometric mean,

$$\phi = r_{ps}^{p}; \tag{1}$$

a parallel combination of the two phases,

$$\phi = p r_{ps} + (1 - p); \tag{2}$$

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and a series combination,

$$\phi = \left(\frac{p}{r_{ps}} + 1 - p \right)^{-1} \tag{3}$$

(Horai 1991 and references therein). All these simple analytical solutions ignore the structure of the porous medium. Moreover, when $r_{ps} \ll 1$, one would not expect the precise value of the low pore conductivity to play a major role (except for p very close to unity); that is, one would expect the effect of pores to converge to that of a vacuum. However, such behavior is obtained only for the parallel combination; on the other hand, the parallel combination is obviously wrong at high porosities.

The first to supply a formula for the thermal conductivity of a *structured* mixed medium—a packed-sphere bed—was Maxwell (1873), based on an analogy between thermal and electrical conductivities. In fact, this approach provides upper and lower limits for the conductivity, obtained by exchanging the roles of the two phases. In terms of the ratio r_{ps} of the pore conductivity to that of the solid, the medium conductivity normalized to that of the solid varies between

$$\phi_L = r_{ps} \frac{2pr_{ps} + (3 - 2p)}{(3 - p)r_{ps} + p} \tag{4}$$

and

$$\phi_U = \frac{(2 - 2p) + (1 + 2p)r_{ps}}{(2 + p) + (1 - p)r_{ps}}. \tag{5}$$

These formulae are valid, however, only at low porosities, where $\phi \sim \phi_U$, and at high ones, where $\phi \sim \phi_L$. The correct answer probably lies in-between these limits. It is noteworthy that for $r_{ps} \ll 1$, the upper limit yields results that are very close to the correction suggested by Smoluchowski (1981) for the conductivity of a porous medium, $\phi = 1 - p^{2/3}$, which was based on geometrical considerations. An extension of Maxwell's formulae, valid to higher orders of the solid concentration $1 - p$, was later provided by Rayleigh (1892). Numerous other formulae followed, based on different assumptions regarding the structure of the porous medium in two or three dimensions (see, e.g., Cheng and Hsu 1999). All these formulae share the property that the effective conductivity is extremely sensitive to porosity either near $p = 0$ or near $p = 1$. The structures considered are simple insofar as the sizes of voids are fixed (although their shapes may vary) and the two phases may be interchanged.

The thermal conductivity of porous ice relevant to comet nuclei was considered by Steiner and Kömle (1991) and, more recently, by Sirono and Yamamoto (1997). The former use a similar, albeit more elaborate version of Maxwell's procedure and arrive at a formula for the effective thermal conductivity of a mixture (ice and void) in terms of the individual conductivities of the components. The latter derive formulae both for the effective conductivity of porous ice and for a mixture of amorphous and crystalline ice, based on the effective medium theory.

None of these studies considers a distribution of pore sizes. A porous medium, however, is characterized by at least two parameters: the porosity and the pore size, or the pore size distribution. How, if at all, does the latter influence the conductivity? The purpose of the present paper is to address this question, using a three-dimensional model of a porous medium. The basic structure assumed is fractal, with the pore size distribution spanning several orders of magnitude. Obviously, in order to model such a structure, a very fine 3-D grid would be required, so as to accommodate the smallest and largest voids. Such an approach is impractical (impossible, in fact, in view of computational constraints!). In order to circumvent this difficulty, we adopt a hierarchical procedure. The porous medium is built by means of a Monte Carlo method (Shoshany *et al*. 1999). Given the bulk conductivities of the constituents, ice and pores, the conductivity of the medium can be modeled as a function of both porosity and temperature. In Section 2 we describe the model and method of computation; results are presented in Section 3 and discussed in Section 4, where we also state our main conclusions and make suggestions for future work.

2. THE MODEL

The first step of each run of the calculation consists of generating the medium. As stated above, in order to take account of pores having largely different sizes, we adopt a hierarchical procedure, which yields a fractal medium. In all cases, we consider a three-dimensional cube composed of cubical cells of two types: "material" (labeled 1) and "void" (labeled 0). In the porous medium of the first generation (referred to as "medium 1"), the material cells are made of ice, while the voids may actually contain gas, such as water vapor. In the second generation medium (referred to as "medium 2"), the material cells are cubes of the first generation; similarly, the material cells of the third generation ("medium 3") are cubes of the second generation. Thus a multitude of pore sizes is accounted for, arising from both the distribution of voids and the fractal structure. This is illustrated in Fig. 1.

An initial porosity p_0 is chosen, and the cells are assigned values of 0 or 1 at random in such a way that the desired porosity is attained. The values are stored in a three-dimensional matrix, to be used in the heat conduction calculations. The same procedure and p_0 is adopted for the media of all generations. However, while the porosity p_1 of medium 1 is obviously $p_1 = p_0$, the actual porosity of medium 2 is

$$p_2 = p_0 + (1 - p_0)p_1 = 2p_0 - p_0^2, \tag{6}$$

and similarly for medium 3,

$$p_3 = p_0 + (1 - p_0)p_2 = 3p_0 - 3p_0^2 + p_0^3. \tag{7}$$

It can be easily shown that, generally, for a hypothetical

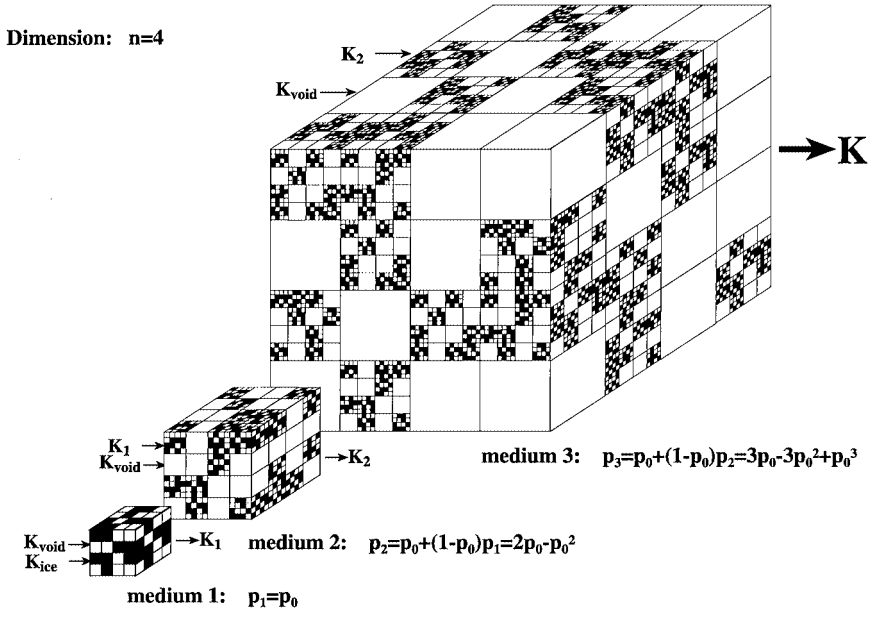

FIG. 1. Schematic representation of the hierarchical (fractal) model of a porous medium. The dimension adopted for illustration is $n = 4$, whereas the calculations were performed for $n = 50$. The basic (fractal) porosity is $p_0 = 0.5$. For further details, see text.

medium n,

$$p_n = 1 - (1 - p_0)^n. \tag{8}$$

The porosity values corresponding to the different cases are listed in Table I. We only consider *fractal* porosity values p_0 that are lower than the percolation threshold $p_c \approx 0.7$, above which there is generally no continuous path of material cells connecting the top to the bottom.

To illustrate the method of computation of the thermal conductivity, we consider medium 1 and denote the cell side by ℓ. The conductivity of the ice is taken to be of the form

$$K_{\text{ice}} = b/T, \tag{9}$$

typical of crystalline ice at low temperatures, with $b = 5.67 \times 10^2$ W m^{-1}, as given by Klinger (1980). For the void cells, we assume that heat is transported by radiation, which yields a thermal conductivity of the form

$$K_{\text{void}} = 4\epsilon\sigma\ell T^3, \tag{10}$$

where σ is the Stefan–Boltzmann constant and the emissivity ϵ is of order unity. Although there are pores of sizes greater than ℓ, the weighted average pore size is close to ℓ for $p < p_c$, and hence Eq. (10) is valid for the entire medium. The conductivity across a 1–0 boundary is taken to be $2(K_{\text{ice}}^{-1} + K_{\text{void}}^{-1})^{-1}$.

We solve the heat conduction equation

$$\rho c \frac{\partial T(x, y, z, t)}{\partial t} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right), \tag{11}$$

where ρ is the density and c is the heat capacity of the ice,

$$c = \alpha T + \beta, \tag{12}$$

with $\alpha = 7.49 \times 10^4$ erg g^{-1} K^{-2} and $\beta = 9 \times 10^5$ erg g^{-1} K^{-1} (Klinger 1981), and

$$F_x = -K(T)\frac{\partial T(x, y, z, t)}{\partial x}, \text{ etc.} \tag{13}$$

For the void cells, the left-hand side of Eq. (11) vanishes, meaning that a steady state prevails. The boundary conditions are constant temperature at the bottom of the cube, $T(x, y, z = 0, t) = T_0$, and constant heat flux at the top ($z = L$), $F(x, y, L, t) = F_0$. The lateral boundary conditions are periodic, $T(L, y, z, t) = T(0, y, z, t)$, $T(x, L, z, t) = T(x, 0, z, t)$. The initial condition is $T(x, y, z, t = 0) = T_0$.

TABLE I
Porosities of the Three Media

$p_1 = p_0$	p_2	p_3
0	0	0
0.05	0.0975	0.143
0.1	0.190	0.271
0.2	0.360	0.488
0.3	0.510	0.657
0.4	0.640	0.784
0.5	0.750	0.875
0.6	0.840	0.936

Under such conditions, the temperature distribution tends to a steady state: for a constant thermal conductivity, the temperature would vary linearly with depth (i.e., in the z direction) and be constant over surfaces of constant z. The thermal conductivity would then be given by

$$K = \frac{F_0 L}{T_s - T_0}, \tag{14}$$

where $T_s (> T_0)$ is the steady state temperature at the top of the cube. Since in our case the conductivity is temperature dependent, we choose F_0 so that the resulting T_s is sufficiently close to T_0 for Eq. (14) to still be valid, but at the same time sufficiently higher than T_0 for the results to be numerically meaningful. The equations are solved numerically over a 50^3 Cartesian grid, using an explicit difference scheme and time steps dictated by the Courant condition ($\Delta t \leq \frac{\ell^2 \rho(\alpha T_0 + \beta) T_0}{2h}$, since T_0 is the minimal temperature within the medium), until the temperature distribution is stabilized, that is, a steady state is attained. The resulting value of K from Eq. (14) is taken to represent medium 1, $K_1(T_0)$. It serves as the thermal conductivity of material cells in medium 2 (replacing $K_{\rm ice}$). In a similar manner, we now obtain $K_2(T_0)$, to be used for the material cells in medium 3. We repeat the calculations for different values of T_0, at intervals of 50 K, over the range relevant to comets. In each case, runs are repeated for many different random distributions (with the same p_0), in order to reduce the standard deviation to less than 30%. Large variations are obtained at porosities near the percolation limit (cf., e.g., Gingold and Lobb 1990). The calculations are repeated for different values of p_0, at intervals of 0.1. We note that for each pair of values (T_0, p_0), $K(T_0)$ is obtained for three different porosity values: p_0, p_1, and p_2.

As a test, we ran the code with $p_0 = 0$, that is, all cells ice, and recovered the thermal conductivity of ice $K_{\rm ice}$. In this case, the three media are identical. As a further test, we assumed a mixture of two phases, of fixed conductivities K_a and $K_b = 1000\ K_a$, occupying fractions p and $1 - p$ of the volume, respectively. For $p = 1$, we obtained $K = K_a$ and for $p = 0$, $K = K_b$, as required.

3. RESULTS

For the first medium, we assume a cell size $\ell = a_1 = 10\ \mu$m and carry out calculations for fractal porosities in the range $0 \leq p_0 \leq 0.6$. Higher values of p_0 would be unrealistic, since they would result in extremely high values for p_3 (see Table I). We note that the actual pore size only affects the heat conductivity through pores (Eq. (10)) and could otherwise be scaled out. The variation of heat conductivity $K_1(T, p)$ with temperature is plotted in Fig. 2 for various porosity values. As expected, higher porosities lead to lower conductivities; the trend of a decreasing thermal conductivity with increased temperature is preserved.

For the second medium, the cell size is $\ell = a_2 = 50\ a_1 = 0.5$ mm, and material cells have the conductivity $K_1(T, p)$ obtained for the first medium. The resulting conductivity $K_2(T, p)$ as a function of temperature is shown in Fig. 3 for various porosity values.

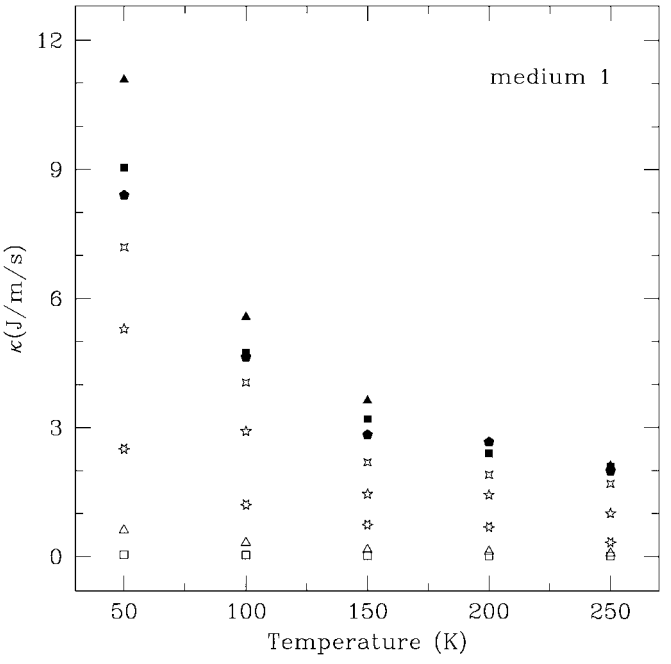

FIG. 2. Thermal conductivity as a function of temperature obtained for medium 1. The various symbols correspond to various porosities: $p = 0$ (filled triangle), $p = 0.05$ (filled square), $p = 0.1$ (filled pentagon), $p = 0.2$ (four-point star), $p = 0.3$ (five-point star), $p = 0.4$ (six-point star), $p = 0.5$ (triangle), and $p = 0.6$ (square).

Finally, for the third medium, the cell size is $\ell = a_3 = 50\ a_2 = 2.5$ cm, and material cells have the conductivity $K_2(T, p)$ obtained for medium 2. The results for $K_3(T, p)$ of medium 3 are shown in Fig. 4. In order to facilitate a comparison among the results obtained for the three media, Figs. 2–4, the same symbol has been assigned to similar porosity values. We find that for the same porosity, the lowest conductivities are those of the lowest rank medium.

In Fig. 5, we show the dependence of thermal conductivity on porosity for various temperatures. It is clearly seen that a high porosity may lower the thermal conductivity by several orders of magnitude. Finally, in Fig. 6, we plot the thermal conductivity divided by the conductivity of the solid material, that is, $K_{\rm ice}(T)$, Eq. (9). The immediate conclusion is that, to a good approximation,

$$\frac{K(T, p)}{K_{\rm ice}(T)} = \phi(p, n).$$

That is, the correction to the thermal conductivity due to porosity is independent of temperature. In addition, for a given fractal porosity p_0, we find $K_1 = K_{\rm ice}\phi(p_0)$, $K_2 = K_1\phi(p_0) = K_{\rm ice}\phi^2(p_0)$, and generally,

$$\phi(p, n) = \phi^n(p_0), \tag{15}$$

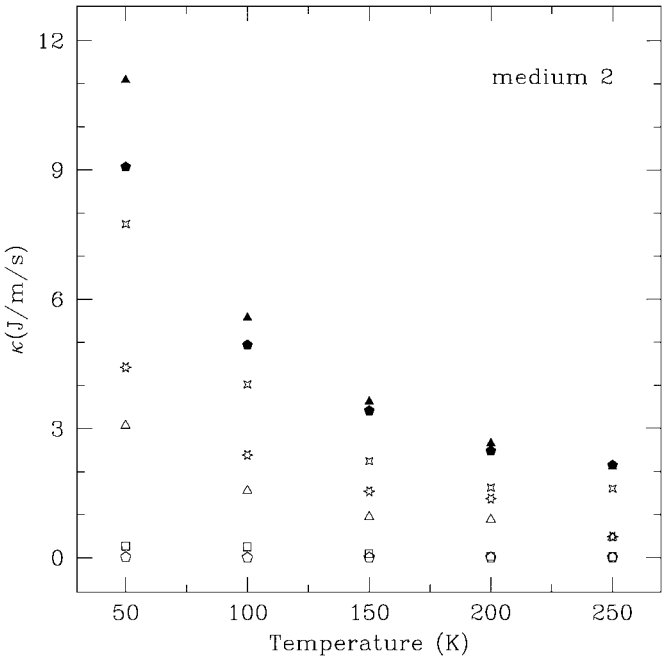

FIG. 3. Thermal conductivity as a function of temperature obtained for medium 2. The various symbols correspond to various porosities: $p = 0$ (filled triangle), $p = 0.0975$ (filled pentagon), $p = 0.19$ (four-point star), $p = 0.36$ (six-point star), $p = 0.51$ (triangle), $p = 0.64$ (square), and $p = 0.75$ (pentagon). Note that the same symbols are used for porosity values similar to those of Fig. 1.

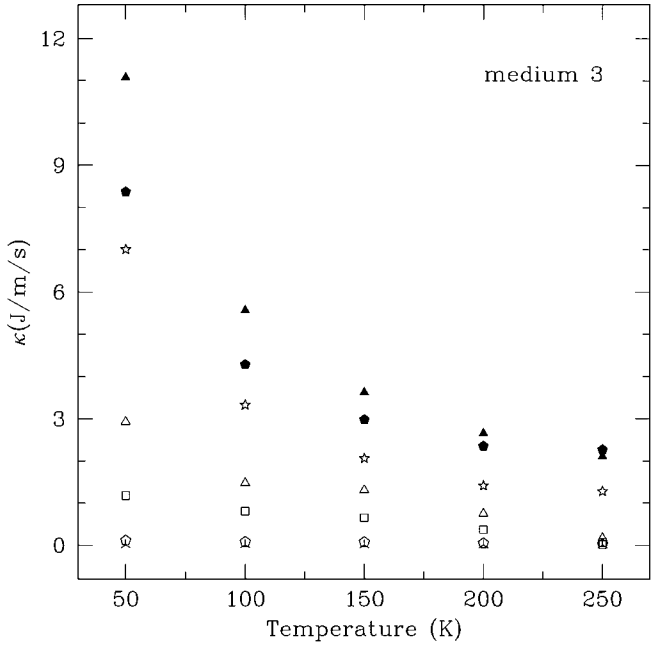

FIG. 4. Thermal conductivity as a function of temperature obtained for medium 3. The various symbols correspond to various porosities: $p = 0$ (filled triangle), $p = 0.143$ (filled pentagon), $p = 0.271$ (five-point star), $p = 0.488$ (triangle), $p = 0.657$ (square), $p = 0.784$ (pentagon), and $p = 0.875$ (skeletal triangle). Note that the same symbols are used for porosity values similar to those of Figs. 1 and 2.

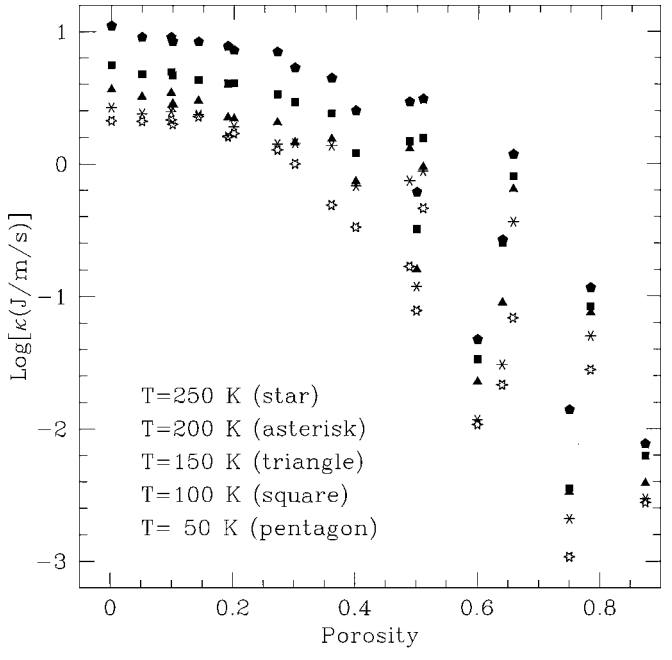

FIG. 5. Thermal conductivity as a function of porosity for all media and temperatures. Various symbols correspond to various temperatures, as indicated.

where $p_0 = p_0(p, n)$. This is correct, however, only so long as the conductivity of the pores is negligible. Since this conductivity increases with pore size and with the third power of the temperature, its effect becomes apparent in the higher order media, at very high porosities and high temperatures. This can be seen in Fig. 6, where the conductivities start deviating from the general trend by increasing with increasing temperature. In fact, as already mentioned, there is a basic difference between the pore conductivity and that of the solid, insofar as the former depends explicitly on the pore size, while the latter is size-independent. It is noteworthy that the effect of pores becomes appreciable at a porosity of $\sim$0.7. In order to test this conclusion, we ran several models adopting arbitrarily high and low values for ϵ (0.1 and 10). The results for $p \leq 0.7$ were not affected. If we regard the pores as a resistive medium to the flow of heat, and the solid as a permeable one, then the percolation limit for the solid in a 3-D medium is about 0.3, which corresponds to $p = 0.7$ in our case (the volume fraction of the solid being $1 - p$). It is reasonable, therefore, that the effect of pore conductivity becomes appreciable below the percolation threshold of the solid, that is for $p > p_c$, where $p_c \approx 0.7$.

If we ignore the effect of pore conductivity and focus on the correction factor to the conductivity of solid ice due to porosity, we find for $\phi(p_0)$ a very good fit of the form

$$\phi(p_0) = (1 - p_0/p_c)^{\alpha(p_0)}, \tag{16}$$

with $\alpha(p) = 4.1p + 0.22$ and $p_c = 0.7$. The fit is not very sensitive to the exact value of p_c; adopting $p_c = 0.69$ (percolation limit of 0.31 for the solid, corresponding to site percolation in a

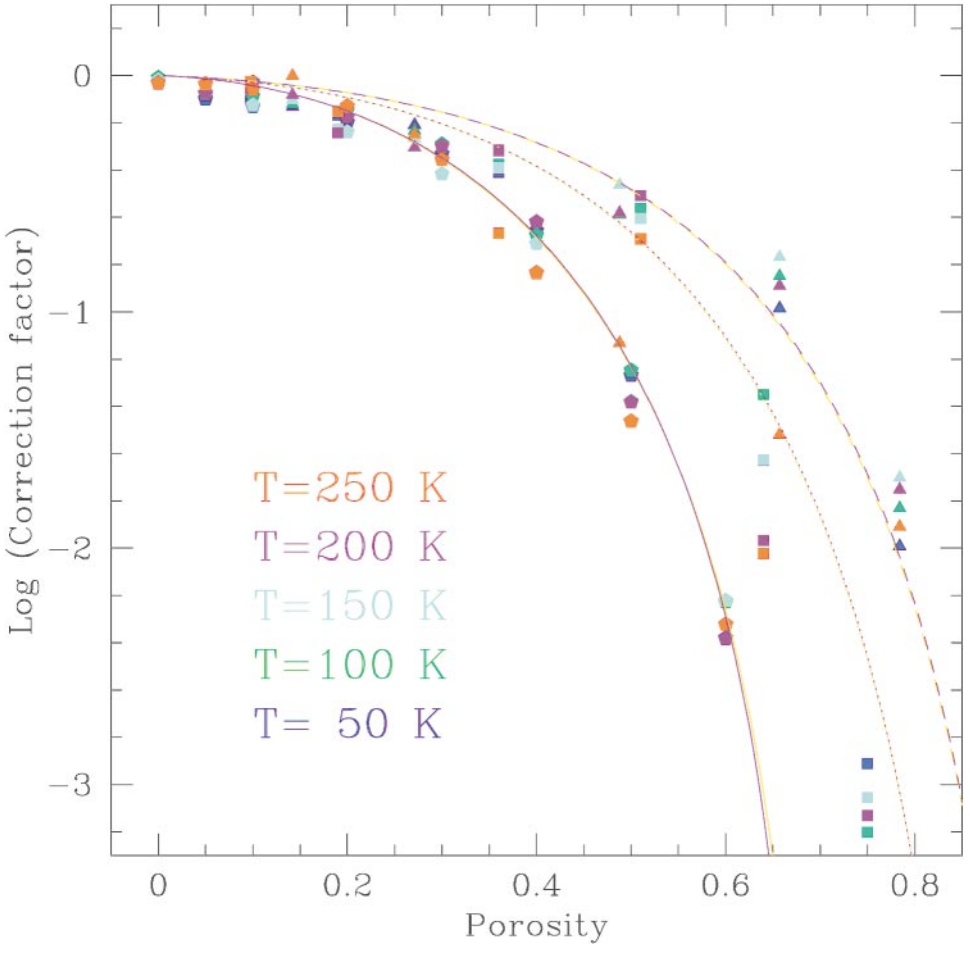

FIG. 6. Normalized thermal conductivity (on a logarithmic scale) as a function of porosity. Various symbols are used for the various media: medium 1 (filled pentagon), medium 2 (filled square), and medium 3 (filled triangle). Lines correspond to the fit formulae for the correction factor: $\phi(p)$ (solid), $\phi^2[p_0(p_2)]$ (dotted), and $\phi^3[p_0(p_3)]$ (dashed).

simple cubic lattice), we obtain an equally good fit by replacing the coefficients 4.1 and 0.22 by 3.9 and 0.23, respectively. Note that $\phi = 1$ for $p_0 = 0$ and $\phi = 0$ for $p_0 = p_c$, as required. In fact, the form of Eq. (16) for the scaled conductivity is in accordance to percolation and effective medium theories (Stauffer and Aharony 1994, Bergman and Stroud 1992).

The medium we are dealing with is fractal, and given the porosity p, it can assume a large (in principle, infinite) number of structures, of increasing complexity, as determined by n, the hierarchical rank of the medium. Thus medium 3 in our case represents a more complicated structure than medium 1. The difference is caused by the size distribution of pores: for a unit size medium, there is a unique pore size in medium 1, given by $\frac{1}{m}$ ($m = 50$ in our calculations); medium 2 has two pore sizes $\frac{1}{m}$ and $\frac{1}{m^2}$, medium 3 has pore sizes $\frac{1}{m}$, $\frac{1}{m^2}$, and $\frac{1}{m^3}$, and so forth. Thus, generally, for medium n of size a, pore sizes range from a minimum a_0 to a maximum, $a_0 m^{n-1}$, and $a = a_0 m^n$. Hence,

$$m = \left(\frac{a}{a_0}\right)^{\frac{1}{n}}. \tag{17}$$

Since the smallest unit size a_0 varies as m^{-n}, while the number of material cubes in each structure N_n varies as $[(1 - p_0)m^3]^n$, the fractal dimension of such a medium is given by

$$D = \lim_{a_0 \to 0} \frac{\ln N}{\ln(1/a_0)} = 3 + \frac{\ln(1 - p_0)}{\ln m}, \tag{18}$$

which, for example, yields $D = 3 - 0.256 \ln(1 - p_0)$ in our calculations. Substituting (8) and (17) into (18), we obtain

$$D = 3 + \frac{\ln(1 - p)}{\ln(a/a_0)}. \tag{19}$$

Thus the structure of this fractal medium is determined by its porosity and by the size of the smallest pore relative to the medium's length scale; for example, $p = 0.5$ and $a/a_0 = 10^3$ yield $D = 2.9$, while for $p = 0.75$ and $a/a_0 = 10$, $D = 2.4$. Although D is independent of n, we found that the correction to the thermal conductivity due to porosity may assume different values depending on the value of n. These, however, are

limited, and the range of variation may be obtained as follows. Suppose that the porosity may assume any value in the interval $[p_{\min}, p_{\max}]$, porosities very close to 0 or 1 being unrealistic. Then, using Eq. (8),

$$p_{\min} \le p_0 \le p = 1 - (1 - p_0)^n \le p_{\max},$$

and the correction factor,

$$\phi(p, n) = \left[(1 - p_0/p_c)^{\alpha(p_0)}\right]^n, \tag{20}$$

where $p_0 = 1 - (1 - p)^{1/n}$, lies between

$$\phi_{\min}(p) = (1 - p/p_c)^{\alpha(p)} \tag{21}$$

and

$$\phi_{\max}(p) = \left[(1 - p_{\min}/p_c)^{\alpha(p_{\min})}\right]^{\ln(1-p)/\ln(1-p_{\min})}. \tag{22}$$

The ultimate lower limit, that is, the most significant correction to the solid phase conductivity, is therefore $\phi_{\min}(p_{\max})$. For the thermal conductivity of a porous medium whose only known property is the porosity, the range of variation of the correction factor is shown in Fig. 7. Future studies will have to test the general validity of this correction factor for other basic thermal conductivity laws.

In order to fix the correction factor within this range, one needs to determine n, noting that both p and D are independent of n. We illustrate this with an example. The volume of the final fractal medium used in our model (medium 3) is larger than that of the smallest pore a_0 by a factor of 1.25×10^5. This was achieved in a three-generation hierarchy, increasing the unit size 50-fold at each later generation. However, roughly the same a/a_0 contrast could be achieved by a greater number of generations with a smaller size folding—for example, five generations with a 10-fold size increase—which would correspond to the same fractal dimension. Even if we choose a similar porosity for the two cases, we predict different correction factors. For example, a porosity of 0.66–0.67 corresponds to $p_0 = 0.3$ in the three-generation medium and $p_0 = 0.2$ in the five generation one, and the correction factors obtained from (15) and (16) are 0.088 and 0.17, respectively. What, then, is the difference between these media—namely, how would we know which case to choose for representing a fractal medium of given porosity and fractal dimension? Obviously, an additional property must be specified in order to achieve a one-to-one correspondence between a given medium and our model, which is based on three independent parameters (p_0, m, and n). We note that the three-generation medium contains a relatively large number of large pores, while the five-generation one contains a variety of medium-size pores instead. Thus, this additional property can be the mean free path of the medium $\bar{a}$ relative to the length scale a, which is easily calculated from the pore sizes

$$a_k = a_0 m^{k-1}, \tag{23}$$

where $1 \le k \le n$, and the weight function $P_k = V_k/V$, where

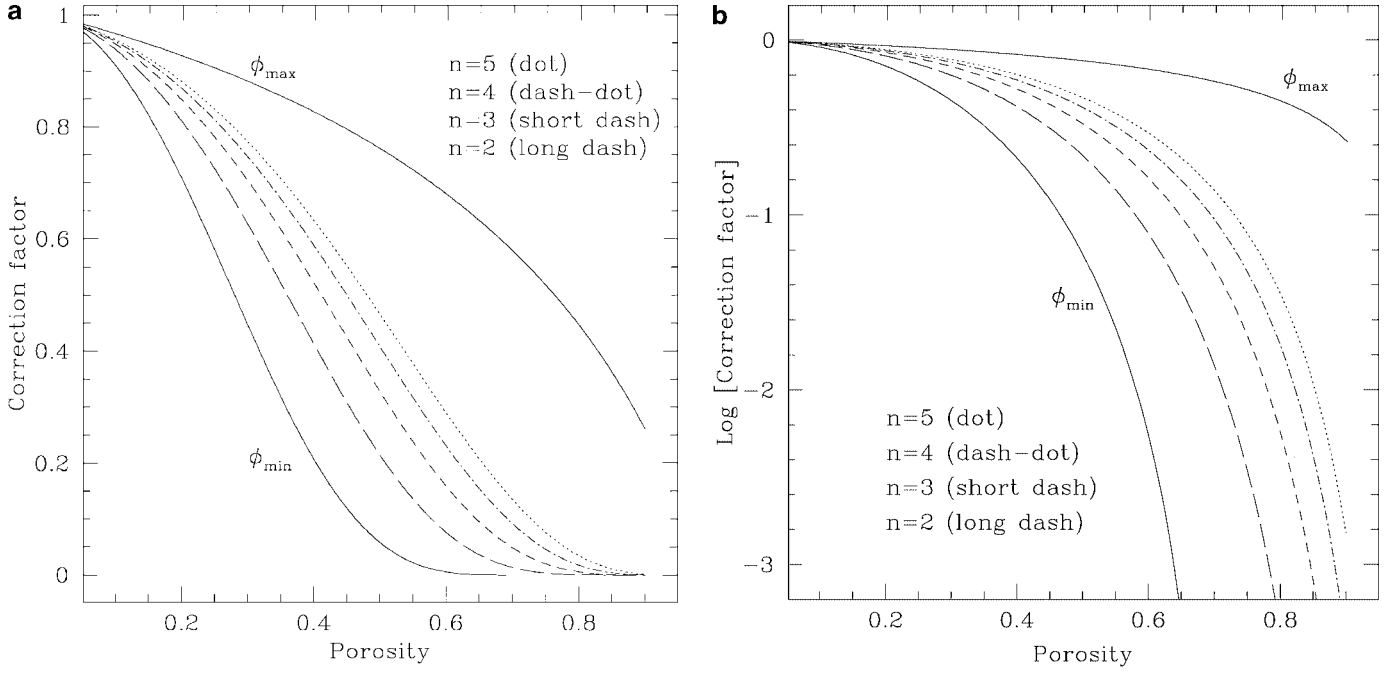

FIG. 7. Correction factor to the thermal conductivity of the solid phase as a function of porosity (a) on a linear scale and (b) on a logarithmic scale. The lower and upper limits for a given p, $\phi_{\min}$, and $\phi_{\max}$, respectively, are indicated. Between them, the various curves correspond to various values of n (generations in the hierarchical structure), as indicated.

V_k is the volume occupied by pores of size a_k,

$$V_k = p_0(1-p_0)^{n-k}m^{3(n+1-k)}(a_0m^{k-1})^3 = a_0^3 p_0(1-p_0)^{n-k}m^{3n}, \quad (24)$$

and V is the total pore volume, $V = \sum V_k = pa_0^3m^{3n}$. Thus

$$\bar{a} = \sum a_k P_k \approx a_0 m^{n-1} p_0/p = (a/m)(p_0/p). \quad (25)$$

In conclusion, given a fractal medium of porosity p, relative size a/a_0 (or, alternatively, fractal dimension D), and relative mean free path $\bar{a}/a$, we derive the relevant model parameters from

$$p = 1-(1-p_0)^n \quad (26)$$

$$a/a_0 = m^n \quad (27)$$

$$pa/\bar{a} = mp_0 \quad (28)$$

and substitute them into (20) to obtain the correction factor to the thermal conductivity of the solid material. We note that (18) may replace (27) if D is known; also, it may be more useful to replace the normalization length a by the largest pore size of the medium $a_{\max}$, in which case (28) becomes $pa_{\max}/\bar{a} = p_0$ and (27) is not required for obtaining ϕ.

4. DISCUSSION

We find that the thermal conductivity is lowered by several orders of magnitude at high porosities. The temperature dependence of the ice conductivity is preserved—the conductivity decreases with increasing temperature—so much so that, to a good approximation, the correction factor is temperature independent. We also find that, for a given basic porosity, the correction factor is the same when one passes from one medium to the next. Thus, as larger and larger pores are added, the conductivity decreases by an increasing power of the basic correction factor. At very high porosities, $p > 0.7$, below the *percolation limit* of the solid through the porous medium, the low radiative conduction through the pores becomes dominant. If the pores were filled with a perfect insulator, the conductivity would tend to zero under these circumstances.

A different correction to the conductivity of a porous medium results from what is known as the *Hertz factor* h. If single grains are considered, this factor takes account of the reduced area of contact between them. In our model, this would be equivalent to replacing the icy cubes by spheres or spheroids. Although this correction can be substantial (Squyres *et al*. 1985), that is $h \ll 1$, it was shown by Kossacki *et al*. (1994) that sintering, which is particularly important for icy grains under cometary conditions, tends to increase the contact area—and hence the Hertz factor resulting from it—quite considerably. In any case, this effect should be included as a correction to $K_{\rm ice}$ itself, before the latter is modified to take account of the porous structure. (Similarly, if the solid material is composed of a mixture of ices or ices and dust, the conductivity of the mixture must first be determined, independently of the porosity.) Generally, however, h is defined as the ratio of contact area to total area if the medium is cut along an arbitrary plane. Thus, a medium such as ours, built of cubes statistically distributed over a given volume, already has a Hertz factor implicitly included. Contact area and void volume are no longer independent parameters in such a medium, as they were considered to be in previous studies (e.g., Steiner and Kömle 1991), and the decrease in thermal conductivity can be attributed to either or to both. It should be mentioned that attempts to determine the Hertz factor by fitting laboratory data yielded a rather wide range of values, between 0.1 and 0.001.

The fractal medium considered here corresponds to a power law pore size distribution of a power close to 3. A normal distribution of pore sizes is not well described by our model of a porous medium and our results may not apply in that case. However, it is rather well established that comets are made of an aggregation of grains and that the grain size distribution follows a power law with a power of order 3.5. We should, therefore, expect the voids between the grains, that is the pores, to have a similar size distribution. As a simple example, if pores and grains are randomly distributed and their mean sizes are r_p and r_g, respectively then $\frac{1}{2} \lesssim r_p/r_g = [p/(1-p)]^{1/3} \lesssim 2$, if $p \le 0.9$. Hence our model is well suited to cometary material. The problem is to find a correspondence between a real porous material and the schematic model we have investigated, in order for our results to be applicable to realistic configurations. To this end we have identified physical characteristics of a porous medium that can be translated into the model parameters we have used. If only the porosity is known, our model provides lower and upper limits to the correction factor by which the conductivity of the solid material should be multiplied,

$$(1-p/p_c)^{\alpha(p)} \le \phi(p) \le (1-p_{\min}/p_c)^{\alpha(p_{\min})\ln(1-p)/\ln(1-p_{\min})}. \quad (29)$$

The range is quite large at high porosity values. It may be reduced, however, if the minimal possible fractal porosity for the material can be estimated, as the upper limit in (29) decreases with increasing $p_{\min}$. For example, if $p = 0.5$, the correction factor varies between a lower limit of 5.8% of the solid conductivity and an upper limit of ~50% for $p_{\min} = 0.1$ to ~20% for $p_{\min} = 0.3$.

If *any* two parameters of the pore size distribution are known as well, a unique correction factor can be derived (provided the distribution may be described by a power law). Correction factors span several orders of magnitude, meaning that porosity has a very significant effect on the thermal conductivity and hence on the behavior—thermal evolution and activity—of comets.

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