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LARGE-N EXPANSIONS IN QUANTUM MECHANICS, ATOMIC PHYSICS AND SOME O(N) INVARIANT SYSTEMS

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Abstract:

Some selected topics on the subject of large-N expansions are reviewed here. A brief discussion of large-N critical phenomena is also presented. Generalizing the Schrödinger equation to N-dimensional space, the idea of the large-N limit is introduced in the context of quantum mechanics and the standard methods of $1/N$ expansion such as perturbed oscillator methods and Riccati equation methods are described in detail, with some interesting applications. Various other methods of $1/N$ expansion, namely the hypervirial $1/N$ expansion, the pseudospin formulation and the collective field method are also highlighted together with some of their important applications. Extension of the large-N scheme to the relativistic domain is presented and the application of the large-N concept to potential scattering is briefly discussed.

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1. Introduction and overview

Many quantum mechanical, statistical and field theoretic models representing interesting physical systems are known to possess global group symmetries. These symmetries are often continuous and may belong to some internal space. Several of these theories admit straightforward generalizations in which the number of internal degrees of freedom N parametrizing the symmetry group of the problem may be treated as a free variable parameter. The essence of the large-N expansion lies in assuming this parameter to be infinitely large, which then entails a dramatic simplification in the analyses of a wide class of such theories, in stark contrast to what our commonplace notions would justify.

Surprising though it may seem, the emerging solution, commonly known as the large-N limit, turns out to be quite impressive in a number of cases. However the $N\to\infty$ contribution may not suffice in general, since the magnitude of the finite-N corrections may not be negligible. It is found that if the $N\to\infty$ limit of a quantum problem is obtainable in an explicit form, then the finite-N corrections can be taken into account by introducing a systematic expansion in inverse powers of N and contact with the original physical finite-N theory can be made by substituting for N its given fixed value at the end of the calculation. This $1/N$ expansion scheme has claimed in recent years considerable attention (for reviews see refs. [1–7]) and has amply proven its efficacy in dealing with a diverse class of theories [2, 3] ranging from a single particle potential problem in quantum mechanics to phase transitions and critical phenomena, multicomponent field theories, quantum spin models, nuclear physics and quantum chromodynamics (QCD) with N of course referring to different objects in different theories. That the simple trick of inflating N to ∞ yields remarkable simplification in a wide variety of problems prompts us to believe that all these problems might possess a common feature in their large-N behaviour. This idea has in turn motivated a number of authors to ponder about the mathematical foundations of the large-N approximation and unravel its basic tenets. Indeed by now it has become well established that in the large-N limit the dynamics of any meaningful large-N theory assumes in some sense a classical character and is therefore simplified.

It is well-known that as we go into the realm of real physical problems, exact treatments are seldom realizable and so one has to frequently turn to approximate methods. The most useful approximation scheme known to us is of course perturbation theory, which consists in expanding the physical quantities to be studied in powers of the coupling parameter of the problem. Obviously to be valid this perturbative expansion requires the coupling parameter to be small and dimensionless. Furthermore, such an expansion completely loses its credibility if the coupling constant can be scaled out of the theory. This is what is really encountered in the power-law potential problems in quantum mechanics [2, 3, 8] and a situation akin to that has cropped up in QCD [9]. A somewhat different but equally delicate issue emanating from the use of perturbation theory was encountered by the theorists working in the area of phase transitions and critical phenomena in the sixties. The typical second-order phase transitions are known to involve fluctuations in their order parameters. In mean field theories, such as the Landau theory, these fluctuations are neglected altogether; in the language of field theory this is the classical or tree approximation. In the gaussian approximation the fluctuations are included to second-order and so this approximation is trustworthy only when the fluctuations are small.

Unfortunately, however, near the critical point the fluctuations are very large. In other words the fluctuations are correlated over a long range i.e. the correlation length diverges and so as a result the perturbation theory breaks down. The insidious aspect of perturbation theory in this context can be explicitly demonstrated by applying it to, say, the Ginzburg–Landau model in d dimensions [10]. If one calculates the self-energy $\Sigma(\boldsymbol{k})$ for $\boldsymbol{k}=0$ in this model by diagrammatic perturbation theory, then a

careful observation reveals that compared to an mth order diagram an $(m+1)$st order diagram contributes for $d<4$ an additional factor of $u\chi^{-(d-4)/2}$ (besides a numerical factor), u measuring the self-interaction strength in the Ginzburg–Landau model and χ being the susceptibility. Thus the actual expansion parameter of the problem is not u but $u\chi^{-(d-4)/2}$. Since the susceptibility (which is proportional to the forward neutron scattering cross section) diverges near the critical point, the expansion parameter will itself blow up at this point for space dimensions less than four and consequently the perturbation theory will fail. The point is that there are situations where the conventional perturbation theory does not work either because of large coupling constant, or because the coupling constant can be scaled out of the theory, or perhaps because of lack of any one-to-one correspondence between the unperturbed and the perturbed states. In such circumstances if variational methods are also not deemed useful owing to the inherent complexities of the problem, an alternative strategy is to devise expansions in terms of some hidden parameter of the theory. $1/N$ serves quite successfully the purpose of being such a parameter.

Since the coupling constant perturbation theory seemed completely inappropriate to deal with the subject of critical phenomena, efforts were aimed at introducing alternative methods. Such efforts culminated in the discovery of the highly sophisticated apparatus of the renormalization group (RG) by Kadanoff, Fisher and Wilson (for a review see ref. [11]) which undoubtedly constitutes an important step forward in the understanding of physical phenomena. An interesting perturbative approach was also proposed by Fisher and Wilson [12] for the determination of the critical exponents. In this approach the expansion parameter is $\varepsilon = 4 - d$, d being the spatial dimensionality. This ε-expansion method, as it is called, hinges on the surprising observation that for $d = 4$ one has exact solutions for the critical exponents which are actually the mean field results. The ε-expansion thus takes care of the deviations from the mean field theory.

Almost simultaneously with the discovery of the RG procedure and the ε-expansion approach, another method was put forward for the study of critical phenomena; this is $1/N$ expansion, where N stands for the order parameter dimensionality. It was observed that in the limit when N was made infinitely large one had an exactly solvable model and then one could organize perturbative expansions about that. (Actually the RG approach offers a logical basis for the $1/N$ expansions of the critical exponents [10].) An attractive feature of the method is that it can be formulated in any dimension. Several significant works on calculating the critical behaviour of physical quantities by envisaging $1/N$ as a small parameter were reported in the early seventies, notably by Abe [13, 14], Abe and Hikami [15], Ferrel and Scalapino [16], Ma [17, 18], Suzuki [19–21], and others [22–24]. The field has been adequately reviewed by Ma [6].

However, the idea of the large-N limit appears to have been mooted much earlier, albeit implicitly, by Berlin and Kac [25]. It seems from Kac's own account [26] that he was inspired by Professor George Uhlenbeck to look into the Ising problem, which is modelled by the hamiltonian [27]

$$H = -J \sum_{\langle ij \rangle} S_i S_j \,, \tag{1.1}$$

where S_i and S_j are the spins located at sites i and j respectively in a three-dimensional lattice. These spins are one-dimensional vectors capable of assuming only discrete values $+1$ and -1 i.e.

$$S_i^2 = 1 \,, \quad i = 1, 2, 3, \ldots, n \,, \tag{1.2}$$

where n is the total number of spins in the lattice. The summation in (1.1) is over the nearest

neighbours and J is the nearest neighbour interaction strength. Soon realizing that the problem was formidable Kac relinquished the idea of solving it and instead proposed to build a somewhat similar model which could be solved. In such an endeavour he proposed a continuum generalization of the Ising model by replacing the constraint (1.2) by

$$\sum_{i=1}^{n} S_i^2 = n \,. \tag{1.3}$$

This interesting new model, known as the spherical model, was solved exactly in three dimensions by Berlin and Kac [25] in 1952 and stands out, to the knowledge of the author, as the only example of an exactly solvable three-dimensional many-body problem. Though the spherical model apparently started as an approximation to the Ising problem, it has won acceptance as a much better approximation to the realistic Heisenberg model. Much later Stanley [28] demonstrated that if the spins are regarded as classical N-dimensional vectors interacting isotropically i.e. if one considers the generalized Heisenberg hamiltonian

$$H = -J \sum_{\alpha=1}^{N} \sum_{\langle ij \rangle} S_i^{(\alpha)} S_j^{(\alpha)} \,, \tag{1.4}$$

then in the limit of infinite spin dimensionality ($N \to \infty$) this generalized model admits an exact treatment and becomes equivalent to the spherical model of Berlin and Kac. In view of the fact that the critical properties of (1.4) can be shown [28] to depend monotonically on the spin dimensionality, the Ising model and the spherical model may be expected to furnish the two bounds of the realistic but hitherto unsolvable Heisenberg model. The spherical model or the N-vector model (as it is often called) of Stanley is the first explicit work on the large-N limit in the field of critical phenomena. $1/N$ corrections to this model were calculated by Abe [13, 14] and Abe and Hikami [15]. Alternative schemes of $1/N$ expansion were also suggested; the methods of Ferrel and Scalapino [16] and Ma [17, 18], coming close on the heels of Abe's work, require special mention, for these works appear to have set the pace in this field.

We have outlined above a historical overview of the early developments of the large-N expansion. This is, however, not the entire story. For instance, in quantum spin models such expansions [29–31] have been known for more than three decades. There N stands for the spin strength or angular momentum and the limit $N \to \infty$ constitutes the leading-order approximation. In this limit the spins can take all orientations and consequently we have a classical model. The $1/N$ expansion then yields the quantum corrections.

Another independent and early work propagating the idea of the large-N approximation has been carried out by Glick, Lipkin and Meshkov [32] in the context of a study of the validity of various approximation schemes in many-body theories. Lipkin et al. have considered the hamiltonian

$$H = \tfrac{1}{2}\varepsilon\left(\sum_{p,\sigma} a_{p\sigma}^{+} a_{p\sigma} + \frac{\alpha}{N} \sum_{p,p',\sigma} a_{p,\sigma}^{+} a_{p',\sigma}^{+} a_{p',-\sigma} a_{p,-\sigma}\right), \tag{1.5}$$

which describes N fermions distributed in two different N-fold degenerate levels which are separated by an energy gap ε. The fermions interact via a monopole–monopole interaction, the strength of the interaction being denoted by α. a^{+} and a are the fermion creation and annihilation operators satisfying the usual anticommutation relations. The index σ takes on values $+1$ and -1, corresponding to the

upper and the lower levels respectively. p enumerates the degenerate states within a particular level and can thus assume values from 1 to N. The monopole–monopole interaction term causes the scattering of pairs of particles between the two levels without effecting any change in their quantum numbers p. The hamiltonian (1.5) is clearly O(N) invariant in the p-space and can be expressed in terms of bilinear products in creation and annihilation operators which close the SU(2) algebra.

By thus mapping their original hamiltonian (1.5) (which may be taken as an idealized model for the study of the vibrational states in nuclei) onto a spin problem, Lipkin et al. were able to obtain the exact eigenvalues. The ranges of validity of various approximate methods were then studied by making a comparison of the eigenvalues obtained by these methods with the exact ones. The results of all these investigations were substantiated in a series of three papers [33, 34, 32]. In their concluding paper [32] the authors employed the diagrammatic Green function formalism to determine the excitation energy of the first excited state above the ground state of the system and confirmed that the random phase approximation (RPA), which consists in summing the ring diagrams, is the leading-order term in an expansion in powers of $1/N$, in which the coefficients are arbitrary functions of $(N\alpha/\varepsilon)$. They also calculated the first correction term to the RPA. The convergence is, however, not very satisfactory unless N is large.

Though the concept of the large-N expansion has been introduced in the literature independently by several authors, the works of Abe [13], Ferrel and Scalapino [16] and Ma [17, 18] apparently gave the initial impetus which triggered significant strides made in the understanding of this method and its applications to other branches of physics. Thus, immediately after its success in dealing with critical phenomena, the idea of the large-N expansion has spread beyond that area to inspire activities in the areas of quantum mechanics [35–108], simple field theories [109–119] and QCD [121–132]. Recently it has found applications in a variety of other fields [133–138].

In QCD the $1/N$ expansion was first applied in 1974 by 't Hooft [121], extending the SU(3) gauge theory to an SU(N) gauge theory; subsequently several other works [122–132] followed. Many subscribe to the belief that $1/N$ is the only meaningful expansion parameter in QCD, N being the number of colours. 't Hooft [121] has shown that in the limit of infinite number of colour degrees of freedom only a special class of diagrams, known as the planar diagrams, survive. Significant though their contribution may have been to the understanding of QCD phenomenology [126], it has not yet been possible to sum the diagrams explicitly to extract quantitative predictions. This is indeed an open problem in large-N QCD.

However, the works of Brezin, Itzykson, Parisi and Zuber [117], Casartelli, Marchesini and Onofri [118] and other authors [119, 120] suggest that the situation may not remain hopelessly bleak for ever. Efforts have also been made to invent alternative routes to the large-N limit. In this context several new methods, such as the collective field technique [57], the methods of constrained classical solutions [41, 59, 108] and the approach based on the Dyson–Schwinger equations [139], have appeared, unleashing a fresh upsurge of activity in the broad subject of large-N expansions [42, 115, 116, 140–147].

In quantum mechanics the large-N expansion was introduced by Ferrel and Scalapino [35] in 1974 by identifying the free energy density of a one-dimensional N-component Ginzburg–Landau model with the quantum mechanical ground state energy of an N-dimensional quartic anharmonic oscillator; since then interest in this subject has continued unabated. Here N denotes the number of space dimensions and the expansion parameter for spherically symmetric potentials is $1/k$ where $k = N + 2l$, l being the angular momentum quantum number. Ferrel and Scalapino [35] obtained the first three terms of the expansion for the ground state energy of the quartic anharmonic oscillator. Here the large-N limit turns out to be analogous to the Hartree approximation. The same results have been obtained by Bray [36]

by summing Feynman diagrams. Hikami and Brezin [37] have suggested a saddle-point expansion method and have investigated the large-order behaviour of the $1/N$ expansion in this problem. A similar technique has later been developed by Mlodinow and Shatz [39].

The mathematical foundations of the large-N scheme have not been very clear, however. Berezin [40] has made a mathematical analysis of this method and has provided further insight into the method. He has considered $O(N)$ vector models and has shown by using a coherent-state method that when N becomes infinitely large the resulting theory is a classical theory. Jevicki and Papanicolaou [41], recognizing the similarity of Berezin's coherent-state formalism with Anderson's approach [148] to the BCS theory of superconductivity, developed the so-called pseudospin formulation of the $1/N$ expansion; this displays the classical nature of the large-N approximation in a rather transparent manner and has a special appeal because of its wide applicability. The classical nature of the large-N limit has subsequently been confirmed by a host of other authors [58, 59, 108, 3, 63].

The work of Yaffe [3] in particular deserves a special mention, for it has discussed this aspect in a rather general framework. Yaffe has shown that if a quantum theory satisfies certain assumptions then it is possible to find a set of generalized coherent states which may be used to construct a classical phase space and obtain a classical hamiltonian such that the resulting dynamics agrees with the large-N quantum dynamics. He has considered various theories and has shown that in each case the large-N limit is a classical limit. Minimizing the classical hamiltonian then yields the large-N solution.

Various other methods of $1/N$ expansion have also been suggested. These include, among others, the perturbed oscillator methods [2, 45], the collective field technique [57], the hypervirial $1/N$ expansion [65] and the Wigner distribution function approach [62]. One palpable advantage of the $1/N$ expansion methods is that they, unlike the usual perturbation theory, do not require any partitioning of the hamiltonian. Moreover, since the expansion is not in terms of the coupling parameter but in terms of $1/N$, the strength of the interaction need not be small. This is why the large-N expansion often receives the status of a nonperturbative method. In fact each coefficient of the expansion may contain several orders of the coupling constant and thus even the leading-order terms may reveal some of the nonperturbative features of the problem. That the method can deal with excited states too is certainly an added advantage which makes it preferable to the variational methods.

However, the $1/N$ expansion's success is not complete; the expansion is at times plagued with slow convergence, particularly for the higher excited states. To ameliorate this problem, Sukhatma and Imbo [67] have developed what they have called the shifted $1/N$ expansion, in which an extra degree of freedom called the shift parameter has been introduced to remove some of the debilitating effects of the higher-order terms on the convergence of the $1/N$ series. The convergence becomes still better if one chooses to work with the supersymmetric partner of the given potential.

Recent years have witnessed a flurry of investigations [2, 89–100] into the large-N helium problem. The helium atom is interesting because it is the simplest atomic system that exhibits both the single-particle and many-body features. The quantitative accuracy and the physical picture that the large-N expansion provides in this case inspires confidence in this method and indicates its promise in the field of atomic and molecular physics.

The study of large-N expansions in relativistic quantum mechanics was initiated by Miramontes and Pajares [102], and Chatterjee [103] has proposed an iterative large-N technique to solve the Klein–Gordon equation for any spherically symmetric potential. Atag [104] has very recently employed this method to solve the Dirac equation. The large-N methodology has been extended to deal with potential scattering problems by Sinha-Roy, Gangopadhyay and Dutta-Roy [105], Gangopadhyay, Dutt and Varshni [106] and Sukhatme, Lauer and Imbo [107].

The most interesting feature of the $1/N$ expansion in quantum mechanics is that even at its worst

($l=0, N=3$), retaining the first few terms of the expansion yields quite accurate results. The pseudospin formulation [42] reveals that for O(N) symmetric problems $1/N$ is indeed a natural expansion parameter, but why the accuracy obtained at $N=3$, which is the dimension of interest and is not really a large number, is so reasonable has remained hitherto elusive and requires more critical investigations at the fundamental level.

It is not possible to cover in this report all of the various aspects of the $1/N$ expansion and discuss all the applications alluded to in the preceding paragraphs. Here we shall restrict our attention to simple quantum mechanics, a few selected problems in atomic physics and some O(N) invariant systems. The choice is influenced mainly by the constraint of space and our personal bias. The organization of the material in this review is as follows. We begin by presenting in section 2 a brief account of $1/N$-expansions in critical phenomena. Here we consider the simple Ginzburg–Landau model and discuss how the critical exponents can be calculated in powers of $1/N$. In section 3 we take up the subject of large-N quantum mechanics. Here the Schrödinger equation is first generalized to N dimensions and next in subsection 3.1 the idea of the large-N limit is introduced. The standard $1/N$-expansion methods of incorporating the finite-N corrections to the large-N limit, namely the perturbed oscillator methods and the Riccati equation methods are then described in subsection 3.2.

In section 4 we review some of the other interesting methods, namely the hypervirial $1/N$ expansion, the pseudospin formulation, and the collective field technique. We do not touch on the elegant coherent-state formalism because this method has been adequately discussed in a very general framework by Yaffe [3]. The $1/N$-expansion methods with improved convergence are included in section 5. Section 6 is devoted to the application of the large-N expansion technique to a few selected problems in atomic and molecular physics. In particular, we consider the strong field Zeeman hamiltonian, the Stark problem, the helium atom and the hydrogen molecular ion. In section 7 we describe the extension of the large-N scheme to the relativistic domain, leaving section 8 for the discussion of the large-N expansion in potential scattering. Concluding remarks are given in section 9.

2. Critical phenomena, the Ginzburg–Landau model and $1/N$ expansion

2.1. Critical phenomena

The subject of critical phenomena (i.e. phenomena observed in the vicinity of a critical point) has come a long way since Van der Waals [149] provided his equation of state for real gases in 1873. In the course of more than a hundred years of activity a vast literature has piled up on this subject. Some of its most interesting aspects, which had been eluding the theorists since their discovery, have become well-understood only in the last decade of this century with the advent of the RG procedure and the related ideas, on which the magnificent edifice of the theories of critical phenomena is now being built. For a detailed review of the subject the reader is referred to the articles of Kadanoff [150], Ma [151], Fisher [152] and Wilson and Kogut [11]. We present here a brief introduction to the subject following refs. [10, 18].

The earliest known transitions in magnetic materials, the normal-to-superconducting phase transitions in various metals, alloys and compounds, the liquid–gas transition at the critical point, the nonferroelectric to ferroelectric transitions, etc., which are commonly referred to as second-order phase transitions (according to the Ehrenfest classification [153]) are some of the familiar examples of the

critical phenomena which have attracted attention. One usually associates with such transitions a quantity called the order parameter which gives a measure of the ordering in a particular phase. This parameter assumes a nonzero value in the ordered phase, i.e. below the critical point, and approaches zero continuously as the critical point is reached from below. Though there is little dispute on the existence of such order parameters in second-order phase transitions, defining them appropriately is not always easy. In the paramagnetic to ferromagnetic transition the net magnetization itself is the order parameter, while in the antiferromagnetic transition the order parameter is the staggered magnetization. In the Heisenberg model these magnetizations are three-dimensional vectors and have three components.

Thus one introduces in the study of critical phenomena the idea of the order-parameter dimensionality which in the case of the Heisenberg model is just three. For the liquid–gas case the order parameter may be defined as the density difference between the liquid phase and the gaseous phase and is thus a scalar having only one component. In superconductivity the complex gap parameter plays the role of the order parameter and has two components. The behaviour of the order parameter (usually as a function of temperature) near the critical point T_c gives rise to the concept of a critical exponent. (The critical point will always be characterized by its critical temperature T_c.)

In the case of the ferromagnetic transition, with which we shall henceforth concern ourselves, the zero-field magnetization m shows for T near T_c the following behaviour

$$m \propto (T_c - T)^{\beta}, \tag{2.1.1}$$

where β is called a critical exponent and has been observed to lie near 0.3 for many materials. The magnetization as a function of an external magnetic field has also been studied. For a small magnetic field h one observes near T_c the following power-law behaviour

$$m \propto h^{1/\delta}, \tag{2.1.2}$$

where δ is another critical exponent whose value is found to be around 4.

Various other physical quantities either approach zero or exhibit a singularity near T_c and from their critical behaviour one can define the corresponding critical exponents. As $T \to T_c$ from above, the magnetic susceptibility $\chi (= \partial m/\partial h)$ for $h = 0$ is found to diverge like

$$\chi \propto (T - T_c)^{-\gamma}, \quad T > T_c, \tag{2.1.3}$$

where the exponent γ can be experimentally determined. For many materials, it is close to 1.3. Another interesting physical quantity is the specific heat C which for $h = 0$ and close to the critical point exhibits a singular behaviour described by the power laws

$$C \propto (T - T_c)^{-\alpha}, \quad T > T_c, \qquad C \propto (T_c - T)^{-\alpha'}, \quad T < T_c. \tag{2.1.4}$$

Experimental data suggest that the exponents α and α' have the same value which is about 0.1. One interesting feature displayed by these critical exponents is that they possess a universal character, i.e., they are independent of the materials involved and of the detailed microscopic interactions. The symmetry properties of materials and the range of the interaction forces in them do, however, exercise some influence in fixing the critical exponents.

Another important feature is the existence of the so-called scaling laws which provide the relationships among various critical exponents. One such scaling law is

$$\alpha + 2\beta + \gamma = 2 . \tag{2.1.5}$$

The scaling hypothesis is based on the idea that the observed singular critical behaviour of physical quantities is solely due to the long-range correlation of the fluctuations in the order parameter and that the length over which the fluctuations are correlated (the correlation length) is the only relevant length scale in these problems. This idea did predict the scaling laws but a rigorous basis of such a powerful hypothesis was lacking. A prospective theory of critical phenomena should thus provide a proper understanding of the scaling hypothesis, prove the scaling laws and predict the correct values of the critical exponents. The RG approach has been quite successful in such an endeavour. In this article, however, we shall not touch on the RG method. We shall describe here the method of $1/N$ expansion as an independent method (though its logical basis lies on the RG ideas) and sketch how the critical exponents can be calculated as a power series in $1/N$, N being the order-parameter dimensionality.

Let us consider for our purpose an N-component real classical field $\phi(r) = \phi_1(r), \phi_2(r), \ldots, \phi_N(r)$ (also called an order-parameter field) at each lattice site of a d-dimensional cubic crystal of volume L^d. Though the model we study may represent a wide variety of physical systems (depending on the values of N and d) we shall for the sake of concreteness restrict our attention to magnetic systems, as we indicated earlier. For such systems $\phi_i(r)$ may be reckoned as the ith component of the local spin density at r. We can also define the Fourier transform ϕ_k of $\phi(r)$ in the usual way

$$\phi_{ik} = L^{-d/2} \int \mathrm{d}^d r \, \mathrm{e}^{-\mathrm{i}k\cdot r} \, \phi_i(r) , \tag{2.1.6}$$

$$\phi_i(r) = L^{-d/2} \sum_{k<\Lambda} \phi_{ik} \, \mathrm{e}^{\mathrm{i}k\cdot r} , \tag{2.1.7}$$

where the cut-off ensures that the summation over k is done only over the L^d discrete points in the first Brillouin zone. The condition of reality of $\phi(r)$ dictates that $\phi^*_{ik} = \phi_{i-k}$. The probability distribution function of spin variables is given by the Boltzmann factor $\exp[-H(\phi)/\kappa_B T]$, where $H(\phi)$ is the model hamiltonian, which is O(N) invariant in the spin space and also invariant under translation in the coordinate space.

An interesting quantity is the correlation function $G(r)$ defined as

$$G(r)\delta_{ij} = \langle \phi_i(r)\phi_j(0) \rangle , \tag{2.1.8}$$

where the isotropy of the spin space has been used and the averaging has to be done in the statistical sense i.e.

$$\langle A \rangle = \frac{\int A \exp[-H(\phi)/\kappa_B T] \, \delta\phi}{\int \exp[-H(\phi)/\kappa_B T] \, \delta\phi} , \tag{2.1.9}$$

where the ϕ-integral is actually a multiple integral over NL^d variables. In terms of the k-space configurations

$$\delta\phi = \prod_{i;k<\Lambda} \mathrm{d}\phi_{ik} \,. \tag{2.1.10}$$

The integral $\int \exp[-H(\phi)/\kappa_B T]\,\delta\phi$ is called the partition function, Z, and is related to the free-energy density F through

$$Z = \exp[-FL^d/\kappa_B T] \,. \tag{2.1.11}$$

Once F is determined the other interesting quantities such as entropy, specific heat, magnetization, susceptibility, etc., can be obtained through the standard thermodynamic relations.

The k-space correlation function $G(k)$ is defined as the Fourier transform of $G(r)$

$$G(k)\delta_{ij} = \int \mathrm{d}^d r_1\, \mathrm{d}^d r_2\, \langle \phi_i(r_1)\phi_j(r_2)\rangle \exp[-ik\cdot(r_1 - r_2)] = \langle |\phi_{ik}|^2\rangle \delta_{ij} \,, \tag{2.1.12}$$

where it has been assumed that the function $\langle \phi(r_1)\phi(r_2)\rangle$ depends upon r_1 and r_2 only through $(r_1 - r_2)$. This follows from the translational invariance of the model hamiltonian (i.e. from the homogeneity of the coordinate space). It is easy to show that the differential cross section (Γ) for scattering of neutrons by the spin system under investigation is proportional to $L^d\langle |\phi_{ik}|^2\rangle$, k denoting the momentum transfer in the scattering process. Thus the correlation function can be determined directly from neutron scattering data. It is observed that for small momentum transfer Γ behaves at $T = T_c$ like $L^d k^{-2+\eta}$ and so we can describe the critical behaviour of the correlation function by

$$G(k) \propto k^{-2+\eta} \,, \tag{2.1.13}$$

η being the corresponding critical exponent. The susceptibility can be shown to be related to the correlation function for $k = 0$

$$\chi/T = G(0) \,. \tag{2.1.14}$$

The fact that $G(k)$ becomes singular at $k = 0$ as T approaches T_c implies a long-range correlation of the spin fluctuations or in other words a divergence in the correlation length ξ at the critical point. One introduces a critical exponent ν to describe this divergence

$$\xi \propto (T - T_c)^{-\nu} \,. \tag{2.1.15}$$

The scaling hypothesis suggests that at $T \neq T_c$ the correlation function G may be expressed as

$$G(k, \xi) = k^{-2+\eta} g(k\xi) \,, \tag{2.1.16}$$

where $g(k\xi)$, which depends only on the product $k\xi$, is known as a scaling function and may be identified as the form factor for the scattering of neutrons from the spin fluctuations at the critical point.

Besides (2.1.5) other interesting scaling laws which are satisfied by critical exponents are

$$\alpha + \beta(\delta + 1) = 2 \,, \qquad \alpha = 2 - \nu d \,, \qquad \beta = \tfrac{1}{2}\nu(d - 2 + \eta) \,, \tag{2.1.17a, b, c}$$

$$\gamma = (2-\eta)\nu\,, \qquad \delta = (d-2-\eta)/(d-2+\eta)\,. \tag{2.1.17d, e}$$

2.2. *The Ginzburg–Landau model and the large-N limit* [6]

To calculate the critical exponents in the large-N limit one has to choose a specific model hamiltonian. The universality of the exponents however gives one the freedom to work with any model. We consider here the simple Ginzburg–Landau (GL) model [10, 6]

$$\mathscr{H} = H/\kappa_B T = \mathscr{H}_0 + \mathscr{H}_1\,, \tag{2.2.1}$$

$$\mathscr{H}_0 = \frac{1}{2}\int \mathrm{d}^d r\,\{[\nabla\phi(r)]^2 + a\phi^2\}\,, \tag{2.2.2}$$

$$\mathscr{H}_1 = \frac{1}{4}\int \mathrm{d}^d r\,\mathrm{d}^d r'\,u(r-r')\phi^2(r)\phi^2(r')\,, \tag{2.2.3}$$

where

$$[\nabla\phi(r)]^2 \equiv \sum_{i=1}^{N} \nabla\phi_i(r)\cdot\nabla\phi_i(r) = \sum_{\alpha=1}^{d}\sum_{i=1}^{N}\left(\frac{\partial\phi_i(r)}{\partial x_\alpha}\right)^2, \tag{2.2.4}$$

$$\phi^2(r) = \sum_{i=1}^{N}\phi_i^2(r)\,. \tag{2.2.5}$$

We shall assume that the interaction is local i.e.

$$u(r-r') = u\delta(r-r')\,. \tag{2.2.6}$$

Using (2.1.7) and

$$u(r) = L^{-d/2}\sum_q u_q\,\mathrm{e}^{-\mathrm{i}q\cdot r}\,, \tag{2.2.7}$$

$$u_q = L^{d/2}\int u(r)\,\mathrm{e}^{\mathrm{i}q\cdot r}\,\mathrm{d}r\,, \tag{2.2.8}$$

one can write $\mathscr{H}$ in terms of the Fourier components ϕ_{ik}

$$\mathscr{H} = \frac{1}{2}\sum_{i=1}^{N}\sum_{k<\Lambda}(a+k^2)|\phi_{ik}|^2 + \tfrac{1}{4}L^{-d/2}\sum_{i,j=1}^{N}\sum_{\substack{k,k',q\\<\Lambda}} u_q\phi_{ik}\phi_{i,-k-q}\phi_{jk'}\phi_{j,k'+q}\,. \tag{2.2.9}$$

In view of (2.2.6) we have $u_q = L^{-d/2}u$. In the absence of the self-interaction term $\mathscr{H}_1$ the probability distribution $\mathrm{e}^{-\mathscr{H}_0}$ would be just gaussian and the calculation of statistical averages would be rather simple. We shall denote such averages by $\langle\cdots\rangle_0$. Using the decomposition $\phi_{ik} = \alpha_{ik} + \mathrm{i}\beta_{ik}$ where $\alpha_{ik} = \alpha_{i,-k}$ and $\beta_{ik} = -\beta_{i,-k}$, one obtains the following results [10],

$$\langle\phi_{ik}\rangle_0 = 0\,, \qquad \langle\phi_{ik}^2\rangle_0 = 0\,, \tag{2.2.10a, b}$$

$$\langle|\phi_{ik}|^2\rangle_0 \equiv G_0(k) = (a + k^2)^{-1}\,, \tag{2.2.10c}$$

$$\langle \phi_{ik}\phi_{jk'}\rangle = \delta_{ij}\delta_{k,-k'}G_0(k)\,. \tag{2.2.10d}$$

The presence of the interaction, however, makes the exact calculation of averages impossible and, as we have mentioned in section 1, the ordinary perturbation theory also breaks down near the critical point because of large fluctuations. Here comes the novelty of the $1/N$ expansion. Let us see how in the large-N limit the GL theory simplifies. We shall mainly follow ref. [6].

The important point to realize is that, in the limit $N\to\infty$, u should be of the order of $1/N$, so that the influence of the interaction term may not dominate. A simple and convincing justification of this N-dependence of u has been given in ref. [154].

In the non-interacting case the coefficient of $\frac{1}{2}\phi_i^2$, which may be called the stiffness parameter, is a. In the presence of the interaction this quantity gets modified and one can define an effective stiffness parameter a_{eff} by figuring out the coefficient of a given $\frac{1}{2}\phi_i^2$ in $\mathcal{H}_1$ and adding it to a. Thus one obtains

$$a_{\text{eff}} = a + u(\phi^2 - \tfrac{1}{2}\phi_i^2)\,. \tag{2.2.11}$$

Since $\phi^2 = \Sigma_{i=1}^N \phi_i^2$, in the large-$N$ limit it seems plausible to assume that the sum ϕ^2 will be much greater than any individual contribution ϕ_i^2 and then eq. (2.2.11) can be approximated by

$$a_{\text{eff}} = a + u\phi^2\,. \tag{2.2.12}$$

Furthermore, though each component ϕ_i may exhibit large fluctuation, the fluctuation in the sum $\Sigma_{i=1}^N \phi_i^2$ is expected to be rather small in the large-N limit. Thus in the limit $N\to\infty$ one can write

$$\phi^2 = \langle\phi^2\rangle + \Delta\phi^2\,, \tag{2.2.13}$$

where $\Delta\phi^2 \ll \langle\phi^2\rangle$ and is of the order $1/N$. Therefore in this approximation the effective stiffness parameter becomes

$$a_{\text{eff}} = a + u\langle\phi^2\rangle \tag{2.2.14}$$

and the GL hamiltonian reduces to

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}^d\boldsymbol{r}\,\{[\nabla\phi(\boldsymbol{r})]^2 + (a + u\langle\phi^2\rangle)\phi^2\} = \frac{1}{2}\sum_{i=1}^{N}\sum_{k<\Lambda}(a_{\text{eff}} + k^2)|\phi_{ik}|^2\,. \tag{2.2.15}$$

The correlation function is given by

$$G(k) = \langle|\phi_{ik}|^2\rangle = (a_{\text{eff}} + k^2)^{-1}\,. \tag{2.2.16}$$

Equations (2.2.14) and (2.2.16) are now to be solved self-consistently for a_{eff}. The large-N approximation thus looks like a self-consistent Hartree approximation.

For $k = 0$ we get $G(0) = a_{\text{eff}}^{-1}$ and therefore we may call a_{eff} the inverse susceptibility. Using (2.1.7) and (2.2.16) in (2.2.14) gives

$$a_{\text{eff}} = a + uNL^{-d} \sum_{k<\Lambda} (a_{\text{eff}} + k^2)^{-1} , \tag{2.2.17}$$

which on substituting $u = 1/N$ and on replacing the summation over $\boldsymbol{k}$ by an integral using the standard prescription $\Sigma_{\boldsymbol{k}} \cdots \to (L/2\pi)^d \int \cdots \mathrm{d}^d\boldsymbol{k}$, reduces to

$$a_{\text{eff}} = a + K_d \int_0^{\Lambda} \frac{k^{d-1}\,\mathrm{d}k}{a_{\text{eff}} + k^2} , \tag{2.2.18}$$

where

$$K_d = 2^{-d+1}\pi^{-d/2}/\Gamma(\tfrac{1}{2}d) \tag{2.2.19}$$

and use has been made of hyperspherical coordinates (to be discussed in detail in the next section). Near the critical point a_{eff} is vanishingly small and so one can take $\Lambda^2 \gg a_{\text{eff}}$. Then (2.2.18) can be shown to yield, for $2 < d \leq 4$,

$$a_{\text{eff}} = \tilde{a} - [J/(\tfrac{1}{2}d - 1)]a_{\text{eff}}^{(d-2)/2} , \tag{2.2.20}$$

$$\tilde{a} = a + K_d \Lambda^{d-2}/(d-2) , \tag{2.2.21}$$

$$J = \tfrac{1}{2}\pi K_d(\tfrac{1}{2}d - 1)\,\mathrm{cosec}\,\pi(\tfrac{1}{2}d - 1) . \tag{2.2.22}$$

From (2.2.20) it follows that as $T \to T_c$, $\tilde{a} \to 0$. Assuming $\tilde{a}$ to be of the form $(T - T_c)$ one can define the critical exponent γ,

$$a_{\text{eff}} \propto \tilde{a}^{\gamma} . \tag{2.2.23}$$

For $d < 4$ and small a_{eff} it then immediately follows from (2.2.20) and (2.2.23) that

$$\gamma = 2/(d-2) . \tag{2.2.24}$$

Writing the correlation function in the form

$$G(\boldsymbol{k}) = k^{-2}[1 + (\tilde{a}^{-1/(d-2)}k)^{-2}]^{-1} \tag{2.2.25}$$

and comparing it with (2.1.16) one gets

$$\eta = 0 , \qquad \xi = \tilde{a}^{-1/(d-2)} , \tag{2.2.26}$$

which gives

$$\nu = 1/(d-2) . \tag{2.2.27}$$

Let us now consider the free energy density and differentiate it twice with respect to a to define a quantity [6]

$$C = -\partial^2 F/\partial a^2 , \tag{2.2.28}$$

which possesses the essential features of the specific heat. By using (2.1.11), eq. (2.2.28) can be written as

$$C = -\tfrac{1}{2}\partial\langle\phi^2\rangle/\partial a , \tag{2.2.29}$$

which with the help of (2.2.14) and (2.2.20) becomes

$$C = \tfrac{1}{2}N(1 - a_{\text{eff}}^{(4-d)/2}) + \text{higher-order terms in } a_{\text{eff}} . \tag{2.2.30}$$

Since $a_{\text{eff}} \propto \tilde{a}^{2/(d-2)}$, the exponent describing the critical behaviour of the specific heat is given by

$$\alpha = (d-4)/(d-2) . \tag{2.2.31}$$

Clearly the specific heat is finite at the critical point but its derivative diverges for $2 < d < 4$. One can now verify that the scaling laws (2.1.17b) and (2.1.17d) are indeed satisfied. Obtaining the exponents β and δ one can verify the other scaling laws also.

To obtain the $1/N$ corrections to the large-N critical exponents Ma [18] has developed a useful $1/N$-expansion technique within the framework of the diagrammatic perturbation theory, to which we now turn.

2.3. *Diagrammatic perturbation theory and* $1/N$ *expansions* [18, 10, 6]

In perturbation theory one treats $\mathscr{H}_1$ as a perturbation and writes the probability distribution as

$$e^{-\mathscr{H}} = e^{-\mathscr{H}_0} e^{-\mathscr{H}_1} = e^{-\mathscr{H}_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathscr{H}_1^n , \tag{2.3.1}$$

so that (2.1.9) becomes

$$\langle A\rangle = \frac{\sum_{n=0}^{\infty}[(-1)^n/n!]\langle A\mathscr{H}_1^n\rangle_0}{\sum_{n=0}^{\infty}[(-1)^n/n!]\langle \mathscr{H}_1^n\rangle_0} . \tag{2.3.2}$$

Thus the correlation function $G(\boldsymbol{k})$ can be written as

$$G(\boldsymbol{k}) = \langle|\phi_{ik}|^2\rangle = \frac{\sum_{n=0}^{\infty}[(-1)^n/n!]\langle \phi_{ik}^*\phi_{ik}\mathscr{H}_1^n\rangle_0}{\sum_{n=0}^{\infty}[(-1)^n/n!]\langle \mathscr{H}_1^n\rangle_0} , \tag{2.3.3}$$

and the free energy density is given by

$$F = F_0 - L^{-d}\ln\Big(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle\mathscr{H}_1^n\rangle_0\Big) , \tag{2.3.4}$$

where F_0 is the free energy density in the non-interacting case i.e. the free energy density corresponding to the unperturbed hamiltonian $\mathscr{H}_0$.

We confine our discussion here to the correlation function only. The problem now boils down to the calculation of the sum of the gaussian averages of the products of the ϕ_{ik}. One can show that the

gaussian average of a product of the ϕ_{ik} can be simplified to a sum of products of pairwise gaussian averages, each way of pairing the ϕ_{ik} contributing a term to the sum. Actually the whole expression vanishes unless the number of the ϕ_{ik} to be gaussian-averaged is even. Each pairwise average is given by (2.2.10d) i.e. each ϕ_{ik} has to be paired with ϕ_{ik}^* to get a nonvanishing contribution.

The correlation function can now be expanded in a power series in u. It is convenient to introduce diagrams to represent these terms. The bare correlation function $G_0(\boldsymbol{k})$ is represented by a solid line carrying a wave vector $\boldsymbol{k}$ and u_q is denoted by a dashed line with a vector $\boldsymbol{q}$. In actual calculations the wave vector dependence of u_q should be ignored because $u(\boldsymbol{r}-\boldsymbol{r}')$ has been assumed a δ-function interaction. It turns out that nth order terms can be represented by all possible topologically nonequivalent diagrams with $2n$ vertices where at each vertex two solid lines and one dashed line meet and the momentum conservation condition is satisfied. The wave vector which is not fixed by the conservation law is to be integrated over, each integral carrying a factor $1/(2\pi)^d$. The lines whose wave vectors are integrated over are called internal lines and those with wave vectors fixed by the ϕ_{ik} in $G(\boldsymbol{k})$ are called external lines. The graphs up to second order are shown in fig. 2.1. Each closed loop of solid lines contributes a factor N, coming from the summation over N spin indices. Hence the prescription $u \sim 1/N$ is just the border line between a free theory and a divergent theory in the large-N limit [154]. It is clear that in the limit $N \to \infty$ only those diagrams which have equal numbers of loops and interaction lines will survive, leading to the spherical limit. To include corrections to order $1/N$ we will have to consider diagrams which have one interaction line more than the number of loops. Similarly the higher-order corrections may be calculated.

One interesting quantity is the self energy Σ defined by

$$G(\boldsymbol{k}) = G_0(\boldsymbol{k}) - G_0(\boldsymbol{k})\Sigma(\boldsymbol{k})G_0(\boldsymbol{k}) + G_0(\boldsymbol{k})\Sigma(\boldsymbol{k})G_0(\boldsymbol{k})\Sigma(\boldsymbol{k})G_0(\boldsymbol{k}) - \cdots$$

$$= G_0(\boldsymbol{k}) - G_0(\boldsymbol{k})\Sigma(\boldsymbol{k})G(\boldsymbol{k}) = [G_0^{-1}(\boldsymbol{k}) + \Sigma(\boldsymbol{k})]^{-1} = [a + k^2 + \Sigma(\boldsymbol{k})]^{-1}\,. \tag{2.3.5}$$

Equation (2.3.5) may be rewritten in the form

$$G^{-1}(\boldsymbol{k}) = a_{\text{eff}} + k^2 + \Sigma(\boldsymbol{k}) - \Sigma(0)\,, \tag{2.3.6}$$

where

$$a_{\text{eff}} = a + \Sigma(0) = G^{-1}(0) \tag{2.3.7}$$

is the inverse susceptibility. At $T = T_c$, a_{eff} vanishes and we have

$$G^{-1}(\boldsymbol{k}) = k^2 + \Sigma(\boldsymbol{k}) - \Sigma(0)\,. \tag{2.3.8}$$

The self energy diagrams up to $O(1/N)$ are given by the graphs without the external lines shown in fig. 2.2. Here a wavy line represents pictorially the sum shown in fig. 2.3 and refers to the dressed interaction. One may note that each term in the right-hand side of fig. 2.3 is of order $1/N$ and therefore while considering diagrams to a particular order in $1/N$ one should dress each bare interaction line by including the bubble corrections.

It may be recalled that also in the Bohm–Pines random-phase approximation theory of plasma oscillations in the electron gas one sums similar bubble diagrams to screen the long-range part of the

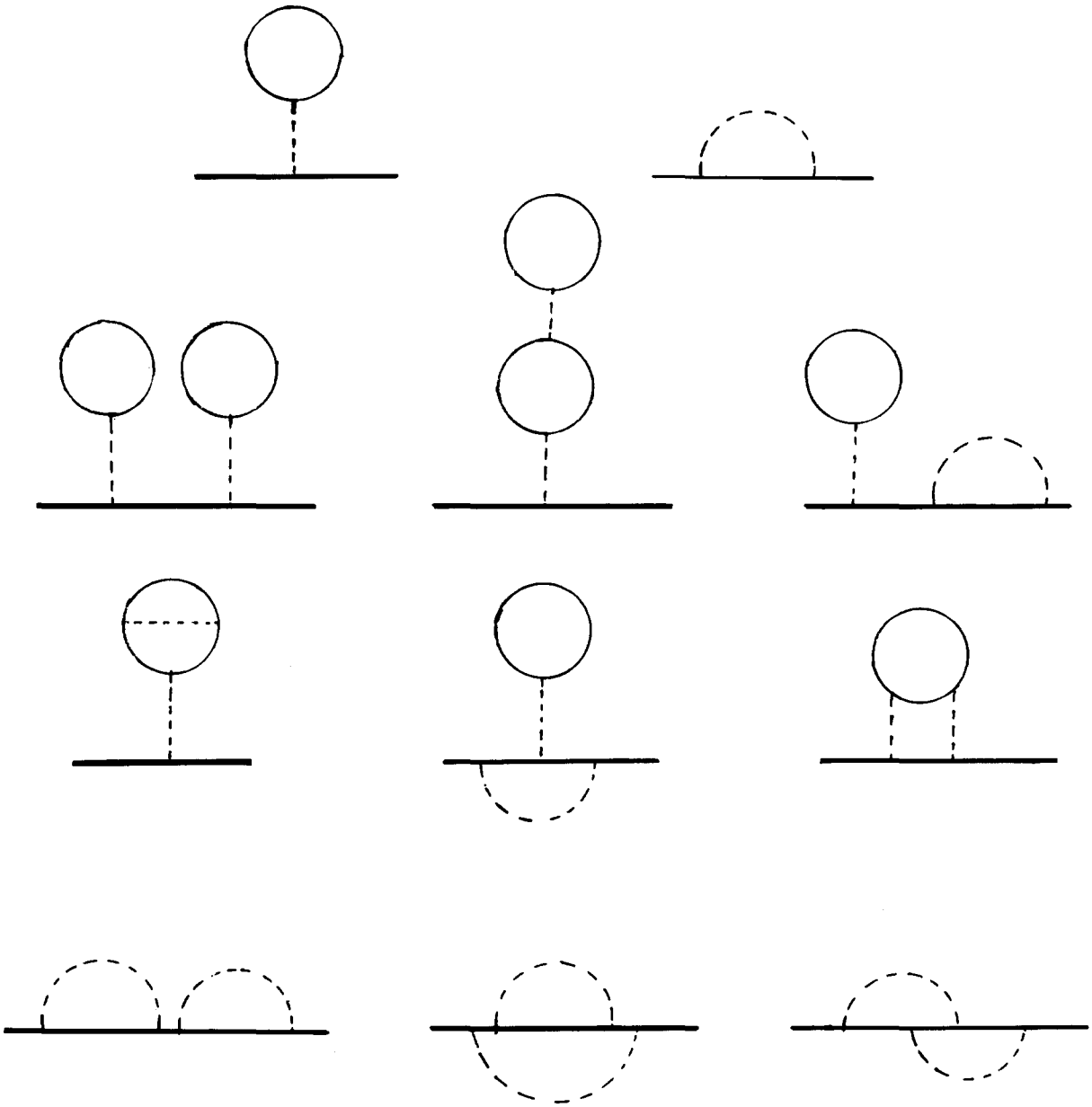

Fig. 2.1

Coulomb interaction. It is because of this analogy that the large-N approximation is sometimes referred to as the screening approximation. Mathematically, a wavy line corresponds to

$$2\{-u + u^2 N\pi(\boldsymbol{k}) - u^3[N\pi(\boldsymbol{k})]^2 + \cdots\} = -2u[1 + uN\pi(\boldsymbol{k})]^{-1}, \tag{2.3.9}$$

where $\pi(\boldsymbol{k})$ represents a bubble and is given by

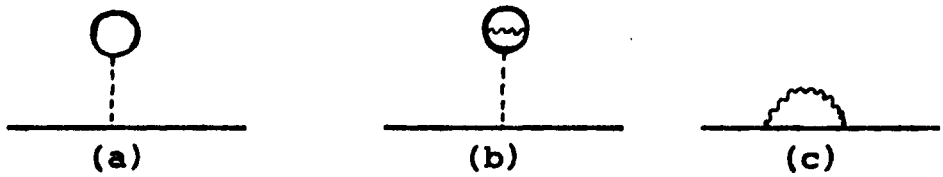

Fig. 2.2

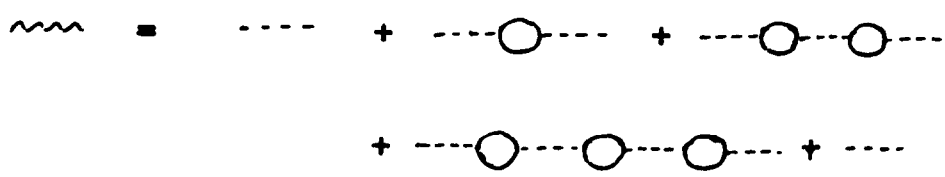

Fig. 2.3

$$\pi(\boldsymbol{k}) = (2\pi)^{-d} \int \mathrm{d}\boldsymbol{k}'\, G_0(\boldsymbol{k}') G_0(\boldsymbol{k}' + \boldsymbol{k}) \equiv \pi(a_{\mathrm{eff}}, k^2) \tag{2.3.10}$$

$$= (2\pi)^{-d} \int \mathrm{d}\boldsymbol{k}'\, (a_{\mathrm{eff}} + k'^2)^{-1} [a_{\mathrm{eff}} + (\boldsymbol{k} + \boldsymbol{k}')^2]^{-1} = \pi(a_{\mathrm{eff}}/k^2, 1) k^{d-4} . \tag{2.3.11}$$

Utilizing the identity

$$\frac{1}{AB} = \int_0^1 \mathrm{d}z\, [(1-z)A + zB]^{-2} , \tag{2.3.12}$$

one can now find

$$\pi(a_{\mathrm{eff}}, 1) = J \int_0^1 \mathrm{d}z\, [a_{\mathrm{eff}} + z(1-z)]^{(d-4)/2} \tag{2.3.13}$$

$$= \pi(0,1)(1 + 4a_{\mathrm{eff}})^{(d-3)/2} + 2J(1 - \tfrac{1}{2}d)^{-1} a_{\mathrm{eff}}^{d/2-1} F(1, \tfrac{1}{2}, \tfrac{1}{2}, d; -4a_{\mathrm{eff}}) , \tag{2.3.14}$$

where J is given by (2.2.22) and $\pi(0,1)$ is given by

$$\pi(0,1) = JB(\tfrac{1}{2}d - 1, \tfrac{1}{2}d - 1) , \tag{2.3.15}$$

F and B being respectively the hypergeometric function and the beta function. Next, substituting (2.3.14) (with a_{eff} replaced by a_{eff}/k^2) into (2.3.11) one can obtain $\pi(\boldsymbol{k})$. At $T = T_c$ we have, however, the simple expression

$$\pi(\boldsymbol{k}) = \pi(0,1) k^{d-4} . \tag{2.3.16}$$

The form of the dressed interaction in the coordinate representation can be easily obtained by taking the Fourier transform of (2.3.9). Though the original (bare) interaction $u(\boldsymbol{r}-\boldsymbol{r}')$ is of the δ-function type, $\sim\delta(\boldsymbol{r}-\boldsymbol{r}')$, the dressed interaction is found to assume a long-range character with the characteristic length ξ.

The $1/N$ diagrammatic expansion presented above can be used to extract $O(1/N)$ corrections to the critical exponents. As an illustration, we shall discuss here the calculation of the lowest-order $1/N$ correction to the large-N limit of η. The logical basis of this calculation is, however, the RG theory, which suggests that at $T=T_c$ and for small k the correlation function is given by

$$G(\boldsymbol{k},\mu)\propto k^{-2+\eta}[1+O(k^{-y_2})]\,, \tag{2.3.17}$$

where μ denotes a set of parameters which specifies the model hamiltonian and defines a space, called the parameter space, on which the RG transformations are defined. The RG analysis in the large-N limit asserts that $\eta=0$ and $y_2=d-4$ when $N\to\infty$. Let us assume $\eta=O(1/N)$ and $y_2=d-4+O(1/N)$. Then expanding k^{η} in powers of η ones gets

$$G(\boldsymbol{k},\mu)\propto k^{-2}[1+\eta\ln k+(1/2!)(\eta\ln k)^2+(1/3!)(\eta\ln k)^3+\cdots+O(k^{4-d},N^{-1}k^{4-d}\ln k,\ldots)]\,. \tag{2.3.18}$$

Thus to $O(1/N)$ we have

$$G^{-1}(\boldsymbol{k})=k^2(1-\eta\ln k+\cdots)\,. \tag{2.3.19}$$

Comparison of (2.3.19) with (2.3.8) shows that the coefficient of the $k^2\ln k$ term of $\Sigma(\boldsymbol{k})-\Sigma(0)$ will give η to $O(1/N)$. Figures 2.2 a, b are independent of $\boldsymbol{k}$; the only diagram contributing to $\Sigma(\boldsymbol{k})-\Sigma(0)$ is fig. 2.2c which gives

$$\Sigma(\boldsymbol{k})-\Sigma(0)=(2\pi)^{-d}\int d\boldsymbol{k}'\,\frac{2u}{1+uN\pi(\boldsymbol{k}')}\left(\frac{1}{(\boldsymbol{k}+\boldsymbol{k}')^2}-\frac{1}{k'^2}\right). \tag{2.3.20}$$

One can easily convince oneself that the above integral can contribute a term of the type $k^2\ln k$ only from its small k' region. For $\boldsymbol{k}'\to 0$. $\pi(\boldsymbol{k}')\sim k'^{d-4}$ is large and therefore (2.3.9) may be approximated as

$$2u/[1+uN\pi(\boldsymbol{k}')]\approx 2\pi^{-1}(\boldsymbol{k}')/N=(2/N)\pi^{-1}(0,1)k'^{4-d}\,. \tag{2.3.21}$$

Substituting (2.3.21) into (2.3.20) and integrating over the d-dimensional wave vector $\boldsymbol{k}'$, one gets

$$\Sigma(\boldsymbol{k})-\Sigma(0)=\eta k^2[\ln k+\text{const.}+O(k^{4-d})]\,, \tag{2.3.22}$$

with

$$\eta=[(4/d)-1]S_d/N\,, \tag{2.3.23}$$

$$S_d=\tfrac{1}{2}K_d/\pi(0,1)\,. \tag{2.3.24}$$

Without going into the calculation of other exponents we quote here the $1/N$-expansion results for α and γ

$$\alpha = \frac{d-4}{d-2}\left(1 + \frac{8(1-d)}{(4-d)}\frac{S_d}{N}\right) + O(N^{-2}), \tag{2.3.25}$$

$$\gamma = [2/(d-2)](1 - 6S_d/N) + O(N^{-2}). \tag{2.3.26}$$

Let us mention before we close this section that the quantitative accuracies of the results predicted by the $1/N$ expansion for the critical exponents are not always very high, but the method is still useful and appealing, for it allows an independent verification of the scaling laws, which are found to be maintained to each order in $1/N$.

3. The N-dimensional Schrödinger equation

Let us consider, for the sake of simplicity, the motion of a particle in a spherically symmetric potential in N dimensions. This problem has been studied by a number of authors [155–160]. Here we shall follow the work of Louck [160]. The Schrödinger equation for this problem (in units of $\hbar = m = 1$) is

$$[-\tfrac{1}{2}\nabla_N^2 + V_N(r)]\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r}). \tag{3.0.1}$$

where $\boldsymbol{r}$ is an N-dimensional position vector having cartesian components $x_1, x_2, \ldots, x_N$ and magnitude $r = (\Sigma_{i=1}^N x_i^2)^{1/2}$, ∇_N^2 is given by

$$\nabla_N^2 = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}, \tag{3.0.2}$$

and $V_N(r)$ is an N-dimensional potential depending only on the radial distance r. Because of the spherical symmetry of the problem it is convenient to introduce the hyperspherical coordinates which are defined as follows

$$\begin{aligned}
x_1 &= r\cos\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{N-1}, \\
x_2 &= r\sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{N-1}, \\
x_3 &= r\cos\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_{N-1}, \\
x_4 &= r\cos\theta_3 \sin\theta_4 \sin\theta_5 \cdots \sin\theta_{N-1}, \\
&\vdots \\
x_j &= r\cos\theta_{j-1} \sin\theta_j \sin\theta_{j+1} \cdots \sin\theta_{N-1}, \\
&\vdots \\
x_{N-1} &= r\cos\theta_{N-2} \sin\theta_{N-1}, \\
x_N &= r\cos\theta_{N-1},
\end{aligned} \tag{3.0.3}$$

where $N = 3, 4, 5, \ldots$ (for $N = 2, x_1 = r\cos\theta_1, x_2 = r\sin\theta_1$), $0 \le r \le \infty$, $0 \le \theta_1 \le 2\pi$, $0 \le \theta_j \le \pi$, $j = 2, 3, \ldots, N-1$.

The laplacian ∇_N^2 can be written as

$$\nabla_N^2 = \frac{1}{h} \sum_{i=0}^{N-1} \frac{\partial}{\partial\theta_i} \left(\frac{1}{h_i^2} \frac{\partial}{\partial\theta_i} \right), \tag{3.0.4}$$

where $\theta_0 \equiv r$. The h_i are known as the scale factors and are related to the fundamental metric tensors g_{ii} for the orthogonal coordinate systems through

$$h_1^2 = g_{ii} = \sum_{k=1}^{N} \left(\frac{\partial x_k}{\partial\theta_i} \right)^2, \tag{3.0.5}$$

and

$$h = \prod_{j=0}^{N-1} h_j . \tag{3.0.6}$$

Expression (3.0.4) reads explicitly

$$\nabla_N^2 = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} r^{N-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{j=1}^{N-2} \frac{1}{\sin^2\theta_{j+1} \sin^2\theta_{j+2} \cdots \sin^2\theta_{N-1}} \left(\frac{1}{\sin^{j-1}\theta_j} \frac{\partial}{\partial\theta_j} \sin^{j-1}\theta_j \frac{\partial}{\partial\theta_j} \right)$$

$$+ \frac{1}{r^2} \left(\frac{1}{\sin^{N-2}\theta_{N-1}} \frac{\partial}{\partial\theta_{N-1}} \sin^{N-2}\theta_{N-1} \frac{\partial}{\partial\theta_{N-1}} \right). \tag{3.0.7}$$

The angular momentum components are defined as skew symmetric tensors

$$L_{ij} = -L_{ji} = x_i p_j - x_j p_i , \quad i = 1, 2, \ldots, j-1 , \quad j = 2, 3, \ldots, N , \tag{3.0.8}$$

where p_i is the momentum operator canonically conjugate to x_i and may be expressed as

$$p_k = -\mathrm{i} \frac{\partial}{\partial x_k} = -\mathrm{i} \sum_{j=0}^{N-1} \left(\frac{\partial\theta_j}{\partial x_k} \right) \frac{\partial}{\partial\theta_j} = -\mathrm{i} \sum_{j=0}^{N-1} \left(\frac{1}{h_j^2} \frac{\partial x_k}{\partial\theta_j} \right) \frac{\partial}{\partial\theta_j} , \tag{3.0.9}$$

where use has been made of the conditions

$$\sum_{l=0}^{N-1} \frac{\partial x_l}{\partial\theta_i} \frac{\partial x_l}{\partial\theta_j} = \delta_{ij} h_i^2 , \qquad \sum_{l=0}^{N-1} \frac{\partial\theta_i}{\partial x_k} \frac{\partial x_l}{\partial\theta_i} = \delta_{kl} , \tag{3.0.10}$$

which follow from the transformation equations (3.0.3) and the orthogonality of the coordinate system. The $N(N-1)/2$ components of the angular momentum defined by (3.0.8) are the generators of rotation in N dimensions and form the basis of the Lie algebra of the O(N) group, the algebra being defined by the commutation relation

$$[L_{ij}, L_{kl}] = \mathrm{i}\delta_{jl}L_{ik} + \mathrm{i}\delta_{ik}L_{jl} - \mathrm{i}\delta_{jk}L_{il} - \mathrm{i}\delta_{il}L_{jk} . \tag{3.0.11}$$

Interesting quantities are the Casimir invariants

$$L_k^2 = \sum_{i,j} L_{ij}L_{ij} , \quad i = 1, 2, \ldots, j-1 , \quad j = 2, 3, \ldots, k+1 . \tag{3.0.12}$$

Thus we can obtain

$$\begin{aligned}
L_1^2 &= -\partial^2/\partial\theta_1^2 , \\
L_2^2 &= -\left(\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\sin\theta_2\frac{\partial}{\partial\theta_2} - \frac{L_1^2}{\sin^2\theta_2}\right), \\
&\vdots \\
L_k^2 &= -\left(\frac{1}{\sin^{k-1}\theta_k}\frac{\partial}{\partial\theta_k}\sin^{k-1}\theta_k\frac{\partial}{\partial\theta_k} - \frac{L_{k-1}^2}{\sin^2\theta_k}\right), \\
&\vdots \\
L_{N-1}^2 &= -\left(\frac{1}{\sin^{N-2}\theta_{N-1}}\frac{\partial}{\partial\theta_{N-1}}\sin^{N-2}\theta_{N-1}\frac{\partial}{\partial\theta_{N-1}} - \frac{L_{N-2}^2}{\sin^2\theta_{N-1}}\right).
\end{aligned} \tag{3.0.13}$$

It is then easy to show that

$$\nabla_N^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r} - \frac{L_{N-1}^2}{r^2} . \tag{3.0.14}$$

Let us now denote the simultaneous eigenfunction of the commuting operators $L_1^2, L_2^2, \ldots$ by $Y_{\lambda_{N-1},\lambda_{N-2},\ldots,\lambda_2,\lambda_1}(\theta_1, \theta_2, \ldots, \theta_{N-1})$. We can then write

$$L_k^2 Y_{\lambda_{N-1},\lambda_{N-2},\ldots,\lambda_2,\lambda_1}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = \lambda_k Y_{\lambda_{N-1},\lambda_{N-2},\ldots,\lambda_2,\lambda_1}(\theta_1, \theta_2, \ldots, \theta_{N-1}) , \tag{3.0.15}$$

where the eigenvalues λ_k can be shown to be all real and non-negative and to furthermore satisfy the condition $\lambda_k \geq \lambda_{k-1}$. The form of the Casimir invariants suggests that the eigenfunction $Y_{\lambda_{N-1},\lambda_{N-2},\ldots,\lambda_2,\lambda_1}(\theta_1, \theta_2, \ldots, \theta_{N-1})$ can be expressed as

$$Y_{\lambda_{N-1},\lambda_{N-2},\ldots,\lambda_2,\lambda_1}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = \prod_{k=1}^{N-1} \Theta_{\lambda_k,\lambda_{k-1}}(\theta_k) , \tag{3.0.16}$$

where $\Theta_{\lambda_k,\lambda_{k-1}}(\theta_1) = \Theta_1(\lambda_1)$. Then we get

$$L_1^2\Theta_1(\theta_1) = \lambda_1\Theta_1(\theta_1) , \tag{3.0.17}$$

$$L_k^2(\lambda_{k-1})\Theta_{\lambda_k,\lambda_{k-1}}(\theta_k) = \lambda_k\Theta_{\lambda_k,\lambda_{k-1}}(\theta_k) , \quad k = 2, 3, \ldots, N-1 , \tag{3.0.18}$$

where

$$L_k^2(\lambda_{k-1}) = -\left(\frac{\partial^2}{\partial\theta_k^2} + (k-1)\cot\theta_k\frac{\partial}{\partial\theta_k} - \frac{\lambda_{k-1}}{\sin^2\theta_k}\right). \tag{3.0.19}$$

Equations (3.0.17) and (3.0.18) with $k=2$ define the ordinary spherical harmonics in 3 dimensions. Using the standard raising and lowering operators one obtains

$$\lambda_1 = l_1^2, \qquad \lambda_2 = l_2(l_2+1), \tag{3.0.20}$$

where for given $l_2 = 0, 1, 2, \ldots, l_1 = l_2, l_2-1, \ldots, l_2+1, -l_2$. From these expressions it appears that λ_k will be of the form

$$\lambda_k = l_k(l_k+k-1), \tag{3.0.21}$$

where l_k is an integer.

This result can in fact be proved by induction. If we assume that

$$\lambda_{k-1} = l_{k-1}(l_{k-1}+k-2), \tag{3.0.22}$$

then constructing by analogy with the three-dimensional case the operators

$$L_k^+(l_{k-1}) = \partial/\partial\theta_k - l_{k-1}\cot\theta_k, \qquad L_k^-(l_{k-1}) = -\partial/\partial\theta_k - (l_{k-1}+k-2)\cot\theta_k, \tag{3.0.23}$$

and performing some operator manipulations we can indeed show

$$\lambda_k = l_k(l_k+k-1), \tag{3.0.24}$$

where for given $l_k = 0, 1, 2, \ldots, l_{k-1} = 0, 1, 2, \ldots, l_k$. Since (3.0.22) is already known to be true for $k=3$, it should also be true for $k=4$ and when it is true for $k=4$ it is also true for $k=5$ and similarly the argument continues. Labelling the eigenfunction belonging to the eigenvalue λ_k by the quantum number l_k we finally have

$$L_{N-1}^2 Y_{l_{N-1},l_{N-2},\ldots,l_2,l_1}(\theta_1,\theta_2,\ldots,\theta_{N-1}) = l_{N-1}(l_{N-1}+N-2)Y_{l_{N-1},l_{N-2},\ldots,l_2,l_1}(\theta_1,\theta_2,\ldots,\theta_{N-1}), \tag{3.0.25}$$

where $Y_{l_{N-1},l_{N-2},\ldots,l_2,l_1}(\theta_1,\theta_2,\ldots,\theta_{N-1})$ is known as the generalized spherical harmonics and $l_{N-1} = 0, 1, 2, \ldots; l_{N-2} = 0, 1, 2, \ldots, l_{N-1}; l_{N-3} = 0, 1, 2, \ldots, l_{N-2}; \ldots; l_3 = 0, 1, 2, \ldots l_4; l_2 = 0, 1, 2, \ldots l_3; l_1 = -l_2, -l_2+1, \ldots, l_2-1, l_2$. In future we shall always use $L_{N-1}^2 \equiv L^2$ and $l_{N-1} \equiv l$ for notational convenience.

Now in (3.0.1) we substitute

$$\Psi(r) = R(r)Y_{l_{N-1},l_{N-2},\ldots,l_2,l_1}(\theta_1,\theta_2,\ldots,\theta_{N-1}), \tag{3.0.26}$$

and using (3.0.14) and (3.0.25) we obtain the radial part of the Schrödinger equation

$$\left[-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{l(l+N-2)}{2r^2} + V_N(r)\right]R(r) = ER(r). \tag{3.0.27}$$

To eliminate the first derivative we make the substitution

$$u(r) = r^{(N-1)/2}R(r)\,, \tag{3.0.28}$$

so that (3.0.27) reduces to

$$-\tfrac{1}{2}\mathrm{d}^2u(r)/\mathrm{d}r^2 + [(k-1)(k-3)/8r^2 + V_N(r)]u(r) = Eu(r)\,, \tag{3.0.29}$$

where $k = N + 2l$. Equation (3.0.29) is the starting point for $1/N$ expansions.

3.1. The large-N limit

After we have obtained the radial part of the N-dimensional Schrödinger equation the next step is to make a proper choice for the large-N limit of the potential $V_N(r)$. The usual prescription is either to rescale the variable r or to redefine the coupling parameter appearing in $V_N(r)$ in such a way that $V_N(r)$ survives with its dominant features in the large-N limit. We shall follow the latter procedure, and so we write eq. (3.0.29) in the form

$$-\tfrac{1}{2}\mathrm{d}^2u/\mathrm{d}r^2 + k^2[(1-1/k)(1-3/k)/8r^2 + \tilde{V}(r)]u = Eu\,, \tag{3.1.1}$$

where $\tilde{V}(r) = V_N(r)/k^2$. The k^2 appearing in the denominator of $\tilde{V}(r)$ is usually absorbed in the redefinition of the coupling parameter and the strategy is to keep this rescaled coupling constant fixed when N goes to ∞; at the end of the calculation, however, everything is written in terms of the original parameter. In the large-N limit $(1-1/k)(1-3/k)$ is of the order of unity and eq. (3.1.1) simplifies to

$$[-(1/2k^2)\mathrm{d}^2u/\mathrm{d}r^2 + V_{\mathrm{eff}}(r)]u(r) = (E/k^2)u(r)\,, \tag{3.1.2}$$

where

$$V_{\mathrm{eff}}(r) = 1/8r^2 + \tilde{V}(r)\,. \tag{3.1.3}$$

Equation (3.1.3) describes the motion of a particle with an effective mass k^2 in an effective potential $V_{\mathrm{eff}}(r)$. Therefore when $k \to \infty$, the effective mass of the particle becomes infinitely large and then the particle may be assumed to remain essentially stationary at the absolute minimum of $V_{\mathrm{eff}}(r)$. Quantum fluctuations are unimportant in this limit and the most significant contribution to the ground state (GS) energy is given by

$$E_\infty = k^2 V_{\mathrm{eff}}(r_0)\,, \tag{3.1.4}$$

where r_0 is to be obtained from

$$\partial V_{\mathrm{eff}}(r)/\partial r|_{r=r_0} = 0\,, \tag{3.1.5}$$

i.e. r_0 gives the position where the wavefunction has a δ-function peak in the large-N limit. Obviously the method fails if the effective potential does not possess a minimum or if it has degenerate minima. The point that may be emphasized here is that the large-N limit as obtained above is essentially a classical limit. Clearly the expression (3.1.4) corresponds to the energy of a classical particle of angular momentum $L = k/2$ executing a circular motion in a potential $V_N(r)$.

An interesting point is that the large-N energy E_∞ provides quite encouraging results in a number of problems even for $N=3$. For the harmonic oscillator $V_N(r)=\frac{1}{2}\omega^2 r^2$, (3.1.4) yields

$$E_\infty = \tfrac{3}{2}\omega\,, \tag{3.1.6}$$

which is the exact GS energy. If one calculates the higher-order corrections in $1/N$ (by methods to be discussed in section 3.2) one is faced with a surprise – all the higher-order corrections turn out to be identically zero. To understand this interesting behaviour in a simple-minded way [63] let us consider the classical energy expression for a particle moving in a potential $V_N(r)$ in N-dimensions

$$E=\sum_{i=1}^{N}\tfrac{1}{2}p_i^2+V_N\Big[\Big(\sum_{i=1}^{N}x_i^2\Big)^{1/2}\Big],\quad m=1\,. \tag{3.1.7}$$

If we now impose the restriction that the uncertainty product of the system is minimum i.e.

$$\Delta x_i\,\Delta p_i=\tfrac{1}{2}\,,\quad \hbar=1\,, \tag{3.1.8}$$

and assume that

$$\Delta x_i\approx x_i\,,\quad \Delta p_i\approx p_i\,, \tag{3.1.9}$$

then we can write

$$E\simeq\sum_{i=1}^{N}\frac{1}{8x_i^2}+V_N\Big[\Big(\sum_{i=1}^{N}x_i^2\Big)^{1/2}\Big]. \tag{3.1.10}$$

If we further assume that the spreads in the position coordinates along different axes are also of the same order, then we obtain

$$E\simeq\tfrac{1}{8}N^2r^{-2}+V_N(r)\,, \tag{3.1.11}$$

which should be minimized with respect to r to give the GS energy thus becoming identical to (3.1.4) for $l=0$.

Thus the large-N ground state energy is obtainable from the classical energy expression when the quantum condition of minimum uncertainty is imposed on it. That the large-N energy E_∞ is exact for the harmonic oscillator GS is therefore not merely coincidental but it is an immediate consequence of the fact that the harmonic oscillator GS wavefunction is a gaussian wave for which the uncertainty is indeed a minimum. Obviously then for most systems the large-N limit alone will not suffice and one will have to incorporate the finite-N corrections to take account of the deviation from the minimum uncertainty. Furthermore, since the minimum uncertainty states are the well-known coherent states, which correspond essentially to classical states, the present derivation does also point to the semiclassical nature of the large-N approximation.

3.2. $1/N$ corrections

3.2.1. The perturbed oscillator methods

The finite-N corrections to the large-N limit (3.1.4) may be obtained by incorporating in the theory the quantum fluctuations around the classical minimum. Let us define the fluctuation by

$$x = (\sqrt{k}/r_0)(r - r_0)\,, \tag{3.2.1}$$

where r_0 is to be obtained from (3.1.5), which gives

$$4r_0^3\tilde{V}'(r_0) = 1\,, \tag{3.2.2}$$

where the prime refers to differentiation with respect to r. The effect of putting the factor k in the numerator of (3.2.1) is two-fold. First, it takes care of the fact that at large k the particle remains in the immediate vicinity of the classical minimum and second, since the domain of x is $-\sqrt{k} \le x < \infty$, for infinitely large k the problem now boils down to that of solving a one-dimensional Schrödinger equation.

Expanding the r-dependent terms in (3.1.1) in a Taylor series around $x = 0$ (i.e. $r = r_0$) and making use of (3.2.2), we obtain, after some rearrangement of terms, a one-dimensional perturbed harmonic oscillator equation of the form

$$[H_0 + \hat{V}(x)]\Psi(x) = \lambda\Psi(x)\,, \tag{3.2.3}$$

where H_0 is a harmonic oscillator hamiltonian given by

$$H_0 = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + \tfrac{1}{2}\omega^2x^2 + \varepsilon_0\,, \tag{3.2.4}$$

with

$$\omega = [\tfrac{3}{4} + r_0^4\tilde{V}''(r_0)]^{1/2}\,, \qquad \varepsilon_0 = \tfrac{1}{8}k - \tfrac{1}{2} + 3/8k + r_0^2k\tilde{V}(r_0)\,. \tag{3.2.5a, b}$$

$\hat{V}(x)$ is the perturbation, which takes the form

$$\begin{aligned}\hat{V}(x) = {} & (1/\sqrt{k})(\varepsilon_1x + \varepsilon_3x^3) + (1/k)(\varepsilon_2x^2 + \varepsilon_4x^4) + (1/k^{3/2})(\delta_1x + \delta_3x^3 + \delta_5x^5) \\ & + (1/k^2)(\delta_2x^2 + \delta_4x^4 + \delta_6x^6) + \ldots\,,\end{aligned} \tag{3.2.6}$$

with

$$\varepsilon_1 = 1\,, \qquad \varepsilon_2 = -\tfrac{3}{2}\,, \qquad \varepsilon_3 = \tfrac{1}{6}r_0^5\tilde{V}'''(r_0) - \tfrac{1}{2}\,, \qquad \varepsilon_4 = \tfrac{1}{24}r_0^6\tilde{V}^{\mathrm{IV}}(r_0) + \tfrac{5}{8}\,, \tag{3.2.7a}$$

$$\begin{aligned}&\delta_1 = -\tfrac{3}{4}\,, \qquad \delta_2 = \tfrac{9}{8}\,, \qquad \delta_3 = 2\,, \qquad \delta_4 = -\tfrac{5}{2}\,, \\ &\delta_5 = \tfrac{1}{120}r_0^7\tilde{V}^{\mathrm{V}}(r_0) - \tfrac{3}{4}\,, \qquad \delta_6 = \tfrac{1}{720}r_0^8\tilde{V}^{\mathrm{VI}}(r_0) + \tfrac{7}{8}\,.\end{aligned} \tag{3.2.7b}$$

The eigenvalue λ in (3.2.3) is related to the original energy E by

$$\lambda = Er_0^2/k\,. \tag{3.2.8}$$

One can now apply the known perturbative methods to (3.2.3) to obtain the energy eigenvalues and the corresponding eigenfunctions and rearrange the final results in powers of $1/k$.

In a perturbation theory one introduces for the nth excited state the expansions

$$\Psi_n = \Psi_n^{(0)} + \Psi_n^{(1)} + \Psi_n^{(2)} + \Psi_n^{(3)} + \ldots , \tag{3.2.9}$$

$$\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)} + \lambda_n^{(2)} + \lambda_n^{(3)} + \ldots , \tag{3.2.10}$$

where an mth order correction contains the mth power of $\hat{V}(x)$, and $\Psi_n^{(0)}$ and $\lambda_n^{(0)}$, the unperturbed wavefunction and eigenvalue, respectively, are given by

$$\Psi_n^{(0)} = (\omega/2\pi)^{1/4}\, \mathrm{e}^{-(\omega/4)x^2}\, H_n(x) , \tag{3.2.11}$$

$$\lambda_n^{(0)} = \varepsilon_0 + (n + \tfrac{1}{2})\omega , \tag{3.2.12}$$

where $H_n(x)$ is a Hermite polynomial $[H_0(x) = 1]$.

Since (3.2.3) is a one-dimensional equation, it is convenient to use the logarithmic perturbation theory which gives, for the GS energy corrections [161],

$$\lambda_0^{(1)} = \int_{-\infty}^{\infty} \hat{V}(x)|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \qquad \lambda_0^{(2)} = \int_{-\infty}^{\infty} C_1^2(x)|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \tag{3.2.13a, b}$$

$$\lambda_0^{(3)} = -\int_{-\infty}^{\infty} 2C_1(x)C_2(x)|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \tag{3.2.13c}$$

$$\lambda_0^{(4)} = -\frac{1}{4}\int_{-\infty}^{\infty} [2C_1(x)C_3(x) + C_2^2(x)]|\Psi_0^{(0)}(x)|^2 , \tag{3.2.13d}$$

etc., where $C_1, C_2, \ldots$ are to be determined from

$$C_1(x)|\Psi_0^{(0)}(x)|^2 = 2\int_{-\infty}^{x} [\hat{V}(x) - \lambda_0^{(1)})|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \tag{3.2.14a}$$

$$C_2(x)|\Psi_0^{(0)}(x)|^2 = -\int [2\lambda_0^{(2)} + C_1^2(x)]|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \tag{3.2.14b}$$

$$C_3(x)|\Psi_0^{(0)}(x)|^2 = -\int_{-\infty}^{x} [2\lambda_0^{(3)} + 2C_1(x)C_2(x)]|\Psi_0^{(0)}(x)|^2\, \mathrm{d}x , \tag{3.2.14c}$$

etc. The GS wavefunction is given by

$$\Psi_0(x) = \exp\Big(\sum_{i=1}^{\infty} S_i(x)\Big)\Psi_0^{(0)}(x) , \tag{3.2.15}$$

where

$$S_i(x) = \int^x C_i(x)\,\mathrm{d}x\,, \quad i = 1, 2, 3, \ldots . \tag{3.2.16}$$

The explicit expressions for (3.2.13), (3.2.14) and (3.2.15) are given in ref. [161].

For excited states (ES) however the logarithmic perturbation theory requires solving partial differential equations. In that case the standard Rayleigh–Schrödinger perturbation theory (RSPT) may prove to be a more economical tool. The RSPT expressions for the energy and the wavefunction to all orders in the perturbation potential are [162]

$$\Psi_n \propto \sum_{m=0}^{\infty} \left(\frac{R}{(\lambda_n^{(0)} - H_0)} [\lambda_n^{(0)} - \lambda_n + \hat{V}(x)] \right)^m \Psi_n^{(0)}\,, \tag{3.2.17}$$

$$\lambda_n = \lambda_n^{(0)} + \sum_{m=0}^{\infty} \langle \Psi_n^{(0)} | \hat{V}(x) \left(\frac{R}{(\lambda_0^{(0)} - H_0)} [\lambda_n^{(0)} - \lambda_n + \hat{V}(x)] \right)^m | \Psi_n^{(0)} \rangle\,, \tag{3.2.18}$$

where R is a projection operator defined by

$$R\Phi = \Phi - \Psi_n^{(0)} \langle \Psi_n^{(0)} | \Phi \rangle\,, \tag{3.2.19}$$

Φ being any function of the same variables as $\Psi_n^{(0)}$. Writing (3.2.18) in the form (3.2.10) one can obtain the standard textbook expressions for the perturbative corrections. Calculating up to $\lambda_n^{(4)}$ and rearranging the terms in powers of $1/k$, one gets the expression for the energy spectrum to order $1/k$

$$\begin{aligned}
E_{nl} = {} & (N+2l)^2[1/8r_0^2 + \tilde{V}(r_0)] + (N+2l)[(n+\tfrac{1}{2})\omega/r_0^2 - 1/2r_0^2] \\
& + \{(1/r_0^2)[\tfrac{3}{8} + (1+2n)\tilde{\varepsilon}_2 + 3(1+2n+2n^2)\tilde{\varepsilon}_4] \\
& - (1/\omega r_0^2)[\tilde{\varepsilon}_1^2 + 6(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_3 + (11+30n+30n^2)\tilde{\varepsilon}_3^2]\} \\
& + [1/(N+2l)]\{(1/r_0^2)[(1+2n)\tilde{\delta}_2 + 3(1+2n+2n^2)\tilde{\delta}_4 + 5(3+8n+6n^2+4n^3)\tilde{\delta}_6] \\
& - (1/\omega r_0^2)[(1+2n)\tilde{\varepsilon}_2^2 + 12(1+2n+2n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4 + 2(21+59n+51n^2+34n^3)\tilde{\varepsilon}_4^2 \\
& + 2(11+30n+30n^2)\tilde{\varepsilon}_3\tilde{\delta}_3 + 10(13+40n+42n^2+28n^3)\tilde{\varepsilon}_3\tilde{\delta}_3] \\
& - (1/\omega^2 r_0^2)[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2 + 36(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3 + 8(11+30n+30n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2 + 24(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4 \\
& + 8(31+78n+78n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4 + 12(57+189n+225n^2+150n^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4] \\
& - (1/\omega^3 r_0^2)[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3 + 108(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2 + 48(11+30n+30n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3 \\
& + 30(31+109n+141n^2+94n^3)\tilde{\varepsilon}_3^4]\} + \mathrm{O}[1/(N+2l)^2]\,,
\end{aligned} \tag{3.2.20}$$

where

$$\tilde{\varepsilon}_j = \varepsilon_j/(2\omega)^{j/2}\,, \qquad \tilde{\delta}_j = \delta_j/(2\omega)^{j/2}\,. \tag{3.2.21}$$

To obtain the realistic three dimensional results one now puts $N = 3$ in (3.2.20).

The method outlined above can be readily applied to any spherically symmetric potential. For the power-law potential $V(r) = Ar^{\nu}$, (3.2.20) yields

$$E_{nl} = (N+2l)^{(\nu-2)/(\nu+2)}(4\nu A)^{2/(\nu+2)}\Big[(N+2l)\Big(\frac{1}{8}+\frac{1}{4\nu}\Big) + [\tfrac{1}{2}(n+\tfrac{1}{2})\sqrt{\nu+2} - \tfrac{1}{2}]$$
$$+ \frac{1}{N+2l}\Big(\frac{3}{8} - \frac{3(n+\frac{1}{2})}{\sqrt{\nu+2}} + \tfrac{1}{16}(\nu^2-8\nu+27)(n^2+n+\tfrac{1}{2}) - \frac{2}{\nu+2} - \frac{\nu-5}{\sqrt{\nu+2}}(n+\tfrac{1}{2})$$
$$- \tfrac{15}{144}(\nu-5)^2(n^2+n+\tfrac{11}{30})\Big) + \mathrm{O}\Big(\frac{1}{(N+2l)^2}\Big)\Big]. \tag{3.2.22}$$

For $\nu = 2$ and $A = \frac{1}{2}\omega_0$ (harmonic oscillator), only the first two terms of (3.2.22) survive and we get

$$E_{nl} = \tfrac{1}{2}\omega_0(4n+2l+N), \tag{3.2.23}$$

which is in fact the exact result.

For $\nu = -1$, $A = -e^2$ (Coulomb potential), (3.2.22) is an infinite series and is identical to the one obtained by expanding the exact N-dimensional Coulomb energy

$$E_{nl} = -e^4/2[n+l+\tfrac{1}{2}(N-1)]^2 \tag{3.2.24}$$

binomially in powers of $1/k$. For $N=3$ and $n=0=l$, the first three terms of the expansion (3.2.22) yield 88.8% of the exact GS energy and the first six terms give 99.3% of the exact result. For states with higher l-values the convergence is even better, as would be expected, since l appears in the denominator of the expansion parameter.

This method has also been used to obtain the energy eigenvalues of an attractive gaussian potential of the form $V(r) = Ae^{-\lambda r^2}$; interest in it was evoked by the work of Buck et al. [163] on the determination of α–α scattering phase shifts using a local gaussian potential as a model for the nucleus–nucleus interaction. Various methods, analytic [164–167] and numerical [168, 169] have been applied to study this problem. The perturbed oscillator method, though it works well for states with low values of the radial quantum number n, cannot claim to have such efficacy when n is large [72]. This also is not surprising; though in the large-k limit the wavefunction has a δ-function peak about the minimum of the effective potential (3.1.3), and the energy spectrum shrinks to the value (3.1.4), for finite k the wavefunction deviates more and more from r_0 as the radial quantum number increases. This makes the large-N approximation efficacious for low lying states only and thus makes it complementary to the WKB approach. This is evident from the energy expression (3.2.20); various powers of n appear only in the numerators and therefore the energy is expected to have poorer convergence for larger values of n.

An interesting variant of the perturbed oscillator method discussed above has been suggested by Ader [45]. In this method which is known as the moment method one writes $\hat{V}(x)$ given by (3.2.6) in a more compact form

$$V(x) = \sum_{n\geq 1} H_n k^{-n/2}, \tag{3.2.25}$$

where

$$H_n = B_{n,n+2}x^{n+2} + B_{n,n}x^n + B_{n,n-2}x^{n-2}\,, \tag{3.2.26}$$

with

$$B_{n,n+2} = \tfrac{1}{8}(-1)^n(n+3) + \tilde{V}^{(n+2)}(r_0)r_0^{n+4}/(n+2)!\,, \qquad B_{n,n} = \tfrac{1}{2}(-1)^{n+1}(n+1)\,, \tag{3.2.27a, b}$$

$$B_{n,n-2} = \tfrac{3}{8}(-1)^n(n-1)\,, \quad \text{for } n \geq 3\,, \qquad B_{n,n-2} = 0\,, \quad \text{for } n \leq 2\,, \tag{3.2.27c}$$

$$\tilde{V}^m(r_0) = \mathrm{d}^m\tilde{V}(r)/\mathrm{d}r^m|_{r=r_0}\,. \tag{3.2.27d}$$

Now one substitutes the expansions

$$\lambda_n = \lambda_n^{(0)} + \sum_{m\geq 1} \bar{\lambda}_n^m/k^{m/2}\,, \tag{3.2.28}$$

$$\Psi_n(x) = \Psi_n^{(0)}(x) + \sum_{m\geq 1} \Psi_n^{(m)}(x)/k^{m/2}\,, \tag{3.2.29}$$

in (3.2.3), which then reduces to

$$[H_0(x) - \lambda_n^{(0)}]\Psi_n^{(p)}(x) + \sum_{m=1}^{\infty} (H_m - \bar{\lambda}_n^{(m)})\Psi_n^{(p-m)}(x) = 0\,. \tag{3.2.30}$$

For the sake of simplicity we shall confine our discussion to the ground state only and use the notation

$$\Psi_0^{(p)}(x) \equiv \Psi^{(p)}(x)\,, \qquad \bar{\lambda}_0^{(m)} \equiv \bar{\lambda}^{(m)}\,. \tag{3.2.31}$$

Multiplying (3.2.30) from the left by $\Psi_n^{(0)*}(x)x^m$ and integrating over x, one gets

$$\sum_{q=1}^{p} (B_{q,q+2}Q_{q+2+m}^{p-q} + B_{q,q}Q_{q+m}^{p-q} + B_{q,q-2}Q_{q+m-2}^{p-q}) + m\omega Q_m^p - \tfrac{1}{2}m(m-1)Q_{m-2}^p - \sum_{q=1}^{p} \bar{\lambda}Q_m^{p-2} = 0\,, \tag{3.2.32}$$

where the definition

$$Q_m^q \equiv \langle \Psi^{(0)}|x^m|\Psi^{(q)}\rangle \tag{3.2.33}$$

has been used. Now putting $m = 0$ in (3.2.32) and using parity considerations we obtain

$$\bar{\lambda}^{(2m)} = \sum_{q=1}^{2m} (B_{q,q+2}Q_{q+2}^{2m-q} + B_{q,q}Q_q^{2m-q} + B_{q,q-2}Q_{q-2}^{2m-q})\,, \qquad \bar{\lambda}^{(2m+1)} = 0\,. \tag{3.2.34a, b}$$

The lowest order nonvanishing correction to $\lambda^{(0)}$ is

$$\bar{\lambda}^{(2)} = B_{13}Q_3^1 + B_{24}Q_4^0 + B_{11}Q_1^1 + B_{22}Q_2^0\,. \tag{3.2.35}$$

The elements of the matrix Q can be calculated in a recursive manner. Q^0_{2n} is given by

$$Q^{(0)}_{2n} = \pi^{-1/2}\Gamma(n+\tfrac{1}{2})\,. \tag{3.2.36}$$

All $Q^{(0)}_{2n+1}$ are zero from parity considerations. Putting $m=0$ in (3.2.32) we can determine all the Q^1_{2n+1}. For instance, $p=1$ leads to

$$Q^1_1 = -(1/\omega)(B_{13}Q^0_4 + B_{11}Q^0_2) \tag{3.2.37a}$$

and $p=2$ gives

$$Q^1_3 = (1/3\omega)(3Q^1_1 - B_{13}Q^0_6 - B_{11}Q^0_4)\,. \tag{3.2.37b}$$

Using (3.2.35)–(3.2.37) one can now calculate $\bar{\lambda}^{(2)}$. The higher order corrections can be calculated in a similar fashion. Generalization of the above analysis to excited states is quite straightforward, and the method can also be extended to obtain the wavefunctions. Ader [45] has applied this method to calculate the GS and the first ES energies and wavefunctions of the rotating harmonic oscillator, a linear plus Coulomb potential and a logarithmic potential. The results for the eigenvalues obtained from the first few terms of the $1/k$ expansion are in excellent agreement with the exact ones, but the eigenfunctions obtained in the lowest order approximation are not generally impressive.

3.2.2. Riccati equation methods

Hikami and Brezin [37] appear to have first suggested a Riccati equation method and later two variants of it were given by Mlodinow and Shatz [39] (MS). In these methods the form of the wavefunction has to be prescribed at the outset. Here we shall follow the versions of MS.

In their first method, MS have chosen for the (nodeless) GS wavefunction the form

$$u_0 = \exp[\Phi_0(x)]\,, \tag{3.2.38}$$

where $x = r - r_0$, r_0 being given by (3.2.2). Equation (3.1.1) then takes on the form of a Riccati equation

$$-\tfrac{1}{2}[\Phi_0''(x) + \Phi_0'(x)] + k^2\bar{V}_{\rm eff}(x) + (-\tfrac{1}{2}k + \tfrac{3}{8})r^{-2}(x) = \mathscr{E}_0\,, \tag{3.2.39}$$

where

$$\Phi_0'(x) = \mathrm{d}\Phi_0(x)/\mathrm{d}x\,, \qquad \Phi_0''(x) = \mathrm{d}^2\Phi_0(x)/\mathrm{d}x^2\,, \tag{3.2.40}$$

$$\bar{V}_{\rm eff}(x) = V_{\rm eff}(r(x)) - V_{\rm eff}(r_0)\,, \tag{3.2.41}$$

$$\mathscr{E}_0 = E_0 - E_\infty\,, \tag{3.2.42}$$

E_0 being the GS energy and $V_{\rm eff}(r)$ and E_∞ being given respectively by (3.1.3) and (3.1.4). In the next step one substitutes the expansions

$$\mathscr{E}_0 = \sum_{n=-1}^{\infty} E_0^{(n)} k^{-n}\,, \tag{3.2.43}$$

$$\Phi_0'(x) = \sum_{n=-1}^{\infty} \Phi_0^{(n)}(x) k^{-n}\,, \tag{3.2.44}$$

in (3.2.39) and collects the terms of the same order in k to generate the following recurrence relations

$$\Phi_0^{(-1)}(x) = -[2\bar{V}_{\text{eff}}(x)]^{1/2}\,, \tag{3.2.45a}$$

$$[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_0^{(0)}(x) = E_0^{(-1)} + \tfrac{1}{2}r^{-2}(x) + \tfrac{1}{2}\Phi_0^{(-1)\prime}(x)\,, \tag{3.2.45b}$$

$$[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_0^{(1)}(x) = E_0^{(0)} - \tfrac{3}{8}r^{-2}(x) + \tfrac{1}{2}[\Phi_0^{(0)\prime}(x) + \Phi_0^{(0)2}(x)]\,, \tag{3.2.45c}$$

$$[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_0^{(n)}(x) = E_0^{(n-1)} + \tfrac{1}{2}\Phi_0^{(n-1)\prime}(x) + \frac{1}{2}\sum_{m=0}^{n-1} \Phi_0^{(m)}(x)\Phi_0^{(n-m-1)}(x)\,, \quad n>1\,. \tag{3.2.45d}$$

To make the wavefunction normalizable one should take the positive square root of $\bar{V}_{\text{eff}}(x)$ in the region $x \geq 0$. At $x=0$, $\bar{V}_{\text{eff}}(x)$ vanishes and in the domain $-r_0 \leq x \leq 0$ the negative square root may be chosen, because in the limit $k \to \infty$ the wavefunction has a δ-function peak at $x=0$.

Since $\bar{V}_{\text{eff}}(x)$ vanishes at the minimum ($r=r_0$) and $\Phi_0^{(0)}(0) < \infty$, higher order corrections to the energy coefficients are as follows:

$$E_0^{(-1)} = -\tfrac{1}{2}r_0^{-2} - \tfrac{1}{2}\Phi_0^{(-1)\prime}(0)\,, \qquad E_0^{(0)} = \tfrac{3}{8}r_0^{-2} - \tfrac{1}{2}\Phi_0^{(0)\prime}(0) - \tfrac{1}{2}\Phi_0^{(0)2}(0)\,, \tag{3.2.46a, b}$$

$$E_0^{(n)} = -\tfrac{1}{2}\Phi_0^{(n)\prime}(0) - \frac{1}{2}\sum_{m=0}^{n} \Phi_0^{(m)}(0)\Phi_0^{(n-m)}(0)\,, \quad n>0\,. \tag{3.2.46c}$$

The calculation is performed in the following way. From the knowledge of $\bar{V}_{\text{eff}}(x)$ one first calculates $\Phi_0^{(-1)}(x)$ using (3.2.45a). Then one finds its derivative, calculates its value at $x=0$ and computes $E_0^{(-1)}$ from (3.2.46a). Next, substituting $E_0^{(-1)}$ and $\Phi_0^{(-1)}(x)$ in (3.2.45b), one determines $\Phi_0^{(0)}(x)$ and obtains the value of $E_0^{(0)}$ from (3.2.46b). The process continues similarly. Thus $E_0^{(n)}$ is to be calculated after $\Phi_0^{(n)}(x)$ but before $\Phi_0^{(n+1)}(x)$. The GS wavefunction is given by

$$u_0(r) = \exp\left(\int^x \sum_{n=-1}^{\infty} \Phi_0^{(n)}(y) k^{-n}\,\mathrm{d}y\right), \tag{3.2.47}$$

and the GS energy is given by

$$E_0 = \sum_{n=-2}^{\infty} E_0^{(n)} k^{-n}\,, \tag{3.2.48}$$

where the definition $E_\infty \equiv k^2 E^{(-2)}$ has been used. In fact the above equations hold for any state with a radial wavefunction which is nodeless.

For states with one node in their radial wavefunctions one may use the following prescription

$$u_1(r) = (r - C_1)\exp[\Phi_1(x)]\,. \tag{3.2.49}$$

As k increases, the wavefunction becomes more and more sharply peaked about r_0; therefore the following expansion may be used:

$$C_1 = r_0 + C_1^{(1)}k^{-1} + C_1^{(2)}k^{-2} + C_1^{(3)}k^{-3} + \cdots . \tag{3.2.50}$$

Equation (3.2.49) then reads

$$u_1(r) = \left(x - \sum_{n=1}^{\infty} C_1^{(n)}k^{-n}\right)\exp[\Phi_1(x)] , \tag{3.2.51}$$

and (3.1.1) reduces to

$$\begin{aligned} &-\tfrac{1}{2}[\Phi_1''(x) + \Phi_1'^2(x)]\left(x - \sum_{n=1}^{\infty} C_1^{(n)}k^{-n}\right) - \Phi_1'(x) \\ &\quad +[k^2\bar{V}_{\text{eff}}(x) + (-\tfrac{1}{2}k + \tfrac{3}{8})r^{-2}(x)]\left(x - \sum_{n=1}^{\infty} C_1^{(n)}k^{-n}\right) = \left(x - \sum_{n=1}^{\infty} C_1^{(n)}k^{-n}\right)\mathscr{E}_1 , \end{aligned} \tag{3.2.52}$$

where

$$\mathscr{E}_1 = E_1 - E_\infty , \tag{3.2.53}$$

E_1 being the exact energy of the state under consideration. Substituting, as before, the expansions

$$\mathscr{E}_1 = \sum_{n=-1}^{\infty} E_1^{(n)}k^{-n} , \tag{3.2.54}$$

$$\Phi_1'(x) = \sum_{n=-1}^{\infty} \Phi_1^{(n)}(x)k^{-n} , \tag{3.2.55}$$

in (3.2.52) and collecting the terms of the same order in k one obtains

$$\Phi_1^{(-1)}(x) = -[2\bar{V}_{\text{eff}}(x)]^{1/2} , \tag{3.2.56a}$$

$$x[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_1^{(0)}(x) = x[E_1^{(-1)} + \tfrac{1}{2}\Phi_1^{(-1)\prime}(x) + \tfrac{1}{2}r^{-2}(x)] + \Phi_1^{(-1)}(x) , \tag{3.2.56b}$$

$$\begin{aligned} x[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_1^{(1)}(x) = {} & x[E_1^{(0)} + \tfrac{1}{2}\Phi_1^{(0)\prime}(x) + \tfrac{1}{2}\Phi_1^{(0)2}(x) - \tfrac{3}{8}r^{-2}(x)] \\ & - C_1^{(1)}[E_1^{(-1)} + \tfrac{1}{2}\Phi_1^{(-1)\prime}(x) + \Phi_1^{(-1)}(x)\Phi_1^{0}(x) + \tfrac{1}{2}r^{-2}(x)] + \Phi_1^{(0)}(x) , \end{aligned} \tag{3.2.56c}$$

$$\begin{aligned} x[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_1^{(2)}(x) = {} & x[E_1^{(1)} + \tfrac{1}{2}\Phi_1^{(1)\prime}(x) + \Phi_1^{(0)}(x)\Phi_1^{(1)}(x)] \\ & - C_1^{(1)}[E_1^{(0)} + \tfrac{1}{2}\Phi_1^{(0)\prime}(x) + \Phi_1^{(1)}(x)\Phi_1^{(-1)}(x) + \tfrac{1}{2}\Phi_1^{(0)2}(x) - \tfrac{3}{8}r^{-2}(x)] \\ & - C_1^{(2)}[E_1^{(-1)} + \tfrac{1}{2}\Phi_1^{(-1)}(x) + \Phi_1^{(-1)}(x)\Phi_1^{(0)}(x) + \tfrac{1}{2}r^{-2}(x)] + \Phi_1^{(1)}(x) , \end{aligned} \tag{3.2.56d}$$

$$x[2\bar{V}_{\text{eff}}(x)]^{1/2}\Phi_1^{(n)}(x) = x\Big(E_1^{(n-1)} + \tfrac{1}{2}\phi_1^{(n-1)\prime}(x) + \frac{1}{2}\sum_{m=0}^{n-1}\Phi_1^{(m)}(x)\Phi_1^{(n-m-1)}(x)\Big)$$
$$- \sum_{m=1}^{n-2} C_1^{(m)}\Big(E_1^{(n-m-1)} + \tfrac{1}{2}\Phi_1^{(n-m-1)\prime}(x) + \frac{1}{2}\sum_{p=-1}^{n-m}\Phi_1^{(p)}(x)\Phi^{(n-m-p-1)}(x)\Big)$$
$$- C_1^{(n-1)}[E_1^{(0)} + \tfrac{1}{2}\Phi_1^{(0)\prime}(x) + \Phi_1^{(-1)}(x)\Phi_1^{(1)}(x) + \tfrac{1}{2}\Phi_1^{(0)2}(x) - \tfrac{3}{8}r^{-2}(x)]$$
$$- C_1^{(n)}[E_1^{(-1)} + \tfrac{1}{2}\Phi_1^{(1)\prime}(x) + \Phi_1^{(-1)}(x)\Phi_1^{(0)}(x) + \tfrac{1}{2}r^{-2}(x)]$$
$$+ \Phi_1^{(n-1)}(x), \quad n>2. \tag{3.2.56e}$$

The above equations are now to be solved to obtain the energy and the wavefunction coefficients. The calculation is now slightly more tedious because of the appearance of a set of new parameters, the $C_1^{(n)}$, which may be called the node coefficients. The calculation proceeds in the following manner. First, $\Phi_1^{(-1)}(x)$ is calculated from (3.2.56a). By differentiating (3.2.56b) with respect to x and setting $x = 0$ one obtains $E_1^{(-1)}$. Then, inserting the value of $E_1^{(-1)}$ in (3.2.56b), one can find $\Phi_1^{(0)}(x)$. Next one considers eq. (3.2.56c). Putting $x = 0$ in this equation one can evaluate $C_1^{(1)}$; differentiating this equation with respect to x and setting $x = 0$ one can compute the value of $E_1^{(0)}$. The wavefunction coefficient $\Phi_1^{(1)}(x)$ is then easily determined. The higher order coefficients are calculated in the same manner.

To obtain the energies for the states with two nodes in their radial functions one may choose

$$u_2(r) = \Big(x - \sum_{n=1}^{\infty} C_1^{(n)}k^{-n}\Big)\Big(x - \sum_{n=1}^{\infty} C_2^{(n)}k^{-n}\Big)\exp[\Phi_2(x)]. \tag{3.2.57}$$

In general, the choice

$$u_n(r) = \prod_{m=1}^{n}\Big(x - \sum_{j=1}^{\infty} C_m^{(j)}\Big)\exp[\Phi_n(x)] \tag{3.2.58}$$

would provide the results for the states with n radial nodes.

This method has been applied to a number of potentials [39, 50–55, 66]. For the harmonic oscillator the exact results are obtained and for the Coulomb potential the $1/N$ series are identical to the ones obtained by binomially expanding the exact N-dimensional Coulomb energies in powers of $1/N$. Let us quote, for example, the $1/N$-expansion result for the GS wavefunction of the Coulomb potential in N dimensions

$$u_0 \sim r^{(N-1)/2}\exp[-(2a/N)(1 + 1/N + 1/N^2 + 1/N^3 + \cdots)r]. \tag{3.2.59}$$

The exact GS wavefunction for this problem is

$$u_0 \sim r^{(N-1)/2}\exp\{-[2a/(N-1)]r\}, \tag{3.2.60}$$

which if expanded in powers of $1/N$ would reproduce exactly the $1/N$ expansion result (3.2.59).

The main advantage of the Riccati equation method outlined above is that in this scheme one ultimately has to solve a set of algebraic equations which are easy to handle. However, one has to

choose suitably the N-dimensional form of the potential under consideration. For example, in the case of the three-dimensional generalized exponential cosine-screened Coulomb potential (GECSCP)

$$V(r) = -(b/r)\exp(-\lambda r)\cos(\varepsilon\lambda r)\,, \tag{3.2.61}$$

a possible choice of the N-dimensional version would be [53]

$$V_N(r) = -(b/r)\exp\{-[(3+2l)^2/k^2]\lambda r\}\cos\{[(3+2l)/k]\varepsilon\lambda r\}\,, \tag{3.2.62}$$

which indeed goes over to (3.2.61) for $N=3$. This choice is by no means unique. Here it has been assumed that for an infinitely large value of N the screened Coulomb potential would behave essentially like a simple Coulomb potential. The advantage of this assumption is that analytic results are obtained for the eigenvalues and eigenfunctions.

For the Yukawa potential ($\varepsilon = 0$) $1/k$-expansion results for the GS energy were first obtained by Moreno and Zepeda [50] by the Riccati equation method. They also compared their results with the numerical ones obtained by them from the Runge–Kutta method. Agreement is good for small values of the screening parameter. The interesting point is that the $1/k$ series, when rearranged in powers of λ, leads exactly to the perturbative results [51–53]. Moreno and Zepeda have shown that for values of the ratio λ/b around 0.22 (or greater) the energies calculated from (3.2.48) show a rapid divergence as one goes on including more and more higher order corrections. However, it is observed [51] that when (3.2.48) is considered as a series in λ/b the situation slightly improves (at least for $\lambda/b = 0.22$). This shows that the $1/N$ expansion may at times be worse than the usual perturbation theory. Another disadvantage of the Riccati equation method in particular is that it becomes highly cumbersome even for states with one node and is probably intractable analytically when the radial quantum number is large.

For cases in which the large distance behaviour of the wavefunction can be sacrificed, MS have suggested a purely algebraic recursion method via the Riccati equation. In this method it is useful to introduce a variable ρ through the coordinate transformation

$$r = \sqrt{k\rho}\,. \tag{3.2.63}$$

Equation (3.1.1) then reduces to

$$-\frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} + k^2\left[V_{\mathrm{eff}}(\rho) + \left(-\frac{1}{2k}\rho^{-2} + \frac{3}{8k^2}\right)\right]u = Eku\,, \tag{3.2.64}$$

where

$$V_{\mathrm{eff}}(\rho) = 1/8\rho^2 + \tilde{V}(\rho)\,, \tag{3.2.65}$$

$$\tilde{V}(\rho) = V_N(r)/k\,. \tag{3.2.65a}$$

The leading-order contribution to the energy in the limit $k\to\infty$ is

$$E_\infty \equiv kE^{(-2)} = kV_{\mathrm{eff}}(\rho_0)\,, \tag{3.2.66}$$

where ρ_0 is obtained from

$$dV_{\text{eff}}(\rho)/d\rho\big|_{\rho=\rho_0} = 0\,. \tag{3.2.67}$$

For the GS one then writes

$$u_0(\rho) = \exp[\Phi_0(x)]\,, \tag{3.2.68}$$

where

$$x = \sqrt{k}(\rho - \rho_0)\,, \tag{3.2.69}$$

to obtain the Riccati equation

$$-\tfrac{1}{2}[\Phi_0''(x) + \Phi_0'(x)]^2 + k\bar{V}_{\text{eff}}(x) + (-\tfrac{1}{2} + \tfrac{3}{8}k^{-1})\rho^{-2}(x) = \mathscr{E}_0\,, \tag{3.2.70}$$

where

$$\bar{V}_{\text{eff}}(x) = V_{\text{eff}}(\rho) - V_{\text{eff}}(\rho_0)\,, \tag{3.2.71}$$

$$\mathscr{E}_0 = E_0 - E_\infty\,. \tag{3.2.72}$$

The potential

$$W(x) \equiv k\bar{V}_{\text{eff}}(x) + (-\tfrac{1}{2} + \tfrac{3}{8}k^{-1})\rho^{-2}(x) \tag{3.2.73}$$

can now be expanded in a Taylor series

$$W(x) = [W_0^{(0)} + W_1^{(0)}x^2] + [W_0^{(1)}x + W_1^{(1)}x^3]k^{-1/2} + \sum (W_0^{(n)}x^{n-2} + W_1^{(n)}x^n + W_2^{(n)}x^{n+2})k^{-n/2}\,, \tag{3.2.74}$$

where

$$\begin{aligned} &W_0^{(0)} = -1/2\rho^2\,, \qquad W_1^{(0)} = V_{\text{eff}}^{(2)}(\rho_0)/2!\,, \qquad W_0^{(1)} = 1/\rho^3\,, \qquad W_1^{(1)} = V_{\text{eff}}^{(3)}(\rho_0)/3!\,, \\ &W_0^{(n)} = (-1)^n 3(n-1)/8\rho_0^n\,, \qquad W_1^{(n)} = (-1)^{n+1}(n+1)/2\rho_0^{(n+2)}\,, \\ &W_2^{(n)} = V_{\text{eff}}^{(n+2)}(\rho_0)/(n+2)!\,, \end{aligned} \tag{3.2.75}$$

with

$$V_{\text{eff}}^{(m)}(\rho_0) \equiv d^m V_{\text{eff}}(\rho)/d\rho^m\big|_{\rho=\rho_0}\,. \tag{3.2.76}$$

Next one substitutes in (3.2.70) the expansions

$$\mathscr{E}_0 = \sum_{n=0}^{\infty} E_0^{(n-1)} k^{-n}, \tag{3.2.77}$$

$$\Phi_0(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left(\frac{D_m^{(n)}}{2m} x^{2m} + \frac{C_m^{(n)} k^{-1/2} x^{2m+1}}{2m+1} \right) k^{-n}, \tag{3.2.78}$$

where

$$D_0^{(n)} = 0, \tag{3.2.79}$$

and one equates the terms of the same orders in k on both sides of the resulting equation to obtain a set of algebraic relations. In each relation terms with like powers in x are then collected, to give finally the energy and the wavefunction coefficients. For example, to order k^0 one gets

$$D_1^{(0)} = -(2W_1^{(0)})^{1/2}, \tag{3.2.80}$$

$$E_0^{(-1)} = -\tfrac{1}{2} D_1^{(0)} + W^{(0)}, \tag{3.2.81}$$

and to order $k^{-1/2}$ the results are

$$C_1^{(0)} = W_1^{(1)}/D_1^{(0)}, \qquad C_0^{(0)} = -(1/D_1^{(0)})(C_1^{(0)} - W_0^{(1)}). \tag{3.2.82}$$

For the excited states the wavefunctions have to be suitably chosen so that the proper node structures are guaranteed. For instance, in the case of the first excited state (one node) one writes

$$u_1(\rho) = \left(\frac{x}{\sqrt{k}} - \sum_{n=1}^{\infty} C_1^{(n)} k^{-n} \right) \exp[\Phi_1(x)]. \tag{3.2.83}$$

The only difference that the purely algebraic recursion method has from the one discussed earlier in this subsection is that it uses a rescaled coordinate. This scaling effects a reorganization of the $1/k$ series for the wavefunction with the energy series remaining unaltered. Because of the scaling the energy calculation becomes easier, but the price paid is that the knowledge of the large distance behaviour of the wavefunction is sacrificed. MS have employed this method to determine the eigenvalues and the nodes of the low-lying states of power-law potentials. The results are, in general, impressive. For highly excited states, however, a Shanks transformation is useful to give better convergence.

This method can also be applied to problems which do not have spherical symmetry. For instance, for a problem with cylindrical symmetry $\Phi(x)$ has to be replaced by $\Phi(x, z)$ and eq. (3.2.39) then reads

$$-\tfrac{1}{2}(\Phi_{0,xx} + \Phi_{0,x}^2 + \Phi_{0,zz} + \Phi_{0,z}^2) + W = \varepsilon, \tag{3.2.84}$$

where W now looks like

$$W(x, z) = (W_{00}^{(0)} + W_{20}^{(0)} x^2 + W_{02}^{(0)} z^2) + (W_{30}^{(1)} x^3 + W_{21}^{(1)} x^2 z + W_{12}^{(1)} x z^2 + W_{03}^{(1)} z^3) k^{-1/2} + \cdots, \tag{3.2.85}$$

and the subscripts x, z, etc., denote the variables with respect to which differentiations have to be performed. Bender, Mlodinow and Papanicolaou [88] have used this method to study the problem of a hydrogen atom in a magnetic field.

4. Other methods and some of their applications

4.1. The hypervirial $1/N$ expansion

In this method [65] the hypervirial perturbation theory [170] is generalized to N dimensions. The hypervirial theorem states that

$$\langle [r^j \, \mathrm{d}/\mathrm{d}r, H] \rangle = 0 \,, \tag{4.1.1}$$

where $j \geq 0$ and the averaging states are the eigenstates of the hamiltonian H. For a particle moving in an N-dimensional potential $V_N(r)$, H may be written as before

$$H = -\tfrac{1}{2} \, \mathrm{d}^2/\mathrm{d}r^2 + k^2[(1-1/k)(1-3/k)/8k^2 + \tilde{V}(r)] \,. \tag{4.1.2}$$

The N-dimensional hypervirial theorem then reads

$$E\langle r^j \rangle = k^2 \langle r^j \tilde{V}(r) \rangle + [k^2/2(j+1)] \langle r^{j+1} \, \mathrm{d}\tilde{V}/\mathrm{d}r \rangle - \{ j[j^2 - (k-2)^2]/8(j+1) \} \langle r^{j-2} \rangle \,. \tag{4.1.3}$$

To concretize the problem let us consider a class of potentials whose N-dimensional versions can be expanded in the form

$$V_N(r) = \sum_{n=0}^{\infty} V_n \bar{\lambda}^n r^{n-1} \,, \tag{4.1.4}$$

where V_n may depend on k and is a negative quantity for $n = 0$ and $\bar{\lambda} = k^2\lambda$, λ being a perturbation parameter contained in the given potential. Equation (4.1.4) encompasses a variety of potentials. For example, it represents the GECSCP

$$V_N(r) = (b/r) \exp(-\lambda r) \cos(\varepsilon \lambda r) = -(b/r) \exp[-(\bar{\lambda}/k^2) r] \cos(\bar{\lambda}/k) \bar{\varepsilon} \lambda] \,, \tag{4.1.5}$$

where $\bar{\varepsilon} = \varepsilon/k$, if

$$\begin{aligned} V_n &= b \sum_{m=0}^{n/2} (-1)^{n+m+1} \frac{\bar{\varepsilon}^{2m}}{(n-2m)!(2m)! k^{2n-2m}} \,, \quad \text{for } n \text{ even} \,, \\ V_n &= b \sum_{m=0}^{(n-1)/2} (-1)^{n+m+1} \frac{\bar{\varepsilon}^{2m}}{(n-2m)!(2m)! k^{2n-2m}} \,, \quad \text{for } n \text{ odd} \,, \end{aligned} \tag{4.1.6}$$

and the Hellman potential

$$V_N(r) = -a/r + (b/r) \exp(-\lambda r) = -a/r + (b/r) \exp[-(\tilde{\lambda}/k^2) r] \,, \tag{4.1.7}$$

if

$$V_n = (-1)^n b/n! k^{2n}, \quad n > 0, \qquad V_0 = -a + b. \tag{4.1.8}$$

Substituting (4.1.4) in (4.1.3) now gives

$$(E - V_1\bar{\lambda})\langle r^j\rangle = \frac{(2j+1)}{2(j+1)} V_0 \langle r^{j-1}\rangle - \tfrac{1}{8} j(j+1)^{-1}[j^2 - (k-2)^2]\langle r^{j-2}\rangle + \sum_{m=2}^{\infty} V_n \bar{\lambda}^n \frac{(2j+n+1)}{2(j+1)} \langle r^{j+n-1}\rangle . \tag{4.1.9}$$

The next step is to introduce the perturbative expansions

$$\langle r^j\rangle = \sum_{n'=0}^{\infty} C_j^{(n')} \bar{\lambda}^{n'}, \tag{4.1.10}$$

and

$$E_n = \sum_{n'=0}^{\infty} E_n^{(n')} \bar{\lambda}^{n'}, \qquad E_n^{(0)} = -\frac{2V_0^2}{(N+2n-3)^2}, \quad n = 1, 2, 3, \ldots, \tag{4.1.11a, b}$$

so that (4.1.9) becomes

$$\sum_{\substack{n'=0\\ n''=0}}^{\infty} E_n^{(n')} C_j^{(n'')} \bar{\lambda}^{(n'+n'')} - V_1 \sum_{n'=0}^{\infty} C_j^{(n')} \bar{\lambda}^{(n'+1)} = \frac{(2j+1)}{2(j+1)} V_0 \sum_{n'=0}^{\infty} C_{j-1}^{(n')} \bar{\lambda}^{n'} - \tfrac{1}{8} j(j+1)^{-1}[j^2 - (k-2)^2] \sum_{n'=0}^{\infty} C_{j-2}^{(n')} \bar{\lambda}^{n'} + \sum_{n'=2}^{\infty} \sum_{n''=0}^{\infty} \frac{(2j+n'+1)}{2(j+1)} V_{n'} C_{j+n'-1}^{(n'')} \bar{\lambda}^{(n'+n'')}. \tag{4.1.12}$$

Collecting the coefficients of $\bar{\lambda}^0$ in (4.1.12) yields

$$C_j^{(0)} = \frac{1}{E_n^{(0)}} \left(\frac{(2j+1)}{2(j+1)} V_0 C_{j-1}^{(0)} - \frac{j[j^2 - (k-2)^2]}{8(j+1)} C_{j-2}^{(0)} \right). \tag{4.1.13}$$

From (4.1.10) it follows that

$$C_0^{(m)} = \delta_{0m}. \tag{4.1.14}$$

Setting $j = 0$ in (4.1.13) and using (4.1.14) for $m = 0$, we then get

$$C_{-1}^{(0)} = 2E_n^{(0)}/V_0. \tag{4.1.15}$$

The values of $C_{-1}^{(0)}$ and $C_0^{(0)}$ having been obtained, the recurrence relation (4.1.13) can be used to

generate

$$C_1^{(0)} = \tfrac{3}{4}V_0/E_n^{(0)} + (k^2 - 4k + 3)/8V_0\,, \tag{4.1.16}$$

$$C_2^{(0)} = \tfrac{5}{8}(V_0/E_n^{(0)})^2 + (3k^2 - 12k + 5)/16E_n^{(0)}\,, \tag{4.1.17}$$

$$C_3^{(0)} = \frac{35}{64}\left(\frac{V_0}{E_n^{(0)}}\right)^3 + \frac{V_0(30k^2 - 120k - 10)}{128(E_n^{(0)})^2} + \frac{3(k^4 - 8k^3 + 14k^2 + 8k - 15)}{256E_n^{(0)}V_0}\,, \tag{4.1.18}$$

etc. Next we use the Hellman–Feynman theorem,

$$\langle \mathrm{d}H/\mathrm{d}\bar{\lambda}\rangle = \mathrm{d}E/\mathrm{d}\bar{\lambda}\,, \tag{4.1.19}$$

to obtain

$$pE_n^{(p)} = \sum_{q=1}^{p} qV_q C_{q-1}^{(p-q)}\,. \tag{4.1.20}$$

Then collecting the coefficients of $\bar{\lambda}^1$ in (4.1.12) and using (4.1.20) we obtain,

$$C_j^{(1)} = \frac{1}{E_n^{(0)}}\left(\frac{(2j+1)}{2(j+1)}V_0 C_{j-1}^{(1)} - \frac{j[j^2 - (k-2)^2]}{8(j+1)}C_{j-2}^{(1)}\right), \tag{4.1.21}$$

which immediately leads to

$$C_p^{(1)} = 0\,, \quad p \geq -1\,. \tag{4.1.22}$$

Similarly collecting the coefficients of $\bar{\lambda}^2$, $\bar{\lambda}^3$, $\bar{\lambda}^4$, etc. in (4.1.12) and using (4.1.20) we can find, respectively, the values of $C_j^{(2)}$, $C_j^{(3)}$, $C_j^{(4)}$ etc. for $j \geq -1$. Using (4.1.11) and (4.1.20) we then obtain the bound state energy spectrum as a power series in $\bar{\lambda}$ which is finally reorganised as a series in $1/k$. This method has been applied to obtain the energy eigenvalues of the GECSCP (eq. 4.1.5). The results are exactly the same as those given by the Riccati equation method [53]. But the present method has one distinct advantage in that it provides the entire energy spectrum in a much simpler way.

As (4.1.11b) indicates, the approach described above treats the potential as a perturbation of a Coulomb potential. As we shall show in section 5.1, it is also possible to base the hypervivial perturbation approach on an unperturbed oscillator potential.

4.2. The pseudospin formulation

The pseudospin formulation of the $1/N$ expansion suggested by Jevicki and Papanicolaou [41] (JP), with a view to advocating the special classical nature of the large-N limit, has some similarity with the coherent state approach of Berezin [40] and with Anderson's pseudo-spin formulation [148] of BCS superconductivity. JP have considered examples of both a Fermi system and a Bose system. These ideas have subsequently been applied to general one-particle quantum mechanics [42] and more complicated

quantal systems by Papanicolaou and collaborators [88, 89, 115, 116, 137]. The advantage of this approach over the effective action method [113] lies in its applicability to planar theories. We shall not, however, touch on this aspect here.

4.2.1. Fermi system: the BCS problem

As an example of a Fermi system JP have considered a modified version of the BCS [41] reduced hamiltonian

$$H = \sum_{i=1}^{N} \sum_{k\sigma} \varepsilon(k) a^{i+}_{k\sigma} a^{i}_{k\sigma} - \sum_{i,j=1}^{N} \sum_{kk'} V_{kk'} a^{i+}_{k'\sigma} a^{i+}_{-k',-\sigma} a^{j}_{-k,-\sigma} a^{j}_{k\sigma} , \tag{4.2.1}$$

where the index i which runs from 1 to N is an additional degree of freedom introduced in the problem and may be called the electron flavour, $\varepsilon(k)$ is the Bloch energy measured from the Fermi level, and a^{+} and a are, respectively, the creation and the annihilation operators satisfying the usual anticommutation relations

$$\{a^{i}_{k\sigma}, a^{j+}_{k'\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}\delta_{ij} , \qquad \{a^{i}_{k\sigma}, a^{j}_{k'\sigma'}\} = \{a^{i+}_{k\sigma}, a^{j+}_{k'\sigma'}\} = 0 . \tag{4.2.2}$$

$V_{kk'}$ is the effective electron–electron interaction which contains both the virtual phonon-mediated attractive interaction and the repulsive screened Coulomb interaction. Following Anderson [148] JP have constructed the bilinear operators

$$\hat{S}_{-}(k) = \sum_{i=1}^{N} a^{i}_{-k,-\sigma} a^{i}_{k\sigma} , \qquad \hat{S}_{+}(k) = \sum_{i=1}^{N} a^{i+}_{k\sigma} a^{i+}_{-k,-\sigma} , \tag{4.2.3a, b}$$

$$\hat{S}_{3}(k) = \frac{1}{2} \sum_{i=1}^{N} (a^{i+}_{k\sigma} a_{k\sigma} - a^{i}_{-k,-\sigma} a^{i+}_{-k,-\sigma}) , \tag{4.2.3c}$$

which are O(N) invariant in the flavour space and close the spin algebra,

$$[\hat{S}_{-}(k), \hat{S}_{+}(k')] = -2\delta_{kk'}\hat{S}_{3}(k) , \tag{4.2.4a}$$

$$[\hat{S}_{3}(k), \hat{S}_{+}(k')] = \delta_{kk'}\hat{S}_{+}(k) , \qquad [\hat{S}_{3}(k), \hat{S}_{-}(k')] = -\delta_{kk'}\hat{S}_{-}(k) . \tag{4.2.4b, c}$$

It is useful to define the real generators

$$\hat{S}_{1}(k) = \tfrac{1}{2}[\hat{S}_{+}(k) + \hat{S}_{-}(k)] , \qquad \hat{S}_{2}(k) = \tfrac{1}{2}[\hat{S}_{+}(k) - \hat{S}_{-}(k)]/\mathrm{i} . \tag{4.2.5a, b}$$

In terms of the O(N) invariant pseudo-spin generators (4.2.3) the hamiltonian (4.2.1) reads

$$H = \sum_{k} 2\varepsilon(k)(\hat{S}_{3}(k) + \tfrac{1}{2}N) - \sum_{kk'} V_{kk'}\hat{S}_{+}(k)\hat{S}_{-}(k') . \tag{4.2.6}$$

It is then easy to verify that the Casimir operators of the ΠSU(2) algebra (4.2.4),

$$\hat{C}(k) = \sum_{i=1}^{3} \hat{S}_{i}^{2}(k) = \hat{S}_{3}^{2}(k) + \tfrac{1}{2}(\hat{S}_{-}(k)\hat{S}_{+}(k) + \hat{S}_{+}(k)\hat{S}_{-}(k)) , \tag{4.2.7}$$

commute with the hamiltonian (4.2.6) and are constants of motion. One can decompose the original Fock space corresponding to the Fermi operators of the BCS hamiltonian into irreducible subspaces in accordance with the representations of the pseudo-spin algebra (4.2.4) and then study the dynamics of the problem within each subspace.

For the sake of simplicity and without any loss of rigour JP have restricted their discussion to the irreducible subspace which contains Cooper pairs only. The values of the Casimir invariants (4.2.7) on this subspace can be obtained by simply operating with them on the Fock vacuum. This immediately gives

$$\hat{C}(\boldsymbol{k}) = \tfrac{1}{2}N(\tfrac{1}{2}N + 1)\,, \tag{4.2.8}$$

for any $\boldsymbol{k}$. The original BCS problem is thus mapped onto a momentum space Heisenberg model with the spin strength $s = N/2$. The path integral representation of this problem is well-known; readers are referred to the articles of Jevicki and Papanicolaou [171] and Klauder [172]. The transition amplitude reads

$$Z = \int \prod_{k'} DS(\boldsymbol{k}) \prod_{k,t} \delta[S^2(\boldsymbol{k}, t) - (\tfrac{1}{2}N)^2] \exp\Big(\mathrm{i} \int_{t_i}^{t_f} \mathscr{L}(S, \dot{S})\, \mathrm{d}t\Big)\,, \tag{4.2.9}$$

where $S(\boldsymbol{k}, t)$ are c-number spin functions, t_i and t_f are respectively the initial and the final times and $\mathscr{L}$ is the classical lagrangian given by

$$\mathscr{L} = \sum_{k} \frac{S_2\dot{S}_1 - S_1\dot{S}_2}{S_3 + \frac{1}{2}N} - H(S)\,. \tag{4.2.10}$$

Various other forms of classical spin lagrangian also exist. Well known among them is the Doering–Gilbert lagrangian [171]

$$\mathscr{L} = S_3(S_1\dot{S}_2 - S_2\dot{S}_1)/[(\tfrac{1}{2}N)^2 - S_3^2] - H(S)\,, \tag{4.2.11}$$

which differs from the form (4.2.10) by a total time derivative. If one now makes a scaling transformation $S \to NS$ and $V_{kk'} \to V_{kk'}/N$, the lagrangian (4.2.10) transforms to

$$\mathscr{L} = N\Big(\sum_{k} \frac{S_2\dot{S}_1 - S_1\dot{S}_2}{S_3 + \frac{1}{2}N} - \frac{1}{N} H(NS)\Big)\,, \tag{4.2.12}$$

in which N^{-1} plays the same role as $\hbar$. This shows that the limit $N \to \infty$ is in effect equivalent to the stationary phase approximation. In this approximation one is then led to the following c-number equations:

$$\mathrm{i}\dot{S}_-(\boldsymbol{k}, t) = 2[\varepsilon(\boldsymbol{k})S_-(\boldsymbol{k}) + \sigma_k S_3(\boldsymbol{k})]\,, \tag{4.2.13}$$

$$\mathrm{i}\dot{S}_3(\boldsymbol{k}, t) = \sigma_k^* S_-(\boldsymbol{k}) - S_+(\boldsymbol{k})\sigma_k\,, \tag{4.2.14}$$

where the definition

$$\sigma_k \equiv \sum_{k'} V_{kk'} S_-(k') \tag{4.2.15}$$

has been used and the c-number Casimir invariant is given by

$$C(k) = S_3^2(k) + S_+(k)S_-(k) = (\tfrac{1}{2}N)^2 . \tag{4.2.16}$$

The interesting point to note is that equations (4.2.13) and (4.2.14) follow quite directly from the hamiltonian (4.2.6) if the Heisenberg equation of motion is used. Accordingly we conclude: the large-N limit is governed by the c-number solutions of the original equations of motion (4.2.13, 4.2.14) subject to the quantum Casimir constraint (4.2.16). (The word "quantum" indicates the fact that if the Planck constant $\hbar$ were retained in the above analysis it would have appeared in the constraint equation.)

The GS of the system will be determined by the static solutions of (4.2.13, 4.2.14). In the static field case eq. (4.2.13) simplifies to

$$\varepsilon(k)S_-(k) + \sigma_k S_3(k) = 0 , \tag{4.2.17}$$

which when solved in conjunction with (4.2.16) leads to

$$S_-(k) = \tfrac{1}{2}N\sigma_k/[\varepsilon^2(k) + \sigma_k\sigma_k^*]^{1/2} . \tag{4.2.18}$$

Multiplying both sides of (4.2.18) by $V_{kk'}$ and summing over k' yields, taking account of eq. (4.2.15), the integral equation

$$\sigma_k = \frac{N}{2}\sum_{k'} \frac{V_{kk'}\sigma_{k'}}{[\varepsilon^2(k) + \sigma_{k'}\sigma_{k'}^*]^{1/2}} , \tag{4.2.19}$$

which is the famous BCS gap equation.

To get the excited states however, one has to consider the time-dependent solutions of the pseudo-spin equations.

4.2.2. Bose system: the anharmonic oscillator

To illustrate the pseudospin method in a bosonic system [41], JP have considered the case of an N-dimensional anharmonic oscillator described by the hamiltonian (in units $m = \hbar = 1$)

$$H = \tfrac{1}{2}(\hat{p}^2 + \omega_0\hat{r}^2) + g^2\hat{r}^4 , \tag{4.2.20}$$

where, as defined earlier,

$$\hat{p}^2 = \sum_{i=1}^{N} p_i p_i , \qquad \hat{r}^2 = \sum_{i=1}^{N} x_i x_i . \tag{4.2.21}$$

One now defines the creation and annihilation operators in the usual manner

$$a_i = (\hat{p}_i/\sqrt{2\omega_0}) - \mathrm{i}\sqrt{\tfrac{1}{2}\omega_0}x_i , \qquad a_i^+ = (\hat{p}_i/\sqrt{2\omega_0}) + \mathrm{i}\sqrt{\tfrac{1}{2}\omega_0}x_i , \tag{4.2.22}$$

and constructs the O(N) invariant bilinear operators

$$\hat{K} = \frac{1}{2}\sum_{i=1}^{N} a_i a_i\,, \qquad \hat{K}^+ = \frac{1}{2}\sum_{i=1}^{N} a_i^+ a_i^+\,, \qquad \hat{K}_0 = \frac{1}{4}\sum_{i=1}^{N}(a_i^+ a_i + a_i a_i^+)\,, \tag{4.2.23}$$

which close an algebra isomorphic to SU(1, 1)

$$[\hat{K}, \hat{K}^+] = 2\hat{K}_0\,, \qquad [\hat{K}_0, \hat{K}] = -\hat{K}\,, \qquad [\hat{K}_0, \hat{K}^+] = \hat{K}^+\,. \tag{4.2.24}$$

The Casimir invariant C can be expressed as

$$\hat{C} \equiv \hat{K}_0^2 - \tfrac{1}{2}(\hat{K}\hat{K}^+ + \hat{K}^+\hat{K}) = \tfrac{1}{4}\hat{L}^2 + \tfrac{1}{4}N(\tfrac{1}{4}N - 1)\,, \tag{4.2.25}$$

where $\hat{L}^2$ is the N-dimensional angular momentum operator defined by (3.0.15) for $k = N-1$. Substituting for $\hat{L}^2$ its eigenvalue then leads to the relations

$$C = k'(k'-1)\,, \quad k' \equiv \tfrac{1}{4}(N+2l)\,. \tag{4.2.26}$$

In terms of the pseudo-spin generators the hamiltonian becomes

$$H = \omega_0^2 \hat{K}_0 + g^2[(1/\omega_0)(\hat{K} + \hat{K}^+ - 2\hat{K}_0)]^2\,. \tag{4.2.27}$$

One may also define hermitian generators

$$\hat{K}_1 = \tfrac{1}{2}(\hat{K} + \hat{K}^+)\,, \qquad \hat{K}_2 = (1/2\mathrm{i})(\hat{K}^+ - \hat{K})\,, \quad [\hat{K}_\alpha, \hat{K}_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma}\hat{K}^\gamma\,, \tag{4.2.28}$$

where

$$\hat{K}_\alpha = g_{\alpha\beta}\hat{K}^\beta\,. \tag{4.2.29}$$

The metric tensor $g_{\alpha\beta}$ is given by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.2.30}$$

and $\varepsilon_{\alpha\beta\gamma}$ is the usual antisymmetric tensor satisfying $\varepsilon_{\alpha\beta\gamma} = -\varepsilon^{\alpha\beta\gamma}$. The Casimir invariant can then be expressed as

$$\hat{C} = \hat{K}_0^2 - \hat{K}_1^2 - \hat{K}_2^2 = \hat{K}^\alpha \hat{K}_\alpha\,. \tag{4.2.31}$$

One may note that the original Fock space F can again be decomposed into irreducible subspaces $F_{k'}$ according to the irreducible representations of the pseudo-spin algebra, each subspace $F_{k'}$ spanning all states corresponding to a given angular momentum l. The transition amplitude can be written in analogy to the spin problem

$$Z = \int \prod_{\alpha} DK_{\alpha} \prod_{t} \delta(K^{\alpha}K_{\alpha}(t) - k'^{2}) \exp\Big(\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L}(K, \dot{K})\Big), \tag{4.2.32}$$

where the lagrangian is given by

$$\mathscr{L}(K, \dot{K}) = (K_1\dot{K}_2 - K_2\dot{K}_1)/(K_0 + k') - H(K). \tag{4.2.33}$$

Making as before the scaling transformations,

$$K_{\alpha} = k'\Lambda_{\alpha}, \qquad \Lambda_{\alpha}\Lambda^{\alpha} = 1, \qquad [\Lambda_{\alpha}, \Lambda_{\beta}] = (\mathrm{i}/k')\varepsilon_{\alpha\beta\gamma}\Lambda^{\gamma}, \tag{4.2.34}$$

which are consistent with the SU(1, 1) algebra, one obtains for the action

$$\int_{t_i}^{t_f} \mathrm{d}t\, \mathscr{L} = k' \int_{t_i}^{t_f} \mathrm{d}t \Big(\frac{\Lambda_1\Lambda_2 - \Lambda_2\Lambda_1}{\Lambda_0 + 1} - \frac{1}{k'} H(k'\Lambda) \Big), \tag{4.2.35}$$

where k' appears in front of the action and so the limit $k' \to \infty$ is effectively a stationary phase approximation. Extremising the classical action (4.2.33) subject to the "quantum" constraint for the c-number Casimir invariant,

$$K^{\alpha}K_{\alpha} = K_0^2 - K_1^2 - K_2^2 = k'^2, \tag{4.2.36}$$

then leads to the equations for the K which can also be obtained from the original classical equations of motion

$$\ddot{\boldsymbol{r}} + (\omega_0^2 + 4g^2r^2)\boldsymbol{r} = 0. \tag{4.2.37}$$

The c-number Casimir invariant (4.2.36) can be shown to be related to the classical invariants p^2, r^2 and $\boldsymbol{r} \cdot \boldsymbol{p}$, and the classical angular momentum $\boldsymbol{L}$ by the expression

$$L^2 \equiv p^2r^2 - (\boldsymbol{p} \cdot \boldsymbol{r})^2 = 4k'^2. \tag{4.2.38}$$

Thus we are again led to the conclusion that the large-N limit is given by the original classical equations of motion subject to some algebraic constraint. Essentially the same features emerge if one considers an N-component complex scalar field theory [41], which we will discuss in 4.3.5 in the context of the collective field method.

In the above two subsections (4.2.1 and 4.2.2) we have restricted our discussion to highlighting the classical nature of the large-N limit using the pseudo-spin approach. Mlodinow and Papanicolaou [42] (MP) have extended the pseudo-spin formulation of JP to obtain the bound state energy spectrum for a spherically symmetric potential and also to deal with the Zeeman problem. In the subsection immediately following we present the work of MP for the spherically symmetric potentials, leaving the Zeeman problem for a later discussion. In 4.2.4 we shall discuss the application of this method to a lattice field theory.

4.2.3. General O(N) symmetric potentials: bound state energy spectrum

Let us consider an N-dimensional spherically symmetric potential problem [42] described by the hamiltonian (in units $m = \hbar = 1$)

$$H = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega_0^2\hat{r}^2 + V(\hat{r}^2)\,. \tag{4.2.39}$$

The creation and annihilation operators are now defined as in (4.2.22) with ω_0 replaced by ω, a new parameter to be chosen later. The pseudo-spin generators and the Casimir invariant are again given by (4.2.23) and (4.2.26) respectively. The hamiltonian reads

$$H = \omega(1 + \omega_0^2/\omega^2)K_0 + \tfrac{1}{2}\omega(1 - \omega_0^2/\omega^2)(K + K^+) + V((1/\omega)(2K_0 - K + K^+))\,. \tag{4.2.40}$$

To obtain the eigenvalues of this hamiltonian one should now eliminate the angular degrees of freedom. As mentioned earlier, the original Fock space F can be decomposed into irreducible subspaces $F_{k',m}$ where m is the azimuthal quantum number. The angular coordinates may be integrated out by restricting the hamiltonian to the subspace $F_{k',m}$. This is accomplished by representing the pseudo-spin generators in terms of Heisenberg operators

$$K_0 = k' + \zeta^+\zeta\,, \qquad K = (2k' + \zeta^+\zeta)^{1/2}\zeta\,, \qquad K^+ = \zeta^+(2k' + \zeta^+\zeta)^{1/2}\,, \qquad [\zeta^+, \zeta] = 1\,. \tag{4.2.41}$$

One may verify that the above transformations preserve the pseudospin algebra and lead to the same Casimir invariant $k'(k'-1)$. (4.2.41) may be called a generalized Holstein–Primakoff representation, in analogy to the well-known Holstein–Primakoff transformation used in the compact SU(2) probem.

To concretize the problem MP have considered

$$V(r^2) = 2\nu e^2 r^{2\nu}\,, \tag{4.2.42}$$

where ν may be positive or negative. The potential $(\frac{1}{2}\omega_0^2 r^2 + 2\nu e^2 r^{2\nu})$ is, however, always attractive. Using the transformation (4.2.41) one then obtains for the present problem

$$H_k = \omega(1 + \omega_0^2/\omega^2)(k' + \zeta^+\zeta) + \tfrac{1}{2}\omega(1 - \omega_0^2/\omega^2)[(2k' + \zeta^+\zeta)^{1/2}\zeta + \zeta^+(2k' + \zeta^+\zeta)^{1/2}]$$
$$+ (2\nu e^2/\omega^\nu)[2k' + 2\zeta^+\zeta - (2k' + \zeta^+\zeta)^{1/2}\zeta - \zeta^+(2k' + \zeta^+\zeta)^{1/2}]^\nu\,. \tag{4.2.43}$$

So far the analysis has been exact. One may notice that in (4.2.43) k' appears as a free parameter and so it may be used as an expansion parameter. MP now introduce the parameter

$$\alpha = 1/(2k')^{1/2} = 1/[(N + 2l)/2]^{1/2} \tag{4.2.44a}$$

and a rescaled coupling constant g^2 (to be treated as k'-independent)

$$e^2 = g^2\alpha^{2(\nu-1)}\,. \tag{4.2.44b}$$

Subsequently they substitute these in (4.2.43) and perform the canonical transformations

$$\zeta = \xi + (\sinh\phi)/\alpha\,, \qquad [\xi, \xi^+] = 1\,, \tag{4.2.45}$$

which are obviously consistent with the pseudo-spin algebra. Finally the hamiltonian is expanded in powers of α. Setting the coefficients of $(\xi^+ + \xi)$ and $(\xi^{+2} + \xi^2)$ to order α^0 equal to zero one now obtains

$$\mu^2 - \omega_0^2 = 4\nu^2 g^2 u^{1-\nu}\,, \qquad \cosh^2\phi \equiv \Omega = [1 + \tfrac{1}{2}(\nu-1)(\mu^2 - \omega_0^2)/\mu^2]^{1/2}\,, \quad \mu = \omega\, e^{2\phi}\,, \tag{4.2.46}$$

which will fix the values of ω and ϕ and may be called the constraint equations. The hamiltonian finally reduces to

$$H_k = h^{(0)} + \alpha h^{(1)} + \alpha^2 h^{(2)}\,, \tag{4.2.47}$$

where $h^{(0)}$, $h^{(1)}$ and $h^{(2)}$ are expressed as follows:

$$h^{(0)} = (1/\alpha^2)T_1 + \alpha^0(T_2\xi^+\xi + T_3)\,, \tag{4.2.48}$$

$$T_1 = \mu(1 + \nu - \Omega^2)/\nu\,, \qquad T_2 = \mu(2\cosh^2\phi)\,, \qquad T_3 = \mu\sinh^2\phi\,,$$
$$h^{(1)} = \mu[A_1(\xi^{+3} + \xi^3) + A_2(\xi^+\xi^2 + \xi^{+2}\xi) + A_3(\xi^+ + \xi)]\,, \tag{4.2.49}$$

$$A_1 = \frac{\tanh^3\phi}{4\cosh\phi} + \frac{\mu^2 - \omega_0^2}{\mu^2}\,\frac{\nu-1}{4\cosh\phi}\left(\tfrac{1}{2}\tanh\phi\,(2 - \tanh^2\phi)\,e^{2\phi} - \frac{\nu-2}{3\cosh^2\phi}\right), \tag{4.2.49a}$$

$$A_2 = \frac{\tanh\phi\,(3\tanh^2\phi - 4)}{4\cosh\phi} + \frac{\mu^2 - \omega_0^2}{\mu^2}\,\frac{\nu-1}{4\cosh\phi}\left((5\tanh\phi - \tfrac{3}{2}\tanh^3\phi - 4)\,e^{2\phi} - \frac{\nu-2}{\cosh^2\phi}\right), \tag{4.2.49b}$$

$$A_3 = 3A_1 - \tfrac{1}{2}(\nu-1)[(\mu^2 - \omega_0^2)/\mu^2]\,e^{2\phi}/\cosh\phi\,, \tag{4.2.49c}$$

$$h^{(2)} = \mu\{V_4 + \tfrac{1}{4}(\nu-1)[(\mu^2 - \omega_0^2)/\mu^2]$$
$$\times [V_2^2 + V_1V_3 + V_3V_1 + \tfrac{1}{3}(\nu-2)(V_1V_2V_1 + V_1^2V_2 + V_2V_1^2) + \tfrac{1}{12}(\nu-2)(\nu-3)V_1^4]\}\,, \tag{4.2.50}$$

$$V_1 = -(\xi + \xi^+)/\cosh\phi\,, \tag{4.2.50a}$$

$$V_2 = e^{2\phi}[(2 - \tanh\phi)\,\xi^+\xi - \tfrac{1}{4}\tanh\phi\,(2 - \tanh^2\phi)\,(\xi + \xi^+)^2 + \tfrac{1}{2}\tanh\phi]\,, \tag{4.2.50b}$$

$$V_3 = (e^{2\phi}/4\cosh^3\phi)[\tfrac{1}{2}\tanh^2\phi\,(\xi + \xi^+)^3 - (\xi + \xi^+)\xi^+\xi - \xi^+\xi(\xi + \xi^+) + (\xi + \xi^+)]\,, \tag{4.2.50c}$$

$$V_4 = \tfrac{1}{4}\tanh^2\phi\,\{\tfrac{1}{2}\tanh^2\phi\,(\tfrac{5}{4}\tanh^2\phi - 1)(\xi + \xi^+)^4 + 2(\xi^+\xi)^2 - 2\xi^+\xi$$
$$+ (\tfrac{3}{2}\tanh^2\phi - 1)[(\xi + \xi^+)^2 - 2(\xi + \xi^+)\xi^+\xi(\xi + \xi^+)] + 2\tanh^2\phi - 1\}\,. \tag{4.2.50d}$$

It is clear that $h^{(0)}$ is diagonal in the harmonic oscillator basis. MP have considered $h^{(0)}$ as the unperturbed hamiltonian and have treated $\alpha h^{(1)}$ plus $\alpha^2 h^{(2)}$ as the perturbation. Since $h^{(1)}$ is an odd function of ξ and ξ^+, the first nonvanishing contribution from $\alpha h^{(1)}$ comes from the second order of perturbation theory and is thus of order α^2. So even for the lowest order correction to the energy corresponding to $h^{(0)}$ one will have to consider both $\alpha h^{(1)}$ and $\alpha^2 h^{(2)}$, the latter's contribution being of order α^2 in the first order perturbation theory. Thus the energy to order α^2 in each angular momentum sector reads

$$E_{nl} = \mu((1/\alpha^2)\varepsilon_n^{(-1)} + \varepsilon_n^{(0)} + \alpha^2 \varepsilon_n^{(1)}) , \tag{4.2.51}$$

where

$$\varepsilon_n^{(-1)} = (1 + \nu - \Omega^2)/\nu , \qquad \varepsilon_n^{(0)} = (2n+1)\Omega - 1 , \tag{4.2.51a, b}$$

$$\varepsilon_n^{(1)} = \frac{1}{\mu} \sum_m{}' \frac{|\langle n|h^{(1)}|m\rangle|^2}{E_n^{(0)} - E_m^{(0)}} + \frac{1}{\mu} \langle n|h^{(2)}|n\rangle , \tag{4.2.51c}$$

$|n\rangle$ being an eigenstate of the unperturbed hamiltonian $h^{(0)}$ belonging to the eigenvalue $E_n^{(0)} = \mu[(\varepsilon_n^{(-1)}/\alpha^2) + \varepsilon_n^{(0)}]$. For simplicity, let us consider $n = 0$. One then obtains

$$\begin{aligned} E_{k',0} = \mu[&(1/\alpha^2)(1 + \nu - \Omega^2)/\nu + (\Omega - 1) \\ &+ \alpha^2\{[(\Omega^2 - 1)/72\Omega^4][9(\Omega - 1)(\Omega - 11) - 6(\nu - 2)(\Omega^2 + 6\Omega - 11) \\ &- 11(\nu - 2)^2(\Omega^2 - 1) + 9(\nu - 2)(\nu - 3)\Omega^2]\}] . \end{aligned} \tag{4.2.52}$$

Special cases

(i) The anharmonic oscillator ($\nu = 2$). The GS ($n = l = 0$) energy now reads

$$E_0 = \mu\{\tfrac{1}{4}(3 - \Omega^2)N + (\Omega - 1) + [(\Omega^2 - 1)/4N\Omega^4](\Omega - 1)(\Omega - 11)\} , \tag{4.2.53}$$

where the parameters μ and Ω are to be determined from the equations

$$\mu^2 - \omega_0^2 = 16g^2/\mu , \qquad \Omega = [1 + (\mu^2 - \omega_0^2)/2\mu^2]^{1/2} = (1 + 8g^2/\mu^3)^{1/2} . \tag{4.2.54}$$

To make a comparison with the work of Ferrel and Scalapino [35] one should set $g^2 = 1/32$. Equation (4.2.53) then agrees completely with the result of Ferrel and Scalapino.

(ii) Power-law potentials ($\omega_0 = 0$). In this case the parameters μ and Ω are given by

$$\mu = (4\nu^2 g^2)^{1/(1+\nu)} , \qquad \Omega = [(1 + \nu)/2]^{1/2} , \tag{4.2.55}$$

and (4.2.52) yields

$$\begin{aligned} E_{k',0} = \left(\frac{4\nu^2 e^2}{(2k')^{1-\nu}}\right)^{1/(1+\nu)} &\left\{\frac{1+\nu}{\nu} k' + \left[\left(\frac{1+\nu}{2}\right)^{1/2} - 1\right]\right. \\ &\left. + \frac{1}{2k'} \frac{1-\nu}{1+\nu}\left[\left(\frac{1+\nu}{2}\right)^{1/2} + \frac{2\nu^2 - 15\nu - 53}{72}\right] + O(k'^{-2})\right\} , \end{aligned} \tag{4.2.56}$$

which is in complete agreement with the expansion (3.2.30) obtained from the perturbed oscillator method.

4.2.4. $O(N)$ ϕ^4 field theory on a lattice

The pseudo-spin formulation has been employed by Mead and Papanicolaou [116] (MEP) to study the elementary excitations of a lattice field theory specified by the O(N) invariant hamiltonian in d-dimensions

$$H = \frac{1}{2}\sum_n\Big(\sum_j \pi_n^j\pi_n^j + \tfrac{1}{2}\kappa^2\sum_n\sum_j\sum_e(\phi_n^j - \phi_{n+e}^j)(\phi_n^j - \phi_{n+e}^j) + V\Big(\sum_j \phi_n^j\phi_n^j\Big)\Big)\,, \tag{4.2.57}$$

where j runs from 1 to N, n is the site index, e extends over neighbouring sites, π_j^n is conjugate to ϕ_j^n and $V(\Sigma_j \phi_n^j\phi_n^j)$ is a single site anharmonic potential. In what follows we shall consider $d=1$ (anharmonic chain).

The creation and annihilation operators are first introduced through the prescription

$$\begin{aligned}\phi_n^j &= \sum_p (2\Lambda\Omega_p)^{-1/2}(a_p^j\,\mathrm{e}^{\mathrm{i}pn} + a_p^{j+}\,\mathrm{e}^{-\mathrm{i}pn})\,,\\ \pi_n^j &= -\mathrm{i}\sum_p\left(\frac{\Omega_p}{2\Lambda}\right)^{1/2}(a_p^j\,\mathrm{e}^{\mathrm{i}pn} - a_p^{j+}\,\mathrm{e}^{-\mathrm{i}pn})\,,\end{aligned} \tag{4.2.58}$$

where

$$p = (2\pi/\Lambda)n\,,\quad n = 0, \pm1, \ldots, \pm K\,,\qquad \Lambda = 2K+1\,. \tag{4.2.59}$$

Λ is the total number of sites in the periodic lattice and is assumed to be odd. Ω_p is unknown a priori and will be chosen later. It will be however assumed that $\Omega_p = \Omega_{-p}$. Defining as before the rotationally invariant operators

$$A_{pq} = \frac{1}{4}\sum_{j=1}^N (a_p^{j+}a_q^j + a_q^j a_p^{j+})\,,\qquad B_{pq} = \frac{1}{2}\sum_{j=1}^N a_p^j a_q^j\,,\qquad B_{pq}^+ = \frac{1}{2}\sum_{j=1}^N a_p^{j+}a_q^{j+}\,, \tag{4.2.60}$$

which close the real symplectic group [Sp(2Λ, R)] algebra, we can rewrite (4.2.57).

$$\begin{aligned}H = \sum_p\Big(&\frac{1}{\Omega_p}(\omega_p^2 + \Omega_p^2)A_{pp} + \frac{1}{2\Omega_p}(\omega_p^2 - \Omega_p^2)(B_{p,-p} + B_{p,-p}^+)\Big)\\ &+ \frac{1}{2}\sum_n V\Big(\frac{1}{\Lambda}\sum_{pq}(\Omega_p\Omega_q)^{1/2}(2A_{pq}\,\mathrm{e}^{-\mathrm{i}(p-q)n} + B_{pq}\,\mathrm{e}^{\mathrm{i}(p+q)n} + B_{pq}^+\,\mathrm{e}^{-\mathrm{i}(p+q)n})\Big)\,,\end{aligned} \tag{4.2.61}$$

where

$$\omega_p = 2\kappa\,|\sin(p/2)|\,. \tag{4.2.62}$$

Here A is a $\Lambda\times\Lambda$ hermitian matrix and B and B^+ are symmetric.

Restriction of the hamiltonian (4.2.61) to the singlet sector may be accomplished by the generalized Holstein–Primakoff transformations

$$A = \tfrac{1}{4}NI + \zeta^+\zeta\,, \qquad B = [\tfrac{1}{2}NI + (\zeta^+\zeta)^{\mathrm{T}}]^{1/2}\zeta\,, \qquad B^+ = \zeta^+[\tfrac{1}{2}NI + (\zeta^+\zeta)^{\mathrm{T}}]^{1/2}\,, \tag{4.2.63}$$

where I is the $\Lambda \times \Lambda$ unit matrix, $\zeta = (\zeta_{pq})$ is a Λ-dimensional symmetric matrix whose components satisfy the commutation relation

$$[\zeta_{pq}, \zeta^+_{kl}] = \tfrac{1}{2}(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk})\,, \tag{4.2.64}$$

and $(\zeta^+\zeta)^{\mathrm{T}}$ is the transpose of $(\zeta^+\zeta)$ and is given by

$$(\zeta^+\zeta)^{\mathrm{T}} = \zeta\zeta^+ - \tfrac{1}{2}(\Lambda + 1)I\,, \tag{4.2.65}$$

which shows that for $\Lambda = 1$, (4.2.63) reduces to (4.2.41). The diagonalization of $H(\zeta, \zeta^+)$ would now provide the singlet spectrum of the original problem.

At this point MEP have turned to the large-N approximation. The idea is to choose Ω_p so that $\zeta = 0 = \zeta^+$ becomes the stationary point of the hamiltonian $H(\zeta, \zeta^+)$. Setting $\zeta = 0 = \zeta^+$ in $H(\zeta, \zeta^+)$ gives

$$E \equiv H(\zeta = 0, \zeta^+ = 0) = \sum_p \frac{N}{4\Omega_p}(\omega_p^2 + \Omega_p^2) + \frac{\Lambda}{2}V(\sigma^2)\,, \tag{4.2.66}$$

where the definition

$$\sigma^2 = \frac{N}{2\Lambda}\sum_p \frac{1}{\Omega_p} \tag{4.2.67}$$

has been used. Minimization of E with respect to Ω_p then yields

$$\Omega_p = [\omega_p^2 + V'(\sigma^2)]^{1/2} = [4\kappa^2 \sin^2(p/2) + V'(\sigma^2)]^{1/2}\,, \tag{4.2.68}$$

where $V'(x) = \mathrm{d}V/\mathrm{d}x$ and σ^2 satisfies the gap equation

$$\sigma^2 = \frac{N}{2\Lambda}\sum_p [4\kappa^2 \sin^2(p/2) + V'(\sigma^2)]^{-1/2}\,. \tag{4.2.69}$$

Expression (4.2.66) with σ^2 and Ω_p obtained from (4.2.69) and (4.2.68), respectively, provides the large-N GS energy, E_∞.

Now expanding $H(\zeta, \zeta^+)$ in a Taylor series and keeping terms up to quadratic in ζ and ζ^+ one obtains

$$H = E_\infty + H_0 + \cdots\,, \tag{4.2.70}$$

where E_∞ is the large-N GS energy discussed above and

$$H_0 = \sum_{pq} 2\Omega_p \zeta^+_{pq}\zeta_{pq} + \tfrac{1}{8}NV''(\sigma^2)\sum_n \left(\frac{1}{\Lambda}\sum_{pq}(\Omega_p\Omega_q)^{-1/2}(\zeta_{pq}\,\mathrm{e}^{\mathrm{i}(p+q)n} + \zeta^+_{pq}\,\mathrm{e}^{-\mathrm{i}(p+q)n})\right). \tag{4.2.71}$$

That $\zeta = 0 = \zeta^+$ is indeed a stationary point of $H(\zeta, \zeta^+)$ is clearly evident from the absence of terms

linear in ζ and ζ^+ from (4.2.71). Higher order $1/N$ corrections may be included by following a procedure parallel to that in section 4.2.3. We are interested here in the collective excitations associated with the quadratic hamiltonian (4.2.70).

MEP first considered the simple example of a harmonic potential

$$V\left(\sum_{j=1}^{N} \phi_n^j \phi_n^j\right) = m^2 \sum_{j=1}^{N} \phi_n^j \phi_n^j \,. \tag{4.2.72}$$

The function $V(x)$ and its derivatives appearing in eqs. (4.2.66)–(4.2.69) are then given by $V(x) = m^2 x$, $V'(x) = m^2$, $V''(x) = 0$. Consequently, the second term of eq. (4.2.71) vanishes and we get

$$H_0 = \sum_{pq} 2\Omega_p \zeta_{pq}^+ \zeta_{pq} \,, \tag{4.2.73}$$

where Ω_p is now given by

$$\Omega_p = [4\kappa^2 \sin^2(p/2) + m^2]^{1/2} \,. \tag{4.2.74}$$

Equation (4.2.69) reduces to the vacuum expectation value

$$\sigma^2 = \left\langle \sum_j \phi_n^j \phi_n^j \right\rangle = \frac{N}{2\Lambda} \sum_p [4\kappa^2 \sin^2(p/2) + m^2]^{-1/2} \,. \tag{4.2.75}$$

Putting $V(\sigma^2) = m^2\sigma^2$ in (4.2.66) and using (4.2.69) and (4.2.74) one obtains for the GS energy

$$E_\infty = \frac{N}{2} \sum_p [4\kappa^2 \sin^2(p/2) + m^2]^{1/2} \,. \tag{4.2.76}$$

The excitation energy E_{pq} of the first excited state $|pq\rangle$ defined by

$$|pq\rangle = \zeta_{pq}^+ |0\rangle \tag{4.2.77}$$

can now be easily obtained

$$E_{pq} = \Omega_p + \Omega_q \,. \tag{4.2.78}$$

In the energy spectrum there thus occur elementary excitations with mass gap m. These excitations may be called quasiphonons. It is convenient to use the crystal momentum s defined by

$$\begin{aligned} &s = p + q + \Lambda \,, \quad \text{if } -2K \le p + q \le -K \,, \qquad s = p + q \,, \quad \text{if } -K \le p + q \le K \,, \\ &s = p + q - \Lambda \,, \quad \text{if } K < p + q \le 2K \,. \end{aligned} \tag{4.2.79}$$

For each s there are $K + 1$ states with energies given by (4.2.78), s taking values in the fundamental Brillouin zone. The dispersion curve for the quasiphonon energies versus s has been obtained by MEP and we reproduce it in fig. 4.1a.

In the case of an anharmonic potential

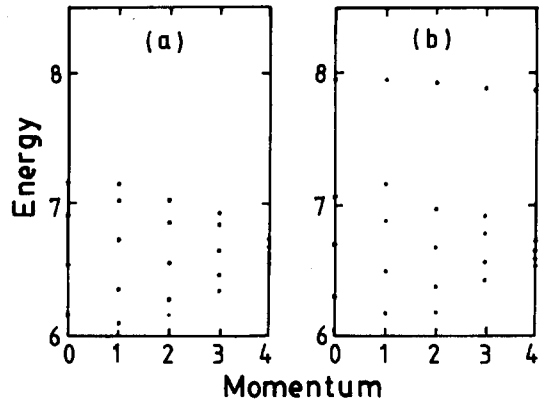

Fig. 4.1. Quasiphonon energies of a one-dimensional lattice with $\Lambda = 9$, $\kappa = 1$, $m = 3$ plotted against the crystal momentum s (in units of $2\pi/\Lambda$) defined in the text. (a) Results for the harmonic potential. (b) Corresponding results for the anharmonic potential (after Mead and Papanicolaou [116], by courtesy of J. Phys.).

$$V\left(\sum_{j=1}^{N} \phi_n^j \phi_n^j\right) = \frac{1}{2} g^2 \left(\sum_{j=1}^{N} \phi_n^j \phi_n^j\right)^2, \quad g^2 > 0, \tag{4.2.80}$$

which is a more interesting and a slightly more realistic case, [the real crystal symmetry is described by a discrete subgroup of O(N)] the quadratic hamiltonian H_0 becomes

$$H_0 = \sum_{pq} 2\Omega_p \zeta_{pq}^+ \zeta_{pq} + \frac{Ng^2}{8\Lambda} \sum_{pqkl} \frac{\Delta(p+q+k+l)}{(\Omega_p \Omega_q \Omega_k \Omega_l)^{1/2}} (\zeta_{pq}^+ + \zeta_{-p,-q})(\zeta_{kl}^+ + \zeta_{-k,-l}), \tag{4.2.81}$$

where

$$\Delta(p+q+k+l) = \frac{1}{\Lambda} \sum_{n=1} e^{-i(p+q+k+l)n}. \tag{4.2.82}$$

With the parametrization $m^2 \equiv g^2\sigma^2$ the gap equation reads

$$m^2 = \frac{Ng^2}{2\Lambda} \sum_p \frac{1}{\Omega_p}, \tag{4.2.83}$$

where

$$\Omega_p = [4\kappa^2 \sin^2(p/2) + m^2]^{1/2}. \tag{4.2.84}$$

Introducing the identity

$$1 = \delta(p+q) + \sum_{s=1}^{K} [\delta(p+q-s) + \delta(p+q+s) + \delta(p+q-s+\Lambda) + \delta(p+q+s-\Lambda)], \tag{4.2.85}$$

in (4.2.81) MEP were able to write this hamiltonian as a sum of commuting pieces

$$H = H_{0,0} + \sum_{s=1}^{K} H_{0,s} , \tag{4.2.86}$$

where

$$H_{0,0} = \sum_{p=-K}^{K} 2\Omega_p \zeta^{+}_{p,-p}\zeta_{p,-p} + \frac{\bar{g}^2}{4\Lambda}\left(\sum_{p=-K}^{K}\frac{1}{\Omega_p}(\zeta^{+}_{p,-p} + \zeta_{p,-p})\right)^2 , \tag{4.2.87}$$

$$H_{0,s} = \sum_{\substack{p+q=s \\ p+q=s-\Lambda}} (\Omega_p + \Omega_q)(\zeta^{+}_{pq}\zeta_{pq} + \zeta^{+}_{-p,-q}\zeta_{-p,-q}) + \tfrac{1}{2}\Gamma^{+}_{s}\Gamma_{s} , \tag{4.2.88}$$

$$\Gamma = \frac{\bar{g}}{\Lambda^{1/2}} \sum_{\substack{p+q=s \\ p+q=s-\Lambda}} \frac{\zeta_{pq} + \zeta^{+}_{-p,-q}}{(\Omega_p\Omega_q)^{1/2}} , \tag{4.2.89}$$

$$\bar{g} = \tfrac{1}{2}Ng^2 , \tag{4.2.90}$$

and s can take values from 1 to K. Each term of (4.2.86) can now be diagonalized separately. The eigenvalues are finally obtained by numerically solving a set of algebraic equations. The energy versus crystal momentum curve is shown in fig. 4.1b. Both figs. 4.1a and b have been plotted for $\Lambda = 9$, $\kappa = 1$ and $m = 3$. Though the total number of normal modes is the same, $\Lambda(\Lambda+1)/2$, in the two cases, the curve for the anharmonic potential contains an interesting feature, namely a collective branch distinctly separated from the two-body quasiphonon continuum.

MEP have also investigated the effect of varying the coupling constant on the energy–momentum curves. In the weak coupling limit i.e. for small m the collective branch is found to be embedded in the two-body quasiphonon continuum. In the limit of strong anharmonicity (large m) the collective branch approaches a horizontal line at a distance $m\sqrt{6}$ and the quasiphonon continuum is also seen to collapse to a horizontal line at a distance $2m$. This strong coupling limit effectively corresponds to the vanishing lattice constant κ, in which case the problem boils down to that of Λ non-interacting particles each moving in the anharmonic potential (4.2.80). The mass parameter is then related to the coupling constant through a simple equation.

$$m^3 \approx \tfrac{1}{2}Ng^2 . \tag{4.2.91}$$

The energy spectrum for this $\kappa = 0$ problem can be directly obtained from (4.2.51) if we replace g^2 appearing in (4.2.51) by $(N+2l)g^2/32$. The fully $1/N$-expanded energy reads

$$E = (\tfrac{1}{2}Ng^2)^{1/3}[\tfrac{3}{8}N + (l + n\sqrt{6} + \sqrt{\tfrac{3}{2}} - 1) + O(N^{-1})] , \tag{4.2.92}$$

which shows that the singlet oscillations have the normal frequency $m\sqrt{6}$ and the angular excitations are associated with a frequency m, thus the frequency for a two-body singlet state being $2m$.

MEP have also extended their calculation to the three-dimensional lattice. The strong coupling mass gap $m\sqrt{6}$ is observed to be independent of the lattice dimensionality, as is expected from the fact that in the limit $\kappa = 0$ the original system decouples into a set of non-interacting anharmonic oscillators.

The pseudo-spin method has enjoyed various other applications. For instance it has been employed

by Papanicolaou [115] to deal with quantum theories of N-component Fermi fields. In particular, the Gross–Neveu model [110] described by the Lagrangian

$$\mathscr{L} = \sum_{j=1}^{N} \bar{\psi}_j (\mathrm{i}\not{\partial}) \psi_j + \tfrac{1}{2} g^2 \Big(\sum_{j=1} \bar{\psi}_j \psi_j \Big)^2 , \tag{4.2.93}$$

where ψ_j is a two-component Dirac spinor, has been discussed in this context.

Papanicolaou [137] has furthermore extended the pseudo-spin approach to study quantum spin systems. A systematic $1/N$ expansion has been obtained for the energy spectrum of a spin-1 ferromagnetic chain, known to exist in $CsNiF_3$. Two kinds of elementary excitations appear in the spectrum, the usual magnon mode and an additional collective branch.

4.3. The collective field method

With the ultimate aim of providing an alternative formulation for large-N QCD Jevicki and Sakita [57] (JS) have developed the so-called collective field method of $1/N$ expansion, which is essentially a generalization of the Bohm–Pines theory of plasma oscillations [173] in an electron gas. To elucidate the basic points of the method JS have considered the examples of a system of N interacting Bose particles and a system of N identical harmonic oscillators. By generalizing the formalism they have also studied an SU(N) symmetric quantum system and have obtained the planar limit of this theory. The method has been subsequently applied to other interesting models by Sakita [140] and Jevicki and collaborators [52–60, 141–145].

The main idea of the collective field method is to make a transformation from the original dynamical operators to an overcomplete set of new operators, called the collective variables, which are the invariant combinations of the original variables of the system, which possess some symmetry, the low energy states being singlets under that symmetry. In the large-N limit the collective variables may be treated as independent variables and one can then construct a field theory with these variables which should simulate the original quantum mechanical system.

In sections 4.3.1–4.3.3 we describe the work of JS and then proceed to discuss two more examples [59, 41] in sections 4.3.4 and 4.3.5, where the connection of the collective field method with the classical equations of motion will be discussed in more explicit terms.

4.3.1. N Bose particles

The quantum mechanical hamiltonian describing a system of N interacting Bose particles in one dimension [57] can be written as

$$H = \frac{1}{2} \sum_{i=1}^{N} \hat{p}_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} v(\hat{x}_i, \hat{x}_j) + \sum_{i=1}^{N} V(\hat{x}_i) , \tag{4.3.1}$$

where the second term represents the interparticle interaction and the third term is some external field applied to the system. For this problem the density operators

$$\hat{\phi}(x) = \sum_{i=1}^{N} \delta(x - \hat{x}_i) , \quad \tfrac{1}{2} L \geq x \geq -\tfrac{1}{2} L , \tag{4.3.2}$$

which are the most general commuting symmetric operators, may be taken as the collective variables. The Schrödinger wavefunction $\psi(x_1, x_2, \ldots, x_N)$ corresponding to the hamiltonian (4.3.1) should be invariant under particle permutations and so it may be expressed as a functional of the density functions $\phi(x)$

$$\psi(x_1, x_2, \ldots, x_N) = \Phi[\phi(\cdots)]\,. \tag{4.3.3}$$

Since x is continuous we have an infinite number of density operators, whereas in the original problem the number of degrees of freedom is N. This implies that not all $\phi(x)$ are independent. Indeed, $\phi(x)$ satisfies the constraint

$$\int \phi(x)\,\mathrm{d}x = N\,. \tag{4.3.4}$$

To obtain a set of independent variables JS use the Fourier components

$$\hat{\phi}_k = \frac{1}{L}\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\,\hat{\phi}(x) = \frac{1}{L}\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}kx_i}\,, \tag{4.3.5}$$

where k can assume discrete values

$$k = 2\pi n/L\,, \quad n = \pm 1, \pm 2, \ldots\,. \tag{4.3.6}$$

If one imposes the restriction

$$|k| \le k_{\max} = \pi N/L\,, \tag{4.3.7}$$

then one has N distinct ϕ_k which can be used as the collective variables. One may notice that in the high density limit, $N/L \to \infty$, one has an infinite number of ϕ_k and in this limit the $\phi(x)$ may be treated as independent variables subject to the constraint (4.3.4). The definition (4.3.2) dictates that the density functions $\phi(x)$ which are the eigenvalues of the density operators $\hat{\phi}(x)$ should always be positive (semidefinite).

Let us now consider the change of variables from x to ϕ_k. Using the chain rule of differentiation

$$\frac{\partial}{\partial x_i}\,\psi(x_i, x_2, \ldots, x_N) = \sum_k \frac{\partial \phi_k}{\partial x_i}\,\frac{\partial}{\partial \phi_k}\,\Phi[\phi]\,, \tag{4.3.8}$$

one obtains

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} = \frac{1}{2}\,\mathrm{i}\sum_k \omega(k;[\phi])\left(-\mathrm{i}\,\frac{\partial}{\partial \phi_k}\right) + \frac{1}{2}\sum_{kk'}\Omega(k,k';[\phi])\left(-\mathrm{i}\,\frac{\partial}{\partial \phi_k}\right)\left(-\mathrm{i}\,\frac{\partial}{\partial \phi_{-k'}}\right), \tag{4.3.9}$$

where

$$\omega(k;[\phi]) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i^2}\,\phi_k = k^2\phi_k\,, \tag{4.3.10a}$$

$$\Omega(k, k'; [\phi]) = \sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} \phi_k\right)\left(\frac{\partial}{\partial x_i} \phi_{-k'}\right) = \frac{kk'}{L} \phi_{k-k'} \,. \tag{4.3.10b}$$

On setting

$$-\mathrm{i}\, \partial/\partial\phi_{-k} = \pi_k \,, \tag{4.3.11}$$

so that

$$[\phi_k, \pi_{k'}] = \mathrm{i}\delta_{k,-k'} \,, \tag{4.3.12}$$

the hamiltonian reads in terms of the collective variables [57]

$$H = \tfrac{1}{2}\,\mathrm{i} \sum_k \omega(k; [\phi])\pi_{-k} + \tfrac{1}{2} \sum_{kk'} \Omega(k, k'; [\phi])\pi_{-k}\pi_{k'} + \frac{1}{2} \sum_{kk'} \phi_{-k}\phi_{k'} v(k, k') + \sum_k [V(k) - \tfrac{1}{2} v(k, k)]\phi_{-k} \,, \tag{4.3.13}$$

where

$$v(k, k') = \int \mathrm{d}x\, \mathrm{d}x'\, \mathrm{e}^{-\mathrm{i}kx + \mathrm{i}k'x'}\, v(x, x') \,, \qquad V(k) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} V(x) \,. \tag{4.3.14a, b}$$

The Fourier transform of π_k may also be defined in the usual way

$$\pi(x) = \frac{1}{L} \sum_{k \neq 0} \mathrm{e}^{\mathrm{i}kx}\, \pi_k \,, \tag{4.3.15}$$

so that

$$[\phi(x), \pi(x')] = \mathrm{i}\, \frac{1}{L} \sum_{k \neq 0} \mathrm{e}^{\mathrm{i}k(x-x')} = \mathrm{i}\left(\delta(x - x') - \frac{1}{L}\right). \tag{4.3.16}$$

The important point to note is that though the given hamiltonian (4.3.1) in the original x-space is hermitian, the hamiltonian (4.3.13) in the space of the collective variables does not appear to be hermitian, if one uses the following hermitian conjugates:

$$\phi_k^{+} = \phi_{-k} \,, \qquad \pi_k^{+} = \pi_{-k} \,, \tag{4.3.17a, b}$$

which follow from the hermiticity of the density operator. This issue may be settled by recognizing that a Jacobian is associated with a coordinate transformation, which has to be considered while defining the inner product of the transformed functionals. Thus

$$(\psi_1, \psi_2) = \int \mathrm{d}x\, \psi_1^{*}(x)\psi_2(x) = \int \prod \mathrm{d}\phi_k\, J[\phi]\Phi_1^{*}[\phi]\Phi_2[\phi] \,. \tag{4.3.18}$$

The rescaled wave functionals,

$$\Psi = (J[\phi])^{1/2}\Phi[\phi(\cdots)]\,, \tag{4.3.19}$$

then have the simple inner product,

$$(\Psi_1, \Psi_2) = \int \prod_{k\neq 0} \mathrm{d}\phi_k\, \Psi_1^*[\phi]\Psi_2[\phi]\,. \tag{4.3.20}$$

The Schrödinger equation for the problem may now be written as

$$H_{\mathrm{eff}}\,\Psi[\phi] = E\Psi[\phi]\,, \tag{4.3.21}$$

where

$$H_{\mathrm{eff}} = (J[\phi])^{1/2}H(J[\phi])^{-1/2}\,, \tag{4.3.22}$$

the Jacobian $J[\phi]$ being a real functional of ϕ. The effective hamiltonian (4.3.22) should now be hermitian. Under the similarity transformation by the Jacobian the variables transform as follows:

$$\phi_k \to (J[\phi])^{1/2}\phi_k(J[\phi])^{-1/2} = \phi_k\,, \qquad \pi_k \to (J[\phi])^{1/2}\pi_k(J[\phi])^{-1/2} = \pi_k + \mathrm{i}C_k[\phi]\,, \tag{4.3.23a, b}$$

where

$$C_k[\phi] = \tfrac{1}{2}\,\partial \ln J[\phi]/\partial\phi_k\,. \tag{4.3.24}$$

$C_k[\phi]$ can be determined by demanding the hermiticity of the effective hamiltonian (4.3.22) i.e.

$$H_{\mathrm{eff}}^{+} - H_{\mathrm{eff}} = 0\,, \tag{4.3.25}$$

which then leads to [174]

$$\omega(k, [\phi]) + 2\sum_{k'\neq 0} \Omega(k, k'; [\phi])C_{k'} = 0\,, \tag{4.3.26}$$

where use has been made of the property

$$(\pi_{-k}\Omega(k, k'; [\phi])) = (kk'/L)\delta_{k,k-k'} = 0\,. \tag{4.3.27}$$

The solution of (4.3.26) can be written formally as

$$C_k = -\frac{1}{2}\sum_{k'} \Omega^{-1}(k, k'; [\phi])\omega(k'; [\phi])\,, \tag{4.3.28}$$

where Ω^{-1} is the inverse matrix of Ω. This inverse clearly exists because the diagonal elements of $\Omega(k, k'; [\phi])$ are c-numbers $[\Omega(k, k; [\phi]) = Nk^2/L^2]$. The hermitian effective hamiltonian finally reduces to [57]

$$H_{\text{eff}} = \frac{1}{2}\sum_{kk'} \pi_{-k}\Omega(k,k';[\phi])\pi_{k'} + \frac{1}{8}\sum_{kk'} \omega(-k;[\phi])\Omega^{-1}(k,k';[\phi])\omega(k';[\phi])$$
$$+ \frac{\mathrm{i}}{4}\sum_{k}(\pi_{-k}\omega(k;[\phi])) + \frac{1}{2}\sum_{kk'}\phi_{-k}\phi_{k'}v(k,k') + \sum_{k}[V(k) - \tfrac{1}{2}v(k,k)]\phi_{-k}\,, \tag{4.3.29}$$

where the term $(\mathrm{i}/4)\Sigma_k(\pi_{-k}\omega(k;[\phi]))$ equals $-\frac{1}{4}\Sigma_k k^2$ and would thus cancel the infinite zero-point energy of the density field. Therefore in the subsequent development this term will be ignored.

In the high density limit ($N/L\to\infty$) the collective field theory may be formulated in the coordinate representation. One gets [57]

$$H_{\text{eff}} = \frac{1}{2}\int \mathrm{d}x\,(\partial_x\pi)\phi(x)(\partial_x\pi) + V_{\text{eff}}(\phi)\,, \tag{4.3.30}$$

where

$$V_{\text{eff}}(\phi) = V_{\text{c}}(\phi) + \mathscr{V}(\phi)\,, \tag{4.3.31}$$

$$V_{\text{c}}(\phi) = \frac{1}{8}\int \frac{[\partial\phi(x)]^2}{\phi(x)}\,\mathrm{d}x\,, \qquad \mathscr{V}(\phi) = \frac{1}{2}\int \mathrm{d}x\int \mathrm{d}y\,\phi(x)v(x,y)\phi(y) + \int \mathrm{d}x\,\phi(x)[V(x) - v(x,x)]\,. \tag{4.3.31a, b}$$

$V_{\text{c}}(\phi)$ is actually a part of the kinetic energy, but it may be reckoned as the centrifugal potential energy. The situation is quite reminiscent of that which is encountered in simple one-particle quantum mechanical problems in three dimensions. Had we retained $\hbar$ throughout the analysis, it would have appeared in the centrifugal potential term showing that this potential has a purely quantum origin.

Rescaling the field variables by the prescription

$$\phi(x)\to N\phi(x)\,, \qquad \pi(x)\to \pi(x)/N\,, \tag{4.3.32}$$

one can immediately show that in the large-N limit it is $\mathscr{V}(\phi)$ which dominates. Thus one should minimize it with respect to $\phi(x)$ subject to the constraint (4.3.4), the imposition of which necessitates the introduction of a Lagrange multiplier λ in this problem. Expanding $\phi(x)$ around the minimum then generates the $1/N$ expansion.

To show the connection of their collective field formulation with the Bohm–Pines theory, JS have considered a system of charged bosonic plasma for which one can take

$$V(x) = 0\,, \tag{4.3.33}$$

$$v(\boldsymbol{x},\boldsymbol{x}') = e^2/|\boldsymbol{x}-\boldsymbol{x}'|\,. \tag{4.3.34}$$

The extension of collective field theory discussed above to three dimensions is straightforward. One should replace x and k by three-dimensional vectors (and consequently the length L is replaced by the volume $\tilde{\tilde{V}}$).

The Fourier transform of $v(\boldsymbol{x},\boldsymbol{x}')$ is given by

$$v(\boldsymbol{k},\boldsymbol{k}') = (4\pi e^2/k^2)\tilde{\tilde{V}}\delta_{\boldsymbol{k},\boldsymbol{k}'}\,. \tag{4.3.35}$$

The large-N solution for ϕ_k can be obtained rather easily in this problem. In view of (4.3.5) ϕ_k may be regarded as a sum of N unit vectors distributed randomly. Thus in the limit of an infinitely large number of particles (high-density limit) $\phi_{k-k'}$ will be nonvanishing only when $k = k'$ i.e.

$$\phi_{k-k'} \sim (N/\tilde{\tilde{V}})\delta_{k-k'} . \tag{4.3.36}$$

Thus the large-N approximation in this problem is exactly the random phase approximation.

Now if we substitute in (4.3.29) the definitions

$$\phi_k = \phi_k^0 + (\sqrt{N}k/\tilde{\tilde{V}})\tilde{\phi}_k , \qquad \pi_k = (\tilde{V}/\sqrt{N}k)\tilde{\pi}_k \tag{4.3.37a, b}$$

(where $\tilde{\phi}_k$ is the fluctuation field) and consider only the quadratic part of the effective hamiltonian, we obtain

$$H_{\text{eff}}^{\text{quad}} \approx \frac{1}{2}\sum_k (\tilde{\pi}_{-k}\tilde{\pi}_k + \omega_k^2\phi_{-k}\phi_k) , \tag{4.3.38}$$

where $\omega_k = [(4\pi e^2 N/\tilde{\tilde{V}}) + \frac{1}{4}k^4]^{1/2}$ is the well-known plasma frequency obtained by Bohm and Pines [173] in the random phase approximation.

4.3.2. N identical harmonic oscillators: the collective modes

The problem of N identical non-interacting harmonic oscillators is exactly solvable but the large-N limit of this problem offers some interesting physics, namely the collective motions of the oscillators [57].

For this problem

$$V(x) = \tfrac{1}{2}\omega^2 x^2 , \qquad v(x, y) = 0 . \tag{4.3.39a, b}$$

The effective potential therefore reads

$$V_{\text{eff}}(\phi) = V_c + \mathscr{V}(\phi) = \frac{1}{8}\int \mathrm{d}x \, \frac{[\partial\phi(x)]^2}{\phi(x)} + \frac{1}{2}\,\omega^2 \int x^2\phi(x)\,\mathrm{d}x , \tag{4.3.40}$$

where the collective field $\phi(x)$ satisfies the constraint

$$\int \mathrm{d}x\, \phi(x) = N . \tag{4.3.41}$$

The large-N limit is obtained by minimizing the effective potential V_{eff} with respect to $\phi(x)$, subject to the constraint (4.3.41). Let $\phi^0(x)$ be the value of $\phi(x)$ where the minimum occurs; then $\phi^0(x)$ is the solution of

$$\partial V_{\text{eff}}(\phi)/\partial\phi(x) + \lambda = 0 , \tag{4.3.42}$$

where λ is a Lagrange multiplier introduced to take care of the constraint equation (4.3.41).

Written explicitly eq. (4.3.42) reads

$$-\tfrac{1}{8}(\partial_x\phi)^2/\phi^2 - \tfrac{1}{4}\partial_x((\partial_x\phi)/\phi) + \tfrac{1}{2}\omega^2x^2 + \lambda = 0\,. \tag{4.3.43}$$

It is easy to see that the solution of (4.3.43) is given by

$$\phi^0(x) = N\sqrt{\omega/\pi}\,\mathrm{e}^{-\omega x^2}\,. \tag{4.3.44}$$

Next JS use the transformations

$$\phi(x) = \phi^0(x) + \eta(x)\,, \qquad \pi(x) = \xi(x)\,, \tag{4.3.45a, b}$$

where η and ξ satisfy the same commutation relation as $\phi(x)$ and $\pi(x)$,

$$[\eta(x), \xi(x)] = \mathrm{i}[\delta(x - x') + \text{constant}]\,. \tag{4.3.46}$$

One obtains in the quadratic approximation

$$H_{\text{eff}}^{\text{quad}} \approx \frac{1}{2}\int \mathrm{d}x\,\phi^0(x)[\partial_x\xi(x)]^2 + \frac{1}{4}\int \frac{[\partial_x\eta(x)]^2}{\phi^0(x)}\,\mathrm{d}x - \frac{1}{2}\,\omega\int \frac{\eta^2(x)}{\phi^0(x)}\,\mathrm{d}x\,, \tag{4.3.47}$$

where the fluctuation η satisfies the constraint

$$\int \mathrm{d}x\,\eta(x) = 0\,, \tag{4.3.48}$$

since $\phi^0(x)$ satisfies the same condition as $\phi(x)$.

Introducing the transformations

$$\xi(x) = [\phi^0(x)]^{-1/2}\tilde{\xi}(x)\,, \qquad \eta(x) = [\phi^0(x)]^{1/2}\tilde{\eta}(x)\,, \tag{4.3.49a, b}$$

and using (4.3.44) then lead to

$$H_{\text{eff}}^{\text{quad}} \approx \frac{1}{2}\left(\int \mathrm{d}x\,\tilde{\xi}(x)(-\partial_x^2 + \omega^2x^2 - \omega)\tilde{\xi}(x) + \frac{1}{2}\int \mathrm{d}x\,\tilde{\eta}(x)(-\partial_x^2 + \omega^2x^2 - \omega)\tilde{\eta}(x)\right). \tag{4.3.50}$$

Finally expanding

$$\tilde{\xi}(x) = \sum_{n\neq 0}\frac{1}{\sqrt{2\omega n}}\,\chi_n(x)\tilde{\xi}_n\,, \qquad \tilde{\eta}(x) = \sum_{n\neq 0}\sqrt{2\omega n}\chi_n(x)\tilde{\eta}_n\,, \tag{4.3.51a, b}$$

where the χ_n satisfy the harmonic oscillator equation

$$(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}\omega^2x^2)\chi_n(x) = (n + \tfrac{1}{2})\chi_n(x)\,, \tag{4.3.52}$$

JS obtain

$$H_{\text{eff}}^{\text{quad}} = \frac{1}{2}\sum_{n\neq 0}(\tilde{\xi}_n^2 + n^2\omega^2\tilde{\eta}_n^2)\,. \tag{4.3.53}$$

That the mode $n = 0$ is absent from (4.3.53) is an immediate consequence of the constraint (4.3.48).

The collective modes of the system now admit a classical interpretation. The $n = 1$ collective mode corresponds to all particles being at $x = 0$ for $t = 0$ with positive initial velocities whereas the $n = 2$ collective mode corresponds to the situation where all particles are at $x = 0$ for $t = 0$ with zero averaged initial velocity.

4.3.3. The general formalism and its application to an SU(N) symmetric quantum mechanical model

Let us consider a general quantum mechanical problem [57] described by the hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} \hat{p}_i^2 + V(\hat{q}_1, \ldots, \hat{q}_N)\,. \tag{4.3.54}$$

The choice of the collective variables depends, however, on the particular problem we are to solve. Let us assume that there exists a set of invariant combinations $\phi(x)$ of the original operators $q_1, q_2, \ldots, q_N$ $[\phi(x) = f(x, q_1, \ldots, q_N)]$ in terms of which the potential V may be expressed, and also that the wavefunction $\psi(q_1, \ldots, q_N)$ of the system can be written as a functional of the collective variables $\phi(x)$

$$\psi(q_1, q_2, \ldots, q_N) = \Phi[\phi(\cdots)]\,. \tag{4.3.55}$$

The kinetic energy can be written as

$$K = \frac{1}{2}\int \mathrm{d}x\, \omega(x;[\phi])\,\frac{\partial}{\partial\phi(x)} - \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}y\, \Omega(x, y;[\phi])\,\frac{\partial^2}{\partial\phi(x)\partial\phi(y)}\,, \tag{4.3.56}$$

where

$$\omega(x;[\phi]) = -\sum_{i=1}^{N}\frac{\partial^2\phi(x)}{\partial q_i^2}\,, \qquad \Omega(x, y;[\phi]) = \sum_{i=1}^{N}\left(\frac{\partial\phi(x)}{\partial q_i}\right)\left(\frac{\partial\phi(y)}{\partial q_i}\right). \tag{4.3.56 a, b}$$

The hamiltonian $H(\phi)$ expressed in terms of the collective coordinates $\phi(x)$ and their conjugate moments $\pi(x) = -\mathrm{i}\,\partial/\partial\phi(x)$ is however not hermitian. As discussed in section 4.3.1 one then makes the following transformations:

$$\Phi[\phi] \to \Psi[\phi] = (J[\phi])^{1/2}\Phi[\phi]\,, \qquad H(\phi) \to H_{\mathrm{eff}}(\phi) = (J[\phi])^{1/2}H(J[\phi])^{-1/2} \tag{4.3.57a, b}$$

and considers the Schrödinger equation

$$H_{\mathrm{eff}}\,\Psi[\phi] = E\Psi[\phi]\,, \tag{4.3.58}$$

where $J[\phi]$ is the Jacobian of the transformation from the original variables to the collective ones. Eq. (4.3.57b) can be calculated by using

$$(J[\phi])^{1/2}\phi(x)(J[\phi])^{-1/2} = \phi(x)\,, \qquad (J[\phi])^{1/2}\pi(x)(J[\phi])^{-1/2} = \pi(x) + \mathrm{i}C(x;[\phi])\,, \tag{4.3.59a, b}$$

where

$$C(x;[\phi]) = \tfrac{1}{2}\,\partial \ln J[\phi]/\partial\phi(x)\,. \tag{4.3.59c}$$

C can be determined by demanding that H_{eff} is hermitian. This gives

$$\omega(x;[\phi]) + \int \mathrm{d}y \, \frac{\partial \Omega(x, y;[\phi])}{\partial \phi(y)} + 2 \int \mathrm{d}y \, \Omega(x, y;[\phi]) C(y;[\phi]) = 0 , \tag{4.3.60}$$

which can be solved formally to yield

$$C = -\tfrac{1}{2} \Omega^{-1} [\omega + \mathrm{i}(\hat{\pi} \Omega)] . \tag{4.3.61}$$

The hermitian effective hamiltonian finally takes the expression [57]

$$\begin{aligned} H_{\text{eff}} = {} & \frac{1}{2} \int \mathrm{d}x \int \mathrm{d}y \, \pi(x) \Omega(x, y;[\phi]) \pi(y) \\ & + \frac{1}{8} \int \mathrm{d}x \int \mathrm{d}y \left(\omega(x;[\phi]) - \mathrm{i} \int \mathrm{d}z \, (\pi(z) \Omega(x, z;[\phi])) \right) \\ & \times J \Omega^{-1}(x, y;[\phi]) \left(\omega(y;[\phi]) - \mathrm{i} \int \mathrm{d}z' \, (\pi(z') \Omega(y, z;[\phi])) \right) \\ & - \frac{1}{4} \int \mathrm{d}x \, \frac{\partial \omega(x;[\phi])}{\partial \phi(x)} - \frac{1}{4} \int \mathrm{d}x \int \mathrm{d}y \, \frac{\partial^2 \Omega(x, y;[\phi])}{\partial \phi(x) \partial \phi(y)} + V[\phi] . \end{aligned} \tag{4.3.62}$$

JS used the above general formalism developed by them to study an SU(N) symmetric quantum mechanical system described by the lagrangian

$$\mathscr{L} = \tfrac{1}{2} \operatorname{Tr}(\dot{M}^2) - \operatorname{Tr}[\tfrac{1}{2} \hat{M}^2 + (g/N) \hat{M}^4] , \tag{4.3.63}$$

where $\hat{M}$ is an $N \times N$ hermitian matrix. The hamiltonian corresponding to (4.3.63) is given by

$$H = -\frac{1}{2} \sum_{a=0}^{N^2-1} \frac{\partial^2}{\partial M_a^2} + \operatorname{Tr}(\tfrac{1}{2} \hat{M}^2 + (g/N) \hat{M}^4) , \tag{4.3.64}$$

where use has been made of the expansion

$$\hat{M} = \sum_{a=0}^{N^2-1} t^a M_a , \tag{4.3.64a}$$

the t^a being $N \times N$ matrices which are an N-dimensional generalization of the Pauli matrices and form the fundamental representation of the U(N) algebra. The discussion which follows will be confined to the subspace which contains only the singlet states.

The transition amplitudes can be expressed in terms of Feynman path integrals which can be expanded in perturbative series, to each term of which one can then associate a Feynman diagram. Considering the singlet to singlet SU(N) invariant transition amplitudes one can show that in the large-N limit only the planar diagrams survive. By mapping the given problem (4.3.63) onto a one-dimensional Fermi gas model, Brezin et al. [117] obtained the large-N solution of this problem which is the planar limit.

To solve the SU(N) symmetric model (4.3.64) by the collective field method JS treated

$$\phi_k = \mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}k\hat{M}} = \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}k\lambda_i} \tag{4.3.65}$$

as the collective variables. Here k is given by (4.3.6) and (4.3.7) and the λ_i ($i = 1, 2, \ldots, N$) are the real eigenvalues of $\hat{M}$ and are restricted within the range

$$\tfrac{1}{2}L \geq \lambda_i \geq -\tfrac{1}{2}L\,. \tag{4.3.66}$$

In the limit $L \to \infty$, k can be treated as a continuous variable and then the Fourier transform of ϕ_k may be defined as

$$\phi(x) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx}\phi_k = \mathrm{Tr}\, \delta(x - \hat{M})\,. \tag{4.3.67}$$

The domain of x is taken as

$$\tfrac{1}{2}L \geq x \geq -\tfrac{1}{2}L\,, \tag{4.3.68}$$

which eventually becomes

$$\infty \geq x \geq -\infty\,, \tag{4.3.68a}$$

since L is increased to ∞. $\phi(x)$ then satisfies the constraint

$$\int \phi(x)\, \mathrm{d}x = \mathrm{Tr}\, \mathbb{1} = N\,. \tag{4.3.69}$$

To be able to treat the $\phi(x)$ as independent variables [of course, subject to the constraint (4.3.69)] one should take the limit $N/L \to \infty$. Thus in the present problem one has to consider two limits, $N/L \to \infty$ and $L \to \infty$.

Using the general formulae (4.3.56a, b), the relation

$$\frac{\partial}{\partial M_a}\, \mathrm{e}^{-\mathrm{i}k\hat{M}} = \int_0^1 \mathrm{d}\alpha\, \mathrm{e}^{-\mathrm{i}\alpha k\hat{M}}\, \frac{\partial(-\mathrm{i}k\hat{M})}{\partial M_a}\, \mathrm{e}^{-\mathrm{i}(1-\alpha)k\hat{M}}\,, \tag{4.3.70}$$

and the property of the U(N) generators

$$\sum_a (t^a)_{\alpha\beta}(t^a)_{\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\beta\alpha'}\,, \tag{4.3.71}$$

one gets

$$\omega(k;[\phi]) = -\sum_a \frac{\partial^2}{\partial M_a^2}\, \phi_k = k^2 \int_0^1 \mathrm{d}\alpha\, \phi_{\alpha k}\phi_{(1-\alpha)k}\,, \tag{4.3.72a}$$

$$\Omega(k, k'; [\phi]) = \left(\frac{\partial}{\partial M_a}\phi_k\right)\left(\frac{\partial}{\partial M_a}\phi_{k'}\right) = kk'\phi_{k-k'}\,, \tag{4.3.72b}$$

which in the coordinate representation read

$$\omega(x; [\phi]) = 2\frac{\partial}{\partial x}[\phi(x)G(x; [\phi])]\,, \qquad \Omega(x, x'; [\phi]) = \frac{\partial}{\partial x}\frac{\partial}{\partial x'}[\delta(x-x')\phi(x)]\,, \tag{4.3.73a, b}$$

where

$$G(x; [\phi]) = \mathrm{P}\int \frac{\phi(x')}{x-x'}\,\mathrm{d}x'\,, \tag{4.3.74}$$

P denoting the principal value of the integral.

The potential in (4.3.64) can be easily expressed in terms of the collective variables $\phi(x)$

$$\mathrm{Tr}\left(\frac{1}{2}\hat{M}^2 + \frac{g}{N}\hat{M}^4\right) = \int\left(\frac{1}{2}x^2 + \frac{g}{N}x^4\right)\phi(x)\,\mathrm{d}x\,. \tag{4.3.75}$$

Finally the effective hamiltonian (4.3.62) for the present problem reduces to

$$H_{\mathrm{eff}} = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}y\, \pi(x)\Omega(x, y; [\phi])\pi(y) + \mathscr{V}_{\mathrm{eff}}[\phi] + \Delta V\,, \tag{4.3.76}$$

where $\mathscr{V}_{\mathrm{eff}}[\phi]$ is the effective potential given by

$$\mathscr{V}_{\mathrm{eff}}[\phi] = \int \mathrm{d}x\left(\frac{1}{2}G^2(x; \phi) + \frac{1}{2}x^2 + \frac{g}{N}x^4\right)\phi(x)\,, \tag{4.3.77a}$$

ΔV is an additional term which is singular and is given by

$$\Delta V = -\frac{1}{4}\int \mathrm{d}x\,\frac{\partial\omega(x; [\phi])}{\partial\phi(x)}\,, \tag{4.3.77b}$$

and $\pi(x)$ is, as usual, the canonical conjugate to $\phi(x)$

$$[\phi(x), \pi(x)] = \mathrm{i}[\delta(x-x') - \text{constant}]\,. \tag{4.3.78}$$

For the detailed derivation of (4.3.76) the reader is referred to ref. [174].

Scaling the variables suitably one can show that the limit $N\to\infty$ suppresses the kinetic energy part in (4.3.76) making the effective potential $\mathscr{V}_{\mathrm{eff}}[\phi]$ all-important. The large-N solution would now be obtained by minimizing the effective potential $\mathscr{V}_{\mathrm{eff}}[\phi]$ with respect to $\phi(x)$, subject to the constraint (4.3.69). One should thus vary the functional

$$E(\varepsilon, \phi) = \mathscr{V}_{\mathrm{eff}}[\phi] + \varepsilon\left(N - \int \mathrm{d}x\,\phi(x)\right)\,, \tag{4.3.79}$$

with respect to $\phi(x)$ and ε, ε being a Lagrange multiplier. The variation with respect to ε gives trivially

the constraint equation (4.3.69) while that with respect to $\phi(x)$ yields

$$\frac{1}{2}\left(\int dy\,\frac{\phi(y)}{y-x}\right)^2 - \int dy\,\frac{\phi(y)}{y-x}\int dy'\,\frac{\phi(y')}{y'-x} + \frac{1}{2}x^2 + \frac{g}{N}x^4 - \varepsilon = 0\,, \tag{4.3.80}$$

which can be solved to give

$$\phi^0(x) = (1/\pi)[2\varepsilon - x^2 - (2g/N)x^4]^{1/2}\,, \quad |x| < \Lambda\,, \qquad \phi^0(x) = 0\,, \quad |x| > \Lambda\,, \tag{4.3.81}$$

where Λ is the solution of

$$2\varepsilon - \Lambda^2 - 2g\Lambda^4 = 0\,, \tag{4.3.82}$$

and ε is obtained from

$$\int_{-\Lambda}^{\Lambda} \phi^0(x)\,dx = N\,. \tag{4.3.83}$$

The GS energy E_0 is finally obtained as [57]

$$E_0 = V_{\text{eff}}[\phi^0] = \varepsilon N - \int_{-\Lambda}^{\Lambda} \frac{dx}{3\pi}\left(2\varepsilon - x^2 - \frac{2g}{N}x^4\right)^{3/2}\,, \tag{4.3.84}$$

where use has been made of (4.3.81), (4.3.82) and the relation

$$\int dx\,\phi^0(x)G(x;[\phi_0])^2 = \frac{1}{3}\int dx\,\phi^0(x)\left(2\varepsilon - x^2 + \frac{2g}{N}x^4\right),$$

which follows from (4.3.80). The GS energy expression (4.3.84) is the same as that obtained by Brezin et al. [117].

The $1/N$ expansion can be generated as before by expanding $\phi(x)$ around the minimum $\phi^0(x)$.

4.3.4. Motion of a particle on a sphere

The large-N quantum mechanics of a particle moving on an $(N+1)$-dimensional sphere has been studied by Jevicki and Levine [59], with a view to establishing the connection of the collective field hamiltonian with the classical equations of motion. The lagrangian of the system is given by

$$\mathscr{L} = \frac{1}{2}\sum_{i=1}^{N+1} \dot{q}_i\dot{q}_i\,, \tag{4.3.85}$$

with the constraint

$$\sum_{i=1}^{N+1} q_i q_i = 1\,. \tag{4.3.86}$$

Equation (4.3.85) can be rewritten as

$$\mathscr{L} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{q}_i\delta_{ij}q_j = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{q}_i\delta_{ij}\dot{q}_j + \frac{1}{2}\dot{q}_{N+1}^2\,, \tag{4.3.87}$$

which on using the time derivative of (4.3.86) reduces to

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{N}\dot{q}_i g_{ij}\dot{q}_j\,, \tag{4.3.88}$$

where the metric tensor g_{ij} is given by

$$g_{ij} = \delta_{ij} + q_i q_j/(1-q^2)\,. \tag{4.3.89}$$

with

$$q^2 = \sum_{i=1}^{N} q_i q_i\,. \tag{4.3.90}$$

The hamiltonian corresponding to (4.3.88) then reads

$$H = \frac{1}{2}\sum_{i,j=1}^{N} g^{-1/2}p_i(g^{-1})_{ij}g^{1/2}p_j\,, \tag{4.3.91}$$

where the p_i are conjugate to the q_i and $g \equiv \det g_{ij}$. To make the hamiltonian hermitian one should make the similarity transformation

$$H \to H' = g^{1/4}Hg^{-1/4}\,. \tag{4.3.92}$$

Using the explicit form of g_{ij} one obtains

$$H(p,q) = \tfrac{1}{2}[p^2 - (\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p})] + [q^2/8(1-q^2)] + \tfrac{1}{4}(N-1)\,, \tag{4.3.93}$$

where the vectors are all N-dimensional. Jevicki and Levine have now applied the collective field method to this O(N) invariant hamiltonian and have restricted the analysis to the singlets of the O(N) symmetry. The O(N) invariant quantities to be obtained as combinations of the original variables q_i are obviously functions of $\rho = \boldsymbol{q}\cdot\boldsymbol{q}$. Rewriting the hamiltonian in terms of ρ and $\partial/\partial\rho$, one gets

$$H = -\left(2\rho\frac{\partial^2}{\partial\rho^2} + N\frac{\partial}{\partial\rho}\right) + (N+2)\rho\frac{\partial}{\partial\rho} + 2\rho^2\frac{\partial^2}{\partial\rho^2} + \mathrm{O}(N)\,, \tag{4.3.94}$$

where $\hbar$ has been set equal to one. In obtaining (4.3.94) use has been made of the following transformation relations

$$p^2 = -\sum_i \frac{\partial^2}{\partial q_i\partial q_i} = -4\rho\frac{\partial^2}{\partial\rho^2} - 2N\frac{\partial}{\partial\rho}\,, \qquad (\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) = -\left(N + 2\rho\frac{\partial}{\partial\rho}\right)\left(2\rho\frac{\partial}{\partial\rho}\right). \tag{4.3.95a, b}$$

To arrive at a hermitian hamiltonian Jevicki and Levine have made, by analogy with the general form (4.3.58b), the transformation

$$\partial/\partial\rho \to \partial/\partial\rho + f(\rho)\,, \tag{4.3.96}$$

with the choice

$$f = -(N-2)/4\rho\,. \tag{4.3.97}$$

The hermitian effective hamiltonian then reads

$$H_{\text{eff}} = 2\hat{p}(\rho - \rho^2)\hat{p} + (N^2/8\rho)(1-\rho) + \mathrm{O}(N)\,. \tag{4.3.98}$$

In the large-N limit the kinetic energy is suppressed and the potential energy becomes the only surviving term; hence the quantum mechanics of (4.3.98) may be discussed via WKB. To leading order one should then solve for $\rho(t)$ and consider the quantization through

$$\int_0^T p\,\mathrm{d}\rho = 2\pi n\,, \tag{4.3.99}$$

where T is the period of the solution.

A first integral of motion of the system (4.3.98) is the conserved total energy

$$E = \dot{\rho}^2/8\rho(1-\rho) - N^2(1-\rho)/8\rho\,, \tag{4.3.100}$$

and the solution for $\rho(t)$ may be written as

$$\rho(t) = 1 - (a^2/\omega^2)\cos^2\omega t\,, \tag{4.3.101}$$

$$\omega^2 = 2(E - \tfrac{1}{8}N^2)\,, \qquad a^2 = \omega^2 - \tfrac{1}{4}N^2\,. \tag{4.3.102}$$

The semiclassical quantization (4.3.99) then yields the energy levels [59]

$$E_n = \tfrac{1}{2}n(n+N)\,, \tag{4.3.103}$$

which for $N = 3$ is in agreement with Charap's exact O(3) singlet subspace result [59, 175, 176],

$$E_n = \tfrac{1}{2}n(n+N-1)\,, \quad N = 3 \tag{4.3.104}$$

in the large-N limit.

Let us now concentrate on the classical solutions of the original system. Introducing $\boldsymbol{q} = r\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{n}}$ being a unit vector, Jevicki and Levine have obtained the first integral corresponding to (4.3.100) of the classical system (4.3.88)

$$E_{\text{cl}} = \tfrac{1}{2}\dot{r}^2/(1-r^2) - L^2/2r^2\,, \tag{4.3.105}$$

where L^2 is the N-dimensional total angular momentum operator [denoted by L^2_{N-1} in (3.0.13)]. Identifying $r^2 = \rho$ and writing (4.3.105) as

$$E_{\rm cl} = \tilde{E}_{\rm cl} + \tfrac{1}{8}N^2 , \tag{4.3.105a}$$

we find that $\tilde{E}_{\rm cl}$ becomes identical to (4.3.100) if $L^2 = N^2/4$. Thus the large-N collective field equations are identical to the constrained classical equations.

4.3.5. Linear σ model in $(1+1)$ dimensions

The linear σ model first studied in the framework of the collective field formalism by JP is described by the lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \tfrac{1}{2}m^2 \phi^* \phi - (g/8N)(\phi^* \phi)^2 , \tag{4.3.106}$$

$$\phi^* \phi = \sum_{i=1}^{N} \phi_i^*(x,t)\phi_i(x,t) . \tag{4.3.107}$$

For convenience the theory will be formulated in a one-dimensional box of length L, following JP.

JP have considered the subspace of U(N) singlet states. The commuting U(N) invariant combinations of the original fields which may be treated as the collective variables are then chosen as

$$\psi(x,y) = \sum_{i=1}^{N} \phi_i^*(x)\phi_i(y) , \tag{4.3.108}$$

where both $\phi_i^*(x)$ and $\phi_i(y)$ are defined at the same time t. Defining

$$\pi(x,y) = -\mathrm{i}\,\partial/\partial\psi(x,y) , \tag{4.3.109}$$

one can obtain the hamiltonian in terms of ψ and π

$$H = 2\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\pi(x,y)\psi(y,z)\pi(z,x) - 2[N - L\delta(0)]\int \mathrm{d}x\,\frac{\partial}{\partial\psi(x,y)} + V[\psi] , \tag{4.3.110}$$

where

$$V[\psi] = \frac{1}{2}\int \mathrm{d}x\Big(-\lim_{y\to x}\partial_x^2\psi(x,y) + m^2\psi(x,x) + \frac{g}{4N}\psi^2(x,x)\Big) , \tag{4.3.111}$$

and $\hbar$ has been set equal to one. The final hermitian effective collective field hamiltonian reads

$$H_{\rm eff} = 2\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\pi(x,y)\psi(y,z)\pi(z,x) + V_{\rm eff}[\psi] , \tag{4.3.112}$$

$$V_{\rm eff}[\psi] = \frac{1}{2}[N - L\delta(0)]^2 \int \mathrm{d}x\,\psi^{-1}(x,x) + V[\psi] . \tag{4.3.113}$$

Making the scaling transformations

$$\psi = N\tilde{\psi} , \qquad \pi = \tilde{\pi} , \tag{4.3.114}$$

one can show that N appears as an overall factor in front of the hamiltonian and so the stationary point of the effective hamiltonian (4.3.112) will give the large-N limit. The effective equations of motion are

$$\dot{\psi}(x, y, t) = 2(\psi\pi + \pi\psi)(x, y; t), \qquad \dot{\pi}(x, y; t) = -\left(2\pi^2(x, y) + \frac{\partial V_{\text{eff}}[\psi]}{\partial\psi(x, y; t)}\right). \tag{4.3.115a, b}$$

The vacuum solution obtained by JP in the static bilocal field approximation is given by

$$\psi_0(x, y) = N\int \frac{\mathrm{d}k}{2\pi}\,\frac{\mathrm{e}^{ik(x-y)}}{\omega(k, \sigma_0)}, \tag{4.3.116}$$

where

$$\omega(k, \sigma_0) = [k^2 + m^2 + (g/2N)\sigma_0]^{1/2}, \tag{4.3.117}$$

σ_0 satisfying the so called gap equation

$$\sigma_0 = N\int \frac{\mathrm{d}k}{2\pi}\,\frac{1}{\omega(k, \sigma_0)}. \tag{4.3.118}$$

This is exactly the same gap equation as obtained by using other $1/N$ methods.

Let us now consider the original classical field equations

$$[\partial_t^2 - \partial_x^2 + m^2 + (g/2N)(\phi^*\phi)]\phi_i(x, t) = 0, \quad i = 1, 2, \ldots, N, \tag{4.3.119}$$

and ask the question: Under which condition will (4.3.119) coincide with the large-N effective equation (4.3.115b)? In order to answer this question JP have introduced the following additional invariants

$$\alpha(x, y) = \sum_{i=1}^{N} \phi_i^*(x)\dot{\phi}_i(y), \qquad \beta(x, y) = \sum_{i=1}^{N} \dot{\phi}_i^*(x)\phi_i(y). \tag{4.3.120a, b}$$

The bilinear expressions ψ, α and β are analogous to the pseudospin generators discussed in section 4.3.2. These variables may be shown to satisfy the classical equations of motion

$$\dot{\psi}(x, y) = \alpha(x, y) + \alpha^*(y, x), \tag{4.3.121a}$$

$$\dot{\alpha}(x, y) = (\beta - 2\psi\,\partial V/\partial\psi)(x, y), \qquad \dot{\beta}(x, y) = 2((\partial V/\partial\psi)\,\alpha + \alpha^*\,\partial V/\partial\psi)(x, y). \tag{4.3.121b, c}$$

In the static field approximation, eq. (4.3.121b) gives

$$(\psi\,\partial V[\psi]/\partial\psi - \tfrac{1}{2}\beta)(x, y) = 0. \tag{4.3.122}$$

On the other hand the effective collective field equation (4.3.115b) in the static field limit simplifies to

$$\psi\,\partial V/\partial\psi = N^2/2\psi, \tag{4.3.123}$$

and will agree with (4.3.122) if the condition

$$\beta\psi = N^2 \tag{4.3.124}$$

is satisfied. Equation (4.3.124) is a matrix equation which when written more explicitly reads

$$\int \mathrm{d}z\, \beta(x, z)\psi(z, y) = N^2\delta(x - y)\,. \tag{4.3.125}$$

Thus we are again led to the conclusion that the large-N collective field solutions may be obtained from the original classical equations subject to some special constraints.

The classical solution may be now chosen as

$$\phi_i^{\mathrm{cl}}(x, t) = A_i\, \mathrm{e}^{-\mathrm{i}(k_i x - \omega_i t)}\,, \quad i = 1, 2, \ldots, N\,, \tag{4.3.126}$$

where A_i, k_i and ω_i are constants which are taken to be real. Substituting (4.3.126) in (4.3.119) yields

$$\omega_i = [k_i^2 + m^2 + (g/2N)A^2]^{1/2} \equiv \omega(k_i, A^2)\,, \tag{4.3.127}$$

$$A^2 \equiv \sum_{i=1}^{N} A_i A_i\,. \tag{4.3.128}$$

Using the constraint equation (4.3.124) and choosing the periodic boundary condition $k_i = 2\pi i/L$, one then obtains

$$A_i^2 = (N/L)[1/\omega(k_i, A^2)]\,. \tag{4.3.129}$$

Defining $\sigma_0 = A^2$, one can obtain from (4.3.129)

$$\sigma_0 = \frac{N}{L} \sum_{i=1}^{N} \frac{1}{\omega(k_i, \sigma_0)}\,, \tag{4.3.130}$$

which in the large-N and infinite-volume ($L \to \infty$) limits agrees with the gap equation (4.3.118). The classical bilocal field

$$\psi_{\mathrm{cl}}(x, y) = \sum_{i=1}^{N} \phi_i^{*\mathrm{cl}}(x, t)\phi_i^{*\mathrm{cl}}(y, t) = \sum_{i=1}^{N} A_i^2\, \mathrm{e}^{\mathrm{i}k_i(x-y)}\,, \tag{4.3.131}$$

also coincides in the limits $N \to \infty$, $L \to \infty$ with the collective field $\psi_0(x, y)$. Thus the classical field equations give directly the equal-time correlation functions of large-N quantum systems.

The GS energy may be obtained by substituting in the classical energy functional

$$H = \int \mathrm{d}x \left[\tfrac{1}{2}\dot{\phi}^*\dot{\phi} + \tfrac{1}{2}\phi^*\left(-\partial_x^2 + m^2 + \frac{g}{2N}\,\phi^*\phi\right)\phi - \frac{g}{8N}\,(\phi^*\phi)^2 \right], \tag{4.3.132}$$

the classical solution $\phi = \phi^{\mathrm{cl}}$. The result is [41]

$$E_{\mathrm{cl}} = L\Big(-\frac{g}{8N}(A^2)^2 + 2N\int\frac{\mathrm{d}k}{2\pi}\,\omega(k, A^2)\Big), \tag{4.3.133}$$

which is in complete agreement with the large-N GS energy obtained by the effective action approach.

Jevicki and Levine [59] have made a similar investigation on the O($2N+1$) nonlinear σ model. The conclusions are essentially the same. The CP^N model [177] has also been studied by these authors.

The general collective field formalism is applicable to Yang–Mills gauge theories. There all possible gauge invariant phase factors may be regarded as the collective variables. This problem has been investigated by Sakita [140]. Jevicki and Sakita [142] have subsequently generalized the ideas of the collective field method to study euclidean field theories. The idea is to consider the euclidean functional integral and make transformation to invariant variables. This transformation gives rise to a jacobian. The final effective action comprises two parts: the classical action and the $\hbar$-dependent quantum contribution coming from the jacobian. The stationary point of the effective action then provides the large-N behaviour of the system under consideration.

The authors have studied the σ model and the U(N) Yang–Mills gauge theory. In the case of the σ-model the bilocals in the original fields serve as the invariant collective fields while in the U(N) Yang–Mills theory the overcomplete set of invariants is given by the general Wilson loop variables. In the latter case the stationary collective field equation turns out to be an integrated form of the Makeenko–Migdal large-N Schwinger–Dyson equation [129].

Andric et al. [60] have studied the large-N limit in Sp(N) invariant quantum mechanical matrix models. Developing a saddle-point technique through the collective field method and following the same procedure as has been done in the SU(N) symmetric matrix model they have shown that the semiclassical approximation yields the collective large-N behaviour. This investigation is important because it counters the disturbing conclusion made by Carvalho and Fateev [61] that the large-N limit of the symplectic model is not of semiclassical origin. Jevicki and Levine [58] have shown that the large-N vacuum of an SU(N) symmetric quantum theory can be described in terms of the classical equation of motion supplemented by some special boundary conditions.

5. $1/N$ expansions with improved convergence

As we have already remarked, despite the proven efficacy of the $1/N$ expansion for a number of problems, the expansion is often found to have poor convergence, particularly for the states with higher values of the radial quantum number n. With a view to circumventing this convergence problem various attempts have been made to modify the standard large-N methodology. In the present section we shall discuss these modified $1/N$ expansion schemes.

5.1. The shifted $1/N$ expansion

In order to achieve better convergence, Sukhatme and Imbo [67] proposed a novel modified approach called the shifted $1/N$ expansion. This brings in an extra degree of freedom a in the expansion parameter, which is thus modified to $1/\bar{k} = 1/(k-a) = 1/(N+2l-a)$. The shift parameter a is chosen on physical grounds and is found to depend linearly on the radial quantum number. Consequently, the shifted $1/N$ expansion entails a drastic improvement on the convergence of the energy series. Sukhatme

and Imbo developed the formalism for the power-law potential $V(r) = Ar^{\nu}$. Later, the method was generalized by Imbo, Pagnamenta and Sukhatme [68] to deal with any spherically symmetric potential.

In terms of the shift parameter a the N-dimensional radial Schrödinger equation (3.1.1) reads

$$-\frac{1}{2}\frac{d^2u}{dr^2} + \bar{k}^2\left(\frac{[1-(1-a)/\bar{k}][1-(3-a)/\bar{k}]}{8r^2} + \tilde{V}(r)\right)u(r) = Eu(r)\,, \tag{5.1.1}$$

where, as we have already mentioned, $\bar{k} = k - a = N + 2l - a$, and $\tilde{V}(r) = V_N(r)/\bar{k}^2$. In the limit of large $\bar{k}$ $(N \to \infty)$, the effective potential $V_{\text{eff}}(r)$, the energy eigenvalue to leading order and the equation determining r_0 are given respectively by (3.1.3), (3.1.4) and (3.2.2) with k replaced by $\bar{k}$. Introducing the variable

$$x = (\sqrt{\bar{k}}/r_0)(r - r_0)\,, \tag{5.1.2}$$

and making a Taylor series expansion around $x = 0$ one can transform (5.1.1) to the perturbed oscillator equation

$$\begin{aligned}\{&-\tfrac{1}{2}\,d^2/dx^2 + \tfrac{1}{2}\omega^2x^2 + \varepsilon_0 + [(1/\sqrt{\bar{k}})(\varepsilon_1 x + \varepsilon_3 x^3) + (1/\bar{k})(\varepsilon_2 x^2 + \varepsilon_4 x^4)\\ &+ (1/\bar{k}^{3/2})(\delta_1 x + \delta_3 x^3 + \delta_5 x^5) + (1/\bar{k}^2)(\delta_2 x^2 + \delta_4 x^4 + \delta_6 x^6) + \cdots]\}\Psi(x) = (Er_0^2/\bar{k})\Psi(x)\,,\end{aligned} \tag{5.1.3}$$

where

$$\omega = (\tfrac{3}{4} - r_0^4 V''(r_0)/\bar{k}^2)^{1/2} = \tfrac{1}{2}(3 + r_0 V''(r_0)/V'(r_0))^{1/2}\,, \tag{5.1.4a}$$

$$\varepsilon_0 = \tfrac{1}{8}\bar{k} - \tfrac{1}{4}(2-a) + [(1-a)(3-a)/8\bar{k}] + r_0^2\bar{k}\tilde{V}(r_0)\,, \tag{5.1.4b}$$

$$\begin{aligned}&\varepsilon_1 = \tfrac{1}{2}(2-a)\,, \qquad \varepsilon_2 = -\tfrac{3}{4}(2-a)\,, \qquad \varepsilon_3 = -\tfrac{1}{2} + r_0^5 V'''(r_0)/6\bar{k}^2\,,\\ &\varepsilon_4 = \tfrac{5}{8} + r_0^6 V^{\text{IV}}(r_0)/24\bar{k}^2\,,\end{aligned} \tag{5.1.4c}$$

$$\begin{aligned}&\delta_1 = -\tfrac{1}{4}(1-a)(3-a)\,, \qquad \delta_2 = \tfrac{3}{8}(1-a)(3-a)\,, \qquad \delta_3 = (2-a)\,, \qquad \delta_4 = -\tfrac{5}{4}(2-a)\,,\\ &\delta_5 = -\tfrac{3}{4} + r_0^7 V^{\text{V}}(r_0)/120\bar{k}^2\,, \qquad \delta_6 = \tfrac{7}{8} + r_0^8 V^{\text{VI}}(r_0)/720\bar{k}^2\,.\end{aligned} \tag{5.1.4d}$$

To eq. (5.1.3) one can now apply the logarithmic perturbation theory or the RSPT (as discussed in section 3.2.1) to generate $1/\bar{k}$ expansions for the energy spectrum and the wavefunctions. The energy spectrum obtained by Imbo et al. [68] by rearranging the fourth-order RSPT results reads

$$\begin{aligned}E_{nl} = \sum_{m=-2}^{\infty} \bar{k}^{-m} E_{nl}^{(m)} &= (N+2l-a)^2[1/8r_0^2 + \tilde{V}(r_0)] + (N+2l-a)[(n+\tfrac{1}{2})\omega/r_0^2 - (2-a)/4r_0^2]\\ &+ (1/r_0^2)\{\tfrac{1}{8}(1-a)(3-a) + (1+2n)\tilde{\varepsilon}_2 + 3(1-2n+2n^2)\tilde{\varepsilon}_4\\ &- (1/\omega)[\tilde{\varepsilon}_1^2 + 6(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_3 + (11+30n+30n^2)\tilde{\varepsilon}_3^2]\}\end{aligned}$$

$$
\begin{aligned}
&+[1/(N+2l-a)]((1/r_0^2)\{[(1+2n)\tilde{\delta}_2+3(1+2n+2n^2)\tilde{\delta}_4+5(3+8n+6n^2+4n^3)\tilde{\delta}_6]\\
&-(1/\omega)[(1+2n)\tilde{\varepsilon}_2^2+12(1+2n+2n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4+2(21+59n+51n^2+34n^3)\tilde{\varepsilon}_4^2\\
&+2(11+30n+30n^2)\tilde{\varepsilon}_3\tilde{\delta}_3+10(13+40n+42n^2+28n^3)\tilde{\varepsilon}_3\tilde{\delta}_3]\\
&+(1/\omega^2)[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n+30n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2+24(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4\\
&+8(31+78n+78n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4+12(57+189n+225n^2+150n^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]\\
&-(1/\omega^3)[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n+30n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3\\
&+30(31+109n+141n^2+94n^3)\tilde{\varepsilon}_3^4]\})+O(1/(3+2l-a)^2)\,,
\end{aligned}
\tag{5.1.5}
$$

where

$$
\tilde{\varepsilon}_j=\varepsilon_j/(2\omega)^{j/2}\,,\qquad \tilde{\delta}_j=\delta_j/(2\omega)^{j/2}\,. \tag{5.1.6}
$$

The wavefunction up to $O(\bar{k}^2)$ for the state $n=0$ was calculated by the same authors by the logarithmic perturbation theory. The result is

$$
\begin{aligned}
\Psi_0(x)\sim e^{-\omega x^2/2}[&1+Q_1+Q_1^2/2!+Q_1^3/3!+Q_1^4/4!+Q_2+Q_3+Q_4\\
&+Q_2^2/2!+Q_1Q_2+Q_1Q_3+\tfrac{1}{2}Q_1^2Q_2+O(\bar{k}^{-5/2})]\,,
\end{aligned}
\tag{5.1.7}
$$

where

$$
Q_1(x)=-\frac{2}{(N+2l-a)^{1/2}}\left[\left(\frac{\varepsilon_1}{2\omega}+\frac{\varepsilon_3}{2\omega^2}\right)x+\frac{\varepsilon_3}{6\omega}x^3\right], \tag{5.1.8a}
$$

$$
Q_2(x)=\frac{1}{(N+2l-a)}\left\{-2\left[\left(\frac{\varepsilon_2}{4\omega}+\frac{3\varepsilon_4}{8\omega^2}\right)x^2+\frac{\varepsilon_4}{8\omega}x^4\right]+4\left[\left(\frac{\varepsilon_1\varepsilon_3}{8\omega^3}+\frac{7\varepsilon_3^2}{32\omega^4}\right)x^2+\frac{\varepsilon_3^2}{32\omega^3}x^4\right]\right\}, \tag{5.1.8b}
$$

$$
\begin{aligned}
Q_3(x)=\frac{1}{(N+2l-a)^{3/2}}\Bigg\{&-2\left[\left(\frac{\delta_1}{2\omega}+\frac{\delta_3}{2\omega^2}+\frac{\delta_5}{\omega^3}\right)x+\left(\frac{\delta_3}{6\omega}+\frac{\partial_5}{3\omega^2}\right)x^3+\frac{\delta_5}{10\omega}x^5\right]\\
&+4\Bigg[\left(\frac{\varepsilon_1\varepsilon_2}{4\omega^3}+\frac{5\varepsilon_1\varepsilon_4}{8\omega^4}+\frac{\varepsilon_2\varepsilon_3}{2\omega^4}+\frac{3\varepsilon_3\varepsilon_4}{2\omega^5}\right)x\\
&+\left(\frac{\varepsilon_1\varepsilon_4}{12\omega^3}+\frac{3\varepsilon_3\varepsilon_4}{8\omega^4}+\frac{\varepsilon_2\varepsilon_3}{12\omega^3}\right)x^3+\frac{\varepsilon_3\varepsilon_4}{20\omega^4}x^5\Bigg]\\
&-8\left[\left(\frac{\varepsilon_1^2\varepsilon_3}{8\omega^5}+\frac{17\varepsilon_1\varepsilon_3^2}{32\omega^6}+\frac{5\varepsilon_3^3}{8\omega^7}\right)x+\left(\frac{\varepsilon_1\varepsilon_3^2}{16\omega^5}+\frac{13\varepsilon_3^3}{96\omega^6}\right)x^3+\frac{\varepsilon_3^3}{80\omega^5}x^5\right]\Bigg\},
\end{aligned}
\tag{5.1.8c}
$$

$$Q_4(x) = \frac{1}{(N+2l-a)^2}\left\{-2\left[\left(\frac{\delta_2}{4\omega}+\frac{3\delta_4}{8\omega^2}+\frac{15\delta_6}{16\omega^3}\right)x^2+\left(\frac{\delta_4}{8\omega}+\frac{5\delta_6}{16\omega^2}\right)x^4+\frac{\delta_6}{12\omega}x^6\right]\right.$$
$$+4\left[\left(\frac{\varepsilon_1\delta_3}{8\omega^3}+\frac{7\varepsilon_1\delta_5}{16\omega^4}+\frac{\delta_1\varepsilon_3}{8\omega^3}+\frac{7\varepsilon_3\delta_3}{16\omega^4}+\frac{49\varepsilon_3\delta_5}{32\omega^5}+\frac{\varepsilon_2^2}{16\omega^3}+\frac{3\varepsilon_2\varepsilon_4}{8\omega^4}+\frac{21\varepsilon_4^2}{32\omega^5}\right)x^2\right.$$
$$\left.+\left(\frac{\varepsilon_1\delta_5}{16\omega^3}+\frac{\varepsilon_3\delta_3}{16\omega^3}+\frac{11\varepsilon_3\delta_5}{32\omega^4}+\frac{\varepsilon_2\varepsilon_4}{16\omega^3}+\frac{11\varepsilon_4^2}{64\omega^4}\right)x^4+\left(\frac{\varepsilon_3\delta_5}{24\omega^3}+\frac{\varepsilon_4^2}{48\omega^3}\right)x^6\right]$$
$$-8\left[\left(\frac{3\varepsilon_1\varepsilon_2\varepsilon_3}{16\omega^5}+\frac{7\varepsilon_2\varepsilon_3^2}{16\omega^6}+\frac{7\varepsilon_1\varepsilon_3\varepsilon_4}{8\omega^6}+\frac{123\varepsilon_3^2\varepsilon_4}{64\omega^7}+\frac{\varepsilon_1^2\varepsilon_4}{16\omega^5}\right)x^2\right.$$
$$\left.+\left(\frac{3\varepsilon_2\varepsilon_3^2}{64\omega^5}+\frac{47\varepsilon_3^2\varepsilon_4}{128\omega^6}+\frac{3\varepsilon_1\varepsilon_3\varepsilon_4}{32\omega^5}\right)x^4+\frac{\varepsilon_3^2\varepsilon_4}{32\omega^5}x^6\right]$$
$$\left.+16\left[\left(\frac{3\varepsilon_1^2\varepsilon_3^2}{32\omega^7}+\frac{29\varepsilon_1\varepsilon_3^3}{64\omega^8}+\frac{308\varepsilon_3^4}{512\omega^9}\right)x^2+\left(\frac{5\varepsilon_1\varepsilon_3^3}{128\omega^7}+\frac{99\varepsilon_3^4}{1024\omega^8}\right)x^4+\frac{5\varepsilon_3^4}{768\omega^7}x^6\right]\right\}. \quad (5.1.8d)$$

To get the final answers for the energy and the wavefunction one will now have to make a choice for the shift parameter. The accuracy of the result will of course depend on this choice. The prescription $a=0$ obviously corresponds to the unshifted $1/N$ expansion. To motivate the choice of a Sukhatme and Imbo [67] considered the power-law potential $V(r)=Ar^\nu$. If one imposes the condition

$$E_{nl}^{(-1)}=0\,, \quad (5.1.9)$$

which dictates in general

$$a=2-2(2n+1)\omega=2-(2n+1)[3+r_0V''(r_0)/V'(r_0)]^{1/2}\,, \quad (5.1.10)$$

then the equation determining r_0 is given by

$$N+2l-2+(2n+1)[3+r_0V''(r_0)/V'(R_0)]^{1/2}=[4r_0^3V'(r_0)]^{1/2}\,, \quad (5.1.11)$$

and the energy spectrum for the power-law potential for which $\omega=\frac{1}{2}\sqrt{\nu+2}$ reads explicitly (for $N=3$)

$$E_{nl}=(3+2l-a)^{(\nu-2)/(\nu+2)}(4\nu A)^{2/(\nu+2)}\left\{\frac{(3+2l-a)(\nu+2)}{8\nu}-\frac{(\nu+1)(\nu-2)}{12^2(3+2l-a)}\right.$$
$$\times\left[(1+6n+6n^2)-\frac{1}{12(3+2l-a)\sqrt{\nu+2}}[(\nu+1)(\nu-2)+(7\nu^2-31\nu-62)n\right.$$
$$\left.\left.+(5\nu^2-29\nu-58)(3n^2+2n^3)]+O(1/(3+2l-a)^2)\right]\right\}, \quad (5.1.12)$$

where

$$a=2-(2n+1)\sqrt{\nu+2}\,, \quad (5.1.13)$$

and which for both the harmonic oscillator ($\nu=2$) and the Coulomb ($\nu=-1$) potentials yields the

exact results directly in the leading order, with all the higher order terms vanishing identically. Thus, Sukhatme and Imbo [67] made the interesting observation that if the parameter a is chosen according to the prescription (5.1.9) then the higher order contributions in the case of the harmonic oscillator and the Coulomb potentials are all squeezed into the zeroth order term. For other potentials the situation is understandably not so exciting, but including the first few terms of the shifted expansion is still expected to give quite accurate results. Furthermore, since a, as given by (5.1.10), depends linearly on the radial quantum number n, a remarkable improvement on the convergence of the energy series is anticipated.

For the $n = 0$ state of the power-law potential Imbo et al. [68] carried out the energy calculation to one more order than that quoted in (5.1.12)

$$E_0 = (3+2l-a)^{(\nu-2)/(\nu+2)}(4\nu A)^{2/(\nu+2)} \times \left\{ \frac{(3+2l-a)(\nu+2)}{8\nu} - \frac{(\nu+1)(\nu-2)}{12^2(3+2l-a)} \left[1 - \frac{(\nu+1)(\nu-2)}{12(3+2l-a)\sqrt{\nu+2}} + \frac{317\nu^4 - 166\nu^3 - 1923\nu^2 - 15364\nu - 17164}{225(12(3+2l-a)\sqrt{\nu+2})^2} + O\left(\frac{1}{(3+2l-a)^3} \right) \right] \right\}. \tag{5.1.14}$$

The authors compared the results of (5.1.12) and (5.1.14) with those obtained by numerically solving the Schrödinger equation for various values of ν. For $-1 \leq \nu \leq 2$ the $n = 0$ results are accurate to one part in 1000. It is apparent from (2.1.12) that as ν increases the higher order corrections become larger and larger invalidating the expansion itself. For $\nu = -2$ one encounters again a similar problem. For $\nu = 10$ the expression (5.1.14) for the GS ($n = l = 0$) gives however an accuracy of 96.6%, the accuracy of the unshifted expansion to the same order being only 79%. Again for $\nu = -1.5$, the GS energy as calculated from (5.1.14) is 99% accurate, the corresponding accuracy in the unshifted case being 52.1%. Expression (5.1.12) is found to yield remarkably accurate results even for fairly large values of n. For detailed results the reader is referred to the original article [68].

Imbo et al. [68] also performed calculations for the logarithmic potential $V(r) = A \ln(r/b)$ which has found important application in quarkonium physics. The energy eigenvalues are given by

$$E_{nl} = A\{\ln(\bar{k}/2b\sqrt{A}) + \tfrac{1}{2} + (1/18\bar{k}^2)(1 + 6n + 6n^2) + (1/108\sqrt{2}\bar{k}^3)[1 + 31n + 29(3n^2 + 2n^3)] + O(1/\bar{k}^4)\}, \tag{5.1.15}$$

where

$$\bar{k} = k - a = N + 2l - a, \quad a = 2 - (2n+1)\sqrt{2}. \tag{5.1.16}$$

If the mass of the particle m were retained in (5.1.1), then it would have appeared only in the ln term of (5.1.15) showing, as noted by the authors [68], that the spacing between energy levels is independent of mass. The low-lying ($c\bar{c}$) and ($b\bar{b}$) states do approximately satisfy this behaviour. This is indeed a strong point in favour of using the logarithmic potential as an interquark potential. Imbo et al. compared their results with the exact numerical ones for $N = 3$. Excellent agreement has been observed.

Imbo et al. [68] further tested the accuracy of their method by applying it to various other potentials, namely (i) the Martin potential [178], $V(r) = 6.8698r^{0.1} - 8.064$, (ii) the Cornell potential [179], $V(r) = r/2.34^2 - 0.52/r$, (iii) the Morse potential [180] $V(r) = 10(e^{-2r} - 2e^{-r})$, (iv) the anharmonic oscillator, $V(r) = r^2 + r^4$. The first two of the above potentials have proved important in charmonium

spectroscopy, and (iii) and (iv) are useful in atomic and molecular physics. For the GS energy the $1/\bar{k}$ expansion results have been compared with the numerically obtained values and the agreement has been satisfactory. The authors finally considered an example $[V(r) = r^2 - r^3/10]$ in which the potential possesses both a maximum and a minimum. The accuracy of the $1/\bar{k}$ expansion results has again turned out to be excellent. In a subsequent paper [69] the $1/\bar{k}$ expansion was employed to study the Yukawa potential.

Besides carrying out energy calculations, Imbo et al. [68] determined the GS wavefunctions for the harmonic oscillator, the anharmonic oscillator and the Morse potential. Though accurate near the minimum r_0, these wavefunctions do not agree well for both small and large r-values with the exact curves obtained by numerically solving the Schrödinger equation. Imbo and Sukhatme [70] have subsequently suggested a modification in the leading-order term of the wavefunction expansion. The idea is to incorporate the known limiting behaviour of the wavefunction with respect to r and to demand that exact expressions be obtained for the harmonic oscillator and the Coulomb potentials. Therefore they again considered the power-law potential $V(r) = \mathrm{sgn}(\nu) A r^{\nu}$, $A > 0$. The N-dimensional radial Schrödinger equation (5.1.1) then admits exact solutions for $\nu = 2$ and $\nu = -1$. The results are, for the harmonic oscillator ($\nu = 2$)

$$u_{nl} \sim r^{(k-1)/2} \, \mathrm{e}^{-(r/r_c)^2/2} \, L_n^{(k-2)/2}((r/r_c)^2) , \tag{5.1.17}$$

$$E_{nl} = E_c \bar{k} ; \tag{5.1.18}$$

for the Coulomb potential ($\nu = -1$)

$$u_{nl} \sim r^{(k-1)/2} \, \mathrm{e}^{-(r/r_c)/\bar{k}} \, L_n^{(k-2)}(2r/r_c\bar{k}) , \tag{5.1.19}$$

$$E_{nl} = -E_c/\bar{k}^2 , \tag{5.1.20}$$

where

$$\bar{k} = k - a , \quad a = 2 - 2(2n+1)\omega , \tag{5.1.21}$$

$$\omega = \tilde{\omega}/2 , \qquad \tilde{\omega} = \sqrt{\nu + 2} , \tag{5.1.21a}$$

$$r_c = (2A)^{-1/(\nu+2)} , \tag{5.1.22}$$

$$E_c = A r_c^{\nu} , \tag{5.1.23}$$

and L_n^m are the generalized Laguerre polynomials.

The limiting behaviour of $u_{nl}(r)$ for any arbitrary value of ν can be easily obtained from the Schrödinger equation

$$u_{nl}(r) \xrightarrow[r \to 0]{} r^{(k-1)/k} , \tag{5.1.24}$$

$$\begin{aligned} u_{nl}(r) &\xrightarrow[r \to \infty]{} \exp\{-[2/(\nu+2)](r/r_c)^{(\nu+2)/2}\} , \quad \nu \geq 0 , \\ &\xrightarrow[r \to \infty]{} \exp(-\sqrt{2|E_{nl}|}\, r) , \quad -2 < \nu \leq 0 . \end{aligned} \tag{5.1.25}$$

The leading-order term in the $1/\bar{k}$ expansion of the wavefunction is given on the other hand by

$$u_{nl}^{\bar{k}\to\infty}(r) \sim e^{-\omega x^2/2} H_n(\sqrt{\omega}x), \tag{5.1.26}$$

where ω is given by (5.1.21a), and x is defined by (5.1.2). r_0 is the value of r where the effective potential $(1/8r^2 + \mathrm{sgn}(\nu)\, Ar^{\nu})$ has the minimum and the H_n are the Hermite polynomials.

The strategy of Imbo and Sukhatme is now to find a function that would yield the exact expressions (5.1.17) and (5.1.19) for $\nu = 2$ and $\nu = -1$ respectively, would show the limiting behaviour (5.1.24) and (5.1.25) and would also reproduce (5.1.26) in the limit $k\to\infty$ and for r near r_0. Their prescription is

$$\tilde{u}_{nl}^{\bar{k}\to\infty} \sim r^{(k-1)/2} e^{-\lambda(r/r_0)^{\tilde{\omega}}} L_n^{(k-2)/\tilde{\omega}}(2\lambda(r/r_0)^{\tilde{\omega}}), \tag{5.1.27}$$

where

$$\lambda = \bar{k}/2\tilde{\omega}. \tag{5.1.28}$$

One can verify that (5.1.27) satisfies all the requirements mentioned above, except the $r\to\infty$ behaviour expressed by (5.1.25). However, the large-r behaviour of (5.1.27) is still much better than that of (5.1.26).

The expectation value of the energy corresponding to the wavefunction (5.1.27) for $n = 0$ is now given by

$$\tilde{E}_{0,l}^{\bar{k}\to\infty} = \frac{\bar{k}}{4r_0^2} \frac{(2\lambda)^{2/\tilde{\omega}}}{\Gamma((\bar{k}+2-\tilde{\omega})/\tilde{\omega})} \left(\frac{\tilde{\omega}}{2} \Gamma(\bar{k}/\tilde{\omega}) + \frac{\bar{k}}{(\tilde{\omega}^2-2)(2\lambda)^{\tilde{\omega}}} \Gamma(\bar{k}/\tilde{\omega} + \tilde{\omega} - 1) \right). \tag{5.1.29}$$

So far the discussion has been confined to the power-law potentials. Generalization to any spherically symmetric potential is rather straightforward. One simply has to replace $\tilde{\omega}$ by $[3 + r_0 V''(r_0)/V'(r_0)]^{1/2}$, where r_0 is to be determined from (5.1.11).

Equation (5.1.27) includes for $n = 0$ some of the contributions of the higher order terms in $1/\bar{k}$ of (5.1.7). The rest of the contributions can also be taken into account and one can then add to (5.1.27) higher order corrections in $1/\bar{k}$. For the power-law potentials Imbo and Sukhatme [70] obtained

$$\begin{aligned} u_{0,l}(r) \sim \tilde{u}_{0,l}^{k\to\infty}(r)(1 - [(\tilde{\omega}-1)(\tilde{\omega}-2)/6\bar{k}^{1/2}] \\ \times \{(x + \tfrac{1}{6}\tilde{\omega}x^3) - [(\tilde{\omega}-2)/24\bar{k}^{1/2}][(\tilde{\omega}-7)x^2 + \tfrac{1}{6}\tilde{\omega}(\tilde{\omega}-19)x^4 + \tfrac{1}{18}\tilde{\omega}^2(\tilde{\omega}-1)x^6] + O(1/\bar{k})\}). \end{aligned} \tag{5.1.30}$$

Applying a similar procedure to (5.1.14) they obtained for the energy

$$\begin{aligned} E_{0,l} = \tilde{E}_{0,l}^{\bar{k}\to\infty} \Bigg\{ 1 - \frac{(\tilde{\omega}-1)^2(\tilde{\omega}-2)^2(\tilde{\omega}^2-2)}{36(\bar{k}\tilde{\omega})^2} \\ \times \left[11 + \frac{(8\tilde{\omega}^4 - 69\tilde{\omega}^3 + 59\tilde{\omega}^2 + 96\tilde{\omega} - 76)}{6\bar{k}\tilde{\omega}} + O\left(\frac{1}{\bar{k}^2}\right) \right] \Bigg\}. \end{aligned} \tag{5.1.31}$$

The GS energies obtained from (5.1.31) for various values of ν have been compared by Imbo and Sukhatme with the numerically obtained results. The leading-order term $E_{0,l}$ as given by (5.1.29) gives always a better result than the corresponding leading-order term of the simple $1/\bar{k}$ expansion (5.1.14) (see fig. 5.1). But when corrections to order $1/\bar{k}$ are included in both cases, (5.1.31) has been found by the authors to be only slightly better for $-1 \leq \nu \leq 4$. The spirit of Imbo and Sukhatme's approach is simply to bring to the leading-term as many higher order contributions as possible so that one can get a fairly accurate result just by calculating the leading term. For the GS ($n = l = 0$) wavefunction for $V(r) = r^3$ they have found that (5.1.27) provides a remarkable improvement over (5.1.7) (see fig. 5.2). For higher excited states ($n > 0$) the results obtained from (5.1.27) have accuracy 90%. The accuracy however increases with increasing l, as expected.

Because of the remarkable accuracy that it provides, the shifted $1/N$ methodology has attracted considerable attention [72–79]. Chatterjee [72] employed this method to find the energy eigenvalues of an attractive radial Gaussian potential and compared the results with those obtained by the unshifted $1/N$ expansion and also with the hypervirial–Padé [166] results and the accurate numerical values [169] (see table 5.1). It is clear that for $n = 0$ states the unshifted $1/N$ expansion is on a par with the shifted $1/N$ expansion and proves to be an excellent approximation. For large n the shifted $1/N$ expansion turns out to be a much better approximation than the unshifted one and yields fairly accurate results. However, for states very close to the continuum, the shifted $1/N$ expansion too, to order $1/\bar{k}$, cannot claim high efficacy. It fails to predict, for instance, that the states $n = 6$, $l = 2$ and $n = 7$, $l = 0$ are bound states as dictated by accurate numerical work [169] and the hypervirial–Padé analysis [166]. This clearly shows that as one goes up the potential well, the shape of the potential becomes increasingly important. Hence to improve the situation for the bound states lying close to the continuum one should probably include more higher order terms in $1/\bar{k}$.

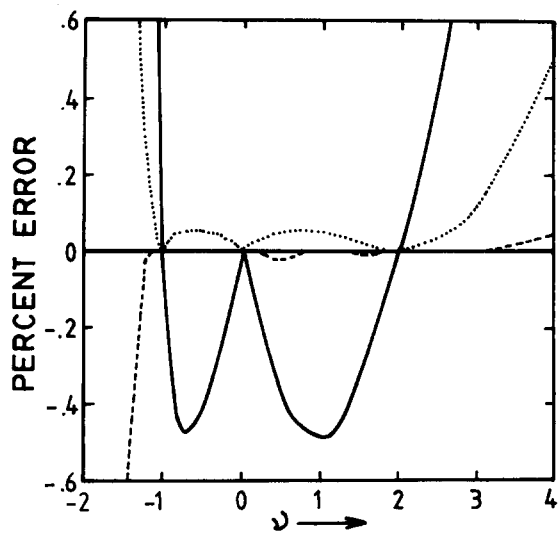

Fig. 5.1. Plots of the percentage error in the calculated value of the GS energy of power-law potentials $V(r) = Ar^{\nu}$ versus the power ν. The solid line comes from energy calculations using the leading term of (5.1.14). The dotted line is drawn using eq. (5.1.29) and corresponds to the improved wavefunction (5.1.27). Equation (5.1.31) yields the dashed line curve (after Imbo and Sukhatme [70], by courtesy of Phys. Rev.).

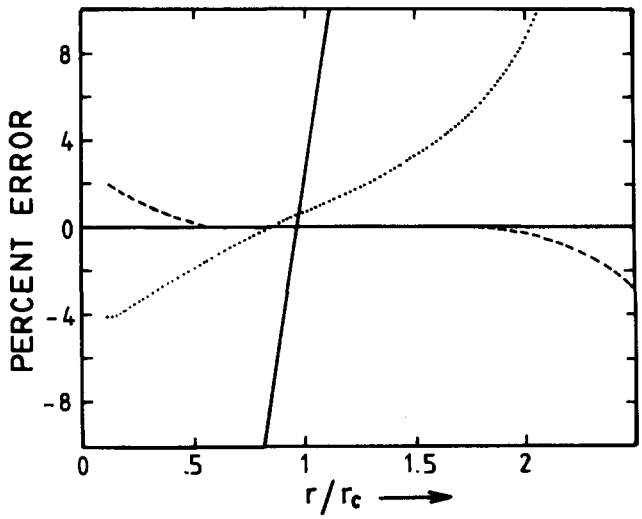

Fig. 5.2. Plots of the percentage error in the calculated GS wavefunction of the potential $V(r) = r^3$ as a function of r/r_c. The solid line corresponds to eq. (5.1.26) and the dotted line corresponds to (5.1.27). Equation (5.1.30) yields the dashed curve (after Imbo and Sukhatme [70], by courtesy of Phys. Rev.).

Dutt, Mukherjee and Varshni [74] studied the bound states of the Hellman potential

$$V(r) = -A/r + B\,e^{-Cr}/r \tag{5.1.32}$$

by using the method of shifted $1/N$ expansion. The potential (5.1.32) represents the electron–core or the electron–ion interaction and has also been used as a model potential for alkali hydride molecules and to study inner-shell ionization problems. Dutt et al. [74] found that, although for small values of the screening parameter C the shifted $1/N$ expansion provides commendable accuracy over a wide range of n and l, the accuracy deteriorates as the strength B and the screening parameter C increase. Moreover, they observed that for certain values of B, C, n and l, the method does not work at all. This happens because for those combinations the large-N effective potential becomes too shallow to contain a local minimum.

Dutt, Mukherjee and Varshni in a subsequent paper [76] studied the exponential cosine screened Coulomb potential $V(r) = -(e^{-\lambda r}/r)\cos(\lambda r)$. The shifted $1/N$ expansion results for the eigenvalues compare favourably with the numerical values of Singh and Varshni [181] for a wide range of n and l and even for large values of the screening parameter λ. Using the normalized wavefunctions, they have also calculated the oscillator strengths for the $1s \rightarrow 2p$ transition for some values of λ. Agreement with the numerically obtained results is fairly good.

Dutt and Varshni [76] computed the shell-binding energies of light to heavy neutral atoms by applying the shifted $1/N$ formalism to the model Yukawa potential with a modified screening parameter. Their results compare very well with those obtained by the hypervirial Padé analysis [182] for the entire range of the atomic number Z up to 84. They also applied this method to the eigenvalue problem of the Lennard-Jones (12, 6) potential [77].

Varshni [78] studied the potential

$$V(r) = \tfrac{1}{2}[r^2 + \lambda r^2/(1 + gr^2)]\,, \tag{5.1.33}$$

Table 5.1

Energy eigenvalues, E_{nl}. First line, the shifted $1/N$ expansion of Imbo et al. [72]; second line, the unshifted $1/N$ expansion [72]; third line, the hypervirial-Padé analysis [166]; fourth line, numerical result [169]; fifth line, the shifted $1/N$ expansion of Maluendes et al. [81].

N	$l=0$	1	2	3	4	5	6	7
0	341.8952	304.4629	268.1108	232.8753	198.7983	165.9283	134.3227	104.0514
	341.8952	304.4628	268.1108	232.8753	198.7983	165.9283	134.3227	104.0513
	341.8952	304.4628	268.1107	232.8753	198.7983	165.9282	134.3226	104.0512
	341.8952	304.4628	268.1107	232.8753	198.7983	165.9282		
	341.8952				198.7983	165.9282		
1	269.6457	235.4513	202.4324	170.6404	140.1362	110.9938	83.3068	57.1971
	269.6415	235.4465	202.4270	170.6340	140.1284	110.9840	83.2935	57.1774
	269.6445	235.4500	202.4313	170.6393	140.1351	110.9929	83.3060	57.1963
	269.6445	235.4500	202.4313	170.6393	140.1351	110.9929		
	269.6445				140.1351	110.9929		
2	203.9969	173.2573	143.8212	115.7649	89.1836	64.2016	40.9905	19.8093
	203.9306	173.1802	143.7299	115.6538	89.0436	64.0163	40.7278	19.3971
	203.9835	173.2443	143.8091	115.7542	89.1750	64.1959	40.9887	19.8128
	203.9835	173.2443	143.8091	115.7542	89.1750	64.1959		
	203.9835			89.1750	64.1959			
3	145.4307	118.4320	92.9186	69.0127	46.8809	26.7680	9.0798	
	145.0832	118.0224	92.4244	68.3977	46.0813	25.6609	7.3964	
	145.3779	118.3840	92.8781	68.9836	46.8681	26.7779	9.1259	
	145.3779	118.3840	92.8781	68.9836	46.8681	26.7779		
	145.3779				46.8681	26.7779		
4	94.5817	71.7197	50.6231	31.5185	14.7685	1.0300		
	93.4194	70.3313	48.9152	29.3284	11.7787	−3.4467		
	94.4577	71.6236	50.5677	31.5211	14.8515	1.2949		
	94.4577	71.6236	50.5677	31.5211	14.8515			
	94.4577				14.8515			
5	52.3080	34.1785	18.3249	5.2989				
	49.2596	30.4652	13.6072	−1.0911				
	52.1436	34.1299	18.4404	5.6729				
	52.1436	34.1299	18.4404	5.6731				
	52.1436			5.6731				
6	19.8693	7.5643	−1.2014					
	12.9244	−1.2173	−13.0949					
	19.9663	8.0833	0.1841					
	19.9663	8.0833	0.2049					
	19.9663		0.2047					
7	−0.4145							
	−15.2660							
	1.3467							
	1.3473							
	1.341							

the one-dimensional version of which is known to occur in various branches of physics. For instance, it serves as a model potential in laser physics [183, 184] and it also occurs in nonlinear field theory [185, 186]. For $\lambda = 0$, (5.1.33) is simply the three-dimensional harmonic oscillator, some of whose states are known to be degenerate. Varshni has shown within the framework of the shifted large-N expansion that for nonzero λ these degeneracies are lifted and, furthermore, when λ is negative the ordering of levels is 1s, 1p, 1d, 2d, 1f, 2p, 1g, 2d, 3s, . . . , where the levels have been labelled by $(n+1)$ and l values, n being the radial quantum number. This is exactly the sequence that a nuclear shell model potential should obey.

Maluendes, Fernandez and Castro [80] suggested an alternative prescription for the choice of the shift parameter a, which in their method becomes order-dependent. They have shown that an order-independent shift parameter leads to asymptotically divergent series. Rewriting the perturbed oscillator equation (5.1.3) in the form

$$\left(-\frac{d^2}{dx^2}+\sum_{m=0}^{\infty}(A_m g^m x^{m+2}+B_m g^{m+1}x^{m+1}+C_m g^{m+2}x^m)\right)\Psi(x)=e\Psi(x)\,, \tag{5.1.34}$$

where

$$g=1/\sqrt{k}\,,\qquad e=[E-k^2V_{\text{eff}}(r_0)]r_0^2g^2\,, \tag{5.1.35a, b}$$

$$A_m=\frac{1}{4}\binom{-2}{m+2}+\frac{g^4r_0^{m+4}}{(m+2)!}V_N^{(m+2)}(r_0),\quad V_N^{(m+2)}(r_0)=\left.\frac{d^{m+2}V_N(r)}{dr^{m+2}}\right|_{r=r_0}, \tag{5.1.35c}$$

$$B_m=\binom{-2}{m+1}(\tfrac{1}{2}a-1)\,,\qquad C_m=\frac{1}{4}\binom{-2}{m}(1-a)(3-a)\,, \tag{5.1.35d, e}$$

and units have been chosen so that $\hbar=2m=1$, the authors have employed the hypervirial perturbation theory using g as the perturbation parameter. They have specifically considered the power-law potential r^ν. The energy eigenvalues can be expanded as

$$e=\sum_{m=0}^{\infty}e_m(a)\bar{k}^m\,, \tag{5.1.36}$$

where $e_0=(2n+1)\sqrt{A_0}$ $(n=0,1,\ldots)$ is the energy spectrum for the unperturbed harmonic oscillator problem. In an actual calculation however, one will have to truncate the infinite series (5.1.36) at some finite order M

$$S_M(a)=\sum_{m=0}^{M}e_m(a)\bar{k}^m\,. \tag{5.1.37}$$

In Sukhatme and Imbo's approach, one sets

$$e_1(a_s)=0\,, \tag{5.1.38}$$

which in the case of the potential r^ν yields

$$a_s=2-(2n+1)\sqrt{\nu+2}\,. \tag{5.1.38a}$$

Maluendes et al. have chosen a by solving the equation

$$e_M(a)=0\,, \tag{5.1.39}$$

for an Mth order calculation. This value of the shift parameter also minimizes $S_M(a)$. The a-value determined in this manner is M-dependent. The energy eigenvalues are finally obtained as

$$S_M E = (\bar{k}/r_0^2) S_M(a_M) + \bar{k}^2 V_{\text{eff}}(r_0) . \tag{5.1.40}$$

The authors have calculated the energy values for $\nu = 1, 4$ taking $M = 25$ and compared their results with the numerical values and also with the unshifted $1/N$ expansion energies and with those obtained by using the order-independent shift of Sukhatme and Imbo. They have found that as the order M is increased the accuracy of their results increases, while that of both the unshifted and the order-independent shifted $1/N$ expansion results deteriorate. However, because of the dependence of A_m on $\bar{k}$ the Hellman–Feynman theorem leads to a very complicated expression in their method. To reduce this difficulty Maluendes et al. in a later paper [81] introduced a dummy perturbation parameter λ in (5.1.34), which now reads

$$\left(-\frac{d^2}{dx^2} + \sum_{m=0}^{\infty} A_m g^m \lambda^m x^{m+2} + B_m g^{m+1} \lambda^{m+1} x^{m+1} + C_m g^{m+2} \lambda^{m+2} x^m\right) \Psi(x) = e\Psi(x) , \tag{5.1.41}$$

where the parameter λ has to be ultimately set equal to unity. The eigenvalues can now be expanded as

$$e = \sum_{m=0}^{\infty} e_m(a) \lambda^m , \tag{5.1.42}$$

where again $e_0 = (2n+1)\sqrt{A_0}$ $(n = 0, 1, \ldots)$. The Hellman–Feyman theorem now looks much simpler and all the coefficients e_m can be easily calculated. Equation (5.1.42) may then be approximated as

$$S_M(a) = \sum_{m=0}^{M} e_m(a) , \tag{5.1.43}$$

and the shift parameter is (as before) taken as that root of

$$e_M(a) = 0 \tag{5.1.44}$$

which gives the smallest absolute value of $\partial S_M/\partial a$. The authors have calculated the first two energy levels of the one-dimensional double-well potential $V(x) = -z^2x^2 + x^4$ and of the attractive radial Gaussian potential, $-400\exp(-r^2)$. In the former case they have obtained results for $z = 0.5$ with various values of M up to 26. The accuracy has been found to increase with M. This is because with increasing M the effective potential tends more and more to the harmonic oscillator type. They have also carried out calculations for several z values (taking $M = 26$) and observed that the accuracy increases with increasing z. This is again expected because as z increases each well looks more and more like a harmonic potential. In fact even the unshifted $1/N$ expansion provides sufficient accuracy for $z^2 \geq 20$. This is an interesting behaviour which makes the large-N expansion complementary to most of the other available methods which are known to work better for smaller values of z. For the Gaussian potential they have included only the first 21 terms and compared their results with the exact numerical ones [169] and with those obtained by the hypervirial–Padé analysis [166].

In table 5.1 we display the results of the Maluendes et al. together with the unshifted $1/N$ expansion results and those determined using the order-independent shift of Sukhatme and Imbo. The hypervirial–Padé results and the accurate numerical values are also included in the table for comparison. The accuracy of the results of Maluendes et al. is clearly evident. For instance the state $n = 6$, $l = 2$ is now correctly predicted to be a bound state.

Papp [82, 83] suggested another prescription for the choice of the shift parameter. He [82] proved that the accuracy of the shifted $1/N$ expansion can be enhanced by choosing the shift parameter such that the sum of corrections to the zeroth order energy vanishes order by order. He [83] then used this idea to obtain the energy eigenvalues of the power-law potential $V_n(r) = \gamma(n)/r^n$ and showed that for $n = 1$ and $n = -2$ the exact results are obtained. However, to make a detailed comparative study of the efficacy of this method, it needs to be applied to several other potentials which have already been studied by the shifted $1/N$ expansion technique of Sukhatme and Imbo [67].

5.2. *Expansion about singular points in $E(N)$*

Doren and Herschbach [84, 85] (DH) developed an interesting and useful version of the expansion about a fixed finite dimension. They observed that there exist special values of the spatial dimensionality for which the dynamics of a quantum mechanical problem becomes remarkably simplified. They considered two examples. (i) The Yukawa potential, which is a singular potential, and (ii) the anharmonic oscillator, which is an example of an analytic potential. For potentials with inverse power-law singularities at the origin, the energy as a function of the dimensionality, $E(N)$, has divergences at certain values of N. These singularities are poles whose locations can be determined by studying the hamiltonian and the order of these poles can be obtained by a scaling analysis. Since these singularities play a dominant role in dictating the behaviour of the energy in their neighbourhood, the expansion about these singular points emerges as a powerful approach. In the case of the anharmonic oscillator, though there is no singularity in the energy, there are certain dimensions for which the energy can be exactly determined. For instance, $E(N)$ has a zero at $N = 0$. An expansion about $N = 0$ then appears to be a useful way of determining the eigenvalues.

5.2.1. *The Yukawa problem*

The N-dimensional radial Schrödinger equation for the Yukawa problem is given by (3.0.29) with

$$V_N(r) = e^{-\lambda r}/r = e^{-\tilde{\lambda} r/N(N-1)}/r\,, \tag{5.2.1}$$

where $\tilde{\lambda} = N(N-1)\lambda$. This redefinition of the screening parameter proves useful in DH's approach. For $l = 0$ states, one has

$$[-\tfrac{1}{2}\,d^2/dr^2 + (N-3)(N-1)/8r^2 + V_N(r)]u_0(r) = E_0 u_0(r)\,. \tag{5.2.2}$$

An analysis of this equation for $N > 1$ reveals that the solution $u_0(r)$ can be written as

$$u_0(r) \sim r^{(N-1)/2} \sum_{i=0}^{\infty} a_i r^i\,, \quad (r \to 0)\,. \tag{5.2.3}$$

Writing $r^{-(N-1)/2} u_0(r) = R_0(r)$, we have

$$\left[-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{N-1}{r}\,\frac{d}{dr}\right) - \frac{e^{-\tilde{\lambda} r/N(N-1)}}{r}\right] R_0 = E_0 R_0\,, \tag{5.2.4}$$

where $R_0(r)$ is finite everywhere. The energy can be written as

$$E_0(N) = \left(\int_0^\infty R_0^*\left[-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr}\right) - \frac{e^{-\tilde{\lambda}r/N(N-1)}}{r}\right]R_0 r^{N-1}\,dr\right)\left(\int_0^\infty R_0^* R_0 r^{N-1}\,dr\right)^{-1}. \tag{5.2.5}$$

Now if $R_0 = \Sigma_{i=0}^{\infty} a_i r^i$ is a solution of (5.2.4) at $N=1$, then as $r\to 0$, $R_0 \to a_0 \neq 0$ and the two integrals in the numerator of (5.2.5) diverge, leading to an infinite energy. Let us now substitute $R_0 = \Sigma_{i=0}^{\infty} a_i r^i$ in (5.2.4) and equate terms with like powers in r, to arrive at the following recurrence relations which determine the coefficients a_i

$$\tfrac{1}{2}(N-1)a_1 + a_0 = 0\,, \qquad Na_2 + a_1 + [E - \tilde{\lambda}/N(N-1)]a_0 = 0\,, \tag{5.2.6a, b}$$

$$\tfrac{1}{2}(N+n)(n+2)a_{n+2} + Ea_n + \sum_{j=0}^{n+1} \frac{-[\tilde{\lambda}/N(N-1)]^{n+1-j}}{(n+1-j)!}\,a_j = 0\,, \quad n \geq 0\,. \tag{5.2.6c}$$

It is evident from (5.2.6a) that since $a_0 \neq 0$, a_1 should be infinite if $D=1$ and the first derivative of the wavefunction R_0 at $r=0$, being equal to a_1, will consequently diverge at the origin. In fact it is easily seen that all the derivatives of R_0 diverge at the origin and furthermore the denominator of (5.2.5) leads to a higher order singularity.

Instead of throwing away such solutions DH have introduced the scaling transformation

$$r \to \xi = [2/(N-1)]r\,, \tag{5.2.7}$$

which makes all the derivatives of R_0 finite. Equation (5.2.4) now reads

$$\left[-\frac{1}{2}\left(\frac{d^2}{d\xi^2} + \frac{N-1}{\xi}\frac{d}{d\xi}\right) - \frac{(N-1)\,e^{-\tilde{\lambda}\xi/2N}}{2\xi}\right]R_0 = \varepsilon_0 R_0\,, \tag{5.2.8}$$

where

$$\varepsilon_0 = [(N-1)/2]^2 E_0\,. \tag{5.2.9}$$

Therefore if one can now solve (5.2.8) in the limit $N\to 1$ and obtain a finite nonzero ε_0 then one can conclude that E has a pole of order two at $N=1$.

DH have then used the identity

$$\int_0^\infty f_1^* \frac{1}{r} f_2 r^{N-1}\,dr = \frac{1}{N-1}\int_0^\infty f_1^* f_2 \delta(r)\,dr + \int_0^1 [f_1^* f_2 - f_1^*(0)f_2(0)]\frac{1}{r}r^{N-1}\,dr + \int_1^\infty f_1^* f_2 \frac{1}{r} r^{N-1}\,dr\,, \quad \mathrm{Re}\,N > 1\,, \tag{5.2.10}$$

where f_1 and f_2 are arbitrary, finite decaying functions and

$$\int_0^a \delta(r)\,dr = 1\,, \quad a > 0\,. \tag{5.2.11}$$

Equation (5.2.10) can be written in shorthand as

$$1/r = \delta(r)/(N-1) + 1/r\,, \tag{5.2.12}$$

which has, however, meaning only in the sense of integrals, as given in (5.2.10), and where $1/r$ represents the finite part of the integral. Using (5.2.12) in (5.2.8) one then gets

$$-\tfrac{1}{2}\{d^2/d\xi^2 + [\delta(\xi) + (N-1)/\xi]\,d/d\xi + [\delta(\xi) + (N-1)/\xi]\,e^{-\tilde{\lambda}\xi/2N}\}R_0 = \varepsilon_0 R_0\,. \tag{5.2.13}$$

One can now see that eq. (5.2.13) can be solved by straightforward RSPT using $(N-1)$ as the perturbation parameter. DH have actually introduced expansion in powers of $(N-1)/N$, since it is bounded between 0 and 1 for the dimensions of interest $(1 \le N < \infty)$ and since $E(N) \sim N^{-2}$ as $N \to \infty$ [85]. Substituting

$$\varepsilon_0 = \varepsilon_0^{(0)} + \frac{N-1}{N}\,\varepsilon_0^{(1)} + \left(\frac{N-1}{N}\right)^2 \varepsilon_0^{(2)} + \cdots\,, \tag{5.2.14}$$

$$R_0 = R_0^{(0)} + \frac{N-1}{N}\,R_0^{(1)} + \left(\frac{N-1)}{N}\right)^2 R_0^{(2)} + \cdots\,, \tag{5.2.15}$$

in (5.2.13), and collecting the zeroth order terms one gets

$$-\tfrac{1}{2}[d^2/d\xi^2 + \delta(\xi)\,d/d\xi + \delta(\xi)]R_0^{(0)} = \varepsilon_0^{(0)} R_0^{(0)}\,, \tag{5.2.16}$$

which has only one bound state solution. Clearly

$$R_0^{(0)}(\xi) = \sqrt{2}\,e^{-\xi} \tag{5.2.17}$$

is the solution of (5.2.16) with

$$\varepsilon_0^{(0)} = \tfrac{1}{2}\,. \tag{5.2.18}$$

Solving the next higher order equation in the perturbative hierarchy one can obtain

$$\varepsilon_0^{(1)} = \ln(1 + \tilde{\lambda}/4)\,, \tag{5.2.19}$$

and the GS energy to first order in $(N-1)/N$ reads

$$E_0(N) = -\frac{2}{(N-1)^2} + \frac{4}{N(N-1)}\ln(1 + \tfrac{1}{4}N(N-1)\lambda) - O(N^{-2})\,. \tag{5.2.20}$$

Let us now quote the $1/N$ expansion energy [51] of the Yukawa potential (5.2.1) for comparison:

$$E_0 = \sum_{n=-2}^{\infty} N^{-n} E_0^{(n)}\,, \tag{5.2.21}$$

$$E_0^{(-2)} = -2/N^4\,, \qquad E_0^{(-1)} = -4/N^4\,, \qquad E_0^{(0)} = -6/N^4 + \lambda\,, \qquad E_0^{(1)} = -8/N^4\,,$$
$$E_0^{(2)} = -10/N^4 - N^4\lambda^2/8\,, \qquad E_0^3 = -12/N^4 + N^4\lambda^2/8\,, \quad \ldots\,. \tag{5.2.22}$$

If one expands the first two terms of (5.2.20) at $N=1$, one can see that they agree with the $1/N$ expansion energy (5.2.21) up to $O(N^3)$.

Going one order further in $(N-1)/N$ DH have obtained

$$\varepsilon_2 = \ln(1+\tfrac{1}{4}\tilde{\lambda}) - \frac{\tilde{\lambda}}{\tilde{\lambda}+4} - \frac{1}{2}\int_0^\infty R_0^{(0)}\,\frac{1}{\xi}\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + \mathrm{e}^{-\tilde{\lambda}\xi/2}\right)R_0^{(1)}\,\mathrm{d}\xi\,, \tag{5.2.23}$$

which they have evaluated numerically. They have also compared their results with the $1/N$ expansion energies, the perturbative results and the accurate numerical values for different values of $\tilde{\lambda}$ (see fig. 5.3). The accuracy of their results is clearly evident.

DH have also considered excited s states, which we do not discuss here; the reader is referred to the original article.

Other potentials which have inverse power-law singularities at the origin can be studied by this approach. The only difference will be in the value of the dimension about which the expansion has to be made. For example, the energy of the $r^{-3/2}$ potential has a singularity at $N=3/2$ and that of the Coulomb plus linear potential has a singularity at $N=1$.

5.2.2. The anharmonic oscillator

As an example of an analytic potential DH have considered

$$V_N(r) = \tfrac{1}{2}r^2 + gr^4\,. \tag{5.2.24}$$

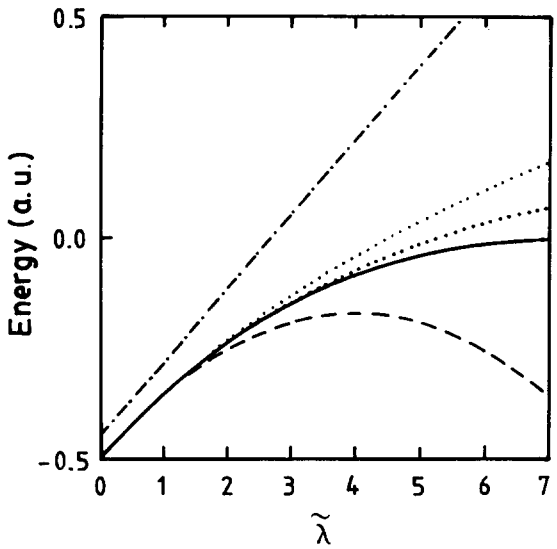

Fig. 5.3. Energy estimates due to two terms in the $(N-1)/N$ series (upper, light dotted curve), three terms in the $(N-1)/N$ series (lower, heavy dotted curve), three terms in the $1/N$ series (dotted–dashed curve), three terms in the $\tilde{\lambda}$ perturbation series (dashed curve), and accurate numerically determined energies (solid curve) as a function of $\tilde{\lambda}$ with $N=3$ fixed (after Doren and Herschbach [84], by courtesy of Phys. Rev.).

The radial equation for the $l = 0$ states then reads

$$HR_0 = E_0 R_0 , \tag{5.2.25}$$

with

$$H = \left(-y\, \mathrm{d}^2/\mathrm{d}y^2 - \tfrac{1}{2}N\, \mathrm{d}/\mathrm{d}y + y + 4gy^2\right), \tag{5.2.26}$$

where the variable y is defined as $y = \frac{1}{2}r^2$. The dimension dependent energy $E_0(N)$ is given by

$$E_0(N) = \frac{\int_0^\infty R_0^*(-y\, \mathrm{d}^2/\mathrm{d}y^2 - \frac{1}{2}N\, \mathrm{d}/\mathrm{d}y + y + 4gy^2)R_0 y^{N/2-1}\, \mathrm{d}y}{\int_0^\infty R_0^* R_0 y^{N/2}\, \mathrm{d}y} . \tag{5.2.27}$$

The denominator of (5.2.27) is clearly singular for Re $N \to 0$ and therefore $E_0(N)$ has a zero at $N = 0$. One can pose a similar argument for other polynomial potentials also. It turns out that the energy of any analytic potential which vanishes at the origin has a zero at $N = 0$. (Thus there does not appear any zero point energy at $N = 0$.) Investigations in this field have been made by several authors [187–189].

DH have used the prescription

$$\int_0^\infty f_1^* \frac{1}{y} f_2 y^{N/2}\, \mathrm{d}y = \frac{2}{N}\int_0^\infty f_1^*(0)\delta(y) f_2(0)\, \mathrm{d}y + \int_0^\infty [f_1^* f_2 - f_1^* f_2 \theta(1-y)]\frac{1}{y}\, y^{N/2}\, \mathrm{d}y , \quad \mathrm{Re}\, N > -2 , \tag{5.2.28}$$

where $\theta(1-y)$ is the Heavyside step function. (5.2.28) can be expressed in shorthand as before

$$1/y = (2/N)\delta(y) + 1/y . \tag{5.2.29}$$

The form of (5.2.28) has prompted the authors to divide equation (5.2.25) throughout by y and consider $y^{-N/2}\, \mathrm{d}y$ as the volume element. Substituting (5.2.29) and introducing the expansions

$$h = y^{-1}H = \sum_{n=0}^{\infty} h^{(n)} N^n , \tag{5.2.30}$$

$$E_0 = \sum_{n=0}^{\infty} E_0^{(n)} N^{n+1} , \tag{5.2.31}$$

$$R_0 = \sum_{n=0}^{\infty} R_0^{(n)} N^n , \tag{5.2.32}$$

in (5.2.25) and then equating the coefficients of equal powers of N on both sides of (5.2.25) one can obtain a series of equations. The first two of these have been considered by DH and are given by

$$h^{(0)} R_0^{(0)} = 2\delta(y) E_0 R_0^{(0)} , \tag{5.2.33a}$$

$$h^{(0)}R_0^{(1)} + h^{(1)}R_0^{(0)} = 2\delta(y)E_0^{(0)}R_0^{(1)} + 2\delta(y)E_0^{(1)}R_0^{(0)} + (1/y)E_0^{(0)}R_0^{(0)}\,, \tag{5.2.33b}$$

where

$$h^{(0)} = -\mathrm{d}^2/\mathrm{d}y^2 - \delta(y)\,\mathrm{d}/\mathrm{d}y + 1 + 4gy\,, \tag{5.2.34a}$$

$$h^{(1)} = -\tfrac{1}{2}(1/y)\,\mathrm{d}/\mathrm{d}y\,. \tag{5.2.34b}$$

One can now easily show that the zeroth order energy is given by

$$E_0^{(0)} = -\tfrac{1}{2}(4g)^{1/3}\,\mathrm{Ai}'[(4g)^{-2/3}]/\mathrm{Ai}[(4g)^{-2/3}]\,, \tag{5.2.35}$$

where $\mathrm{Ai}(z)$ is the Airy function and $\mathrm{Ai}'(z)$ its derivative. For the first-order correction DH have obtained

$$E_1 = -\frac{1}{4}\int_0^\infty \left|\frac{R_0^{(0)}(y)}{R_0^{(0)}(0)}\right|^2 \left(\frac{\mathrm{d}\ln R_0^{(0)}}{\mathrm{d}y} - \frac{\mathrm{d}\ln R_0^{(0)}(0)}{\mathrm{d}y}\right)\frac{\mathrm{d}y}{y}\,, \tag{5.2.36}$$

which they have evaluated numerically. They have obtained results for both $N=1$ and $N=3$ and compared their results with those of the $1/N$ expansion and the perturbative expansion and also with the numerically obtained energies (see fig. 5.4). It is apparent from fig. 5.4 that the accuracy is better when N is smaller. This is indeed in conformity with our expectation. For $N=1$ the energies calculated from the two leading terms of the N-series are much more accurate than those calculated from the two leading terms of the $1/N$ series. Though the accuracy of the N-expansion results decreases with increasing coupling constant, the decrease is slow and is actually slower than that in the case of the perturbative results.

Besides the zero-dimensional case, DH have studied the expansion about $N=-2$, since the quartic anharmonic oscillator should be solvable when $N=-2$, the energy then being independent of the anharmonicity. However, the solutions are not useful in predicting the energies at $N=1$ or 3.

5.3. The supersymmetric $1/N$ expansion

In ordinary large-N expansions one considers the N-dimensional radial Schrödinger equation

$$Hu_n = E_n u_n\,, \tag{5.3.1}$$

where

$$H = -\tfrac{1}{2}\mathrm{d}^2/\mathrm{d}r^2 + \bar{V}(r) = -\tfrac{1}{2}\mathrm{d}^2/\mathrm{d}r^2 + [(k-1)(k-3)/8r^2 + V_N(r)]\,. \tag{5.3.2}$$

Let us rewrite equation (5.3.1) in the form

$$H_+ u_n = \bar{E}_n u_n\,, \tag{5.3.3}$$

where

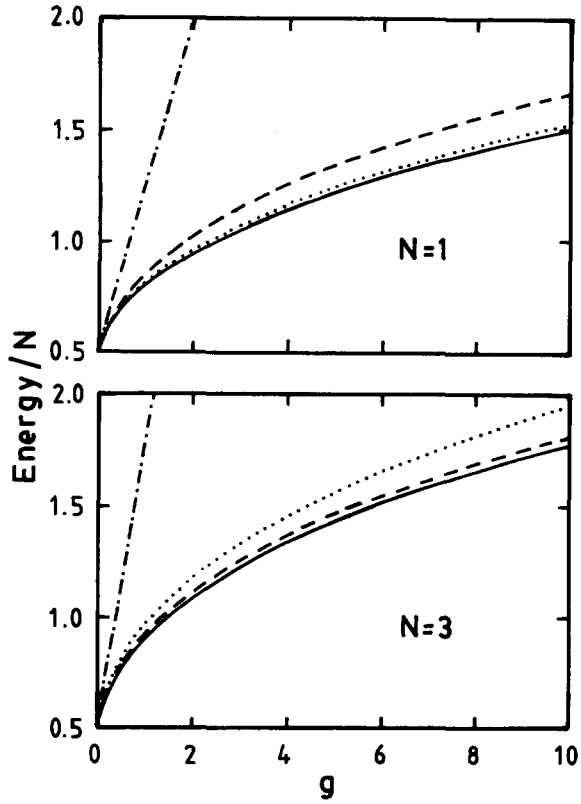

Fig. 5.4. Energy estimates at $N=1$ and $N=3$ from two terms of the expansion at $N=0$ (dotted curve) compared with two terms in the $1/N$ series (dashed curve), two terms in perturbation series (dotted–dashed curve) and numerically determined exact energies (solid curve) (after Doren and Herschbach [84], by courtesy of Phys. Rev.).

$$H_+ = H - E_0 = \tfrac{1}{2}\mathrm{d}^2/\mathrm{d}r^2 + V_+ , \tag{5.3.4}$$

$$V_+ = (k-1)(k-3)/8r^2 + V(r) - E_0 , \tag{5.3.4a}$$

$$\bar{E}_n = E_n - E_0 . \tag{5.3.5}$$

It follows from (5.3.3) that

$$H_+ u_0 = 0 , \tag{5.3.6}$$

which immediately dictates

$$H_+ = \tfrac{1}{2}(-\mathrm{d}^2/\mathrm{d}r^2 + u_0''/u_0) , \tag{5.3.7}$$

or

$$V_+ = \tfrac{1}{2}u_0''/u_0 . \tag{5.3.8}$$

If one now defines the operators

$$Q_{\pm} = (1/\sqrt{2})(\pm \mathrm{d}/\mathrm{d}r - u_0'/u_0)\,, \tag{5.3.9}$$

then one gets

$$Q_+ Q_- = H_+\,, \tag{5.3.10}$$

$$Q_- Q_+ \equiv H_- = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}r^2 + V_-\,, \tag{5.3.11}$$

where

$$V_- = V_+ - (\mathrm{d}/\mathrm{d}r)(u_0'/u_0) = -V_+ + (u_0'/u_0)^2\,. \tag{5.3.11a}$$

One can then show that

$$H_-(Q_- u_n) = \bar{E}_n(Q_- u_n)\,, \tag{5.3.12}$$

which can be rewritten as

$$H_1(Q_- u_n) = E_N(Q_- u_n)\,, \tag{5.3.13}$$

where

$$H_1 = H_- + E_0 = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}r^2 + \bar{V}_1(r)\,, \tag{5.3.14}$$

$$V_1(r) = (\tilde{k}-1)(\tilde{k}-3)/8r^2 + [V_N(r) - (\mathrm{d}/\mathrm{d}r)(R_0'/R_0)]\,, \tag{5.3.14a}$$

$$\tilde{k} = k+2\,, \qquad R_0(r) = r^{-(k-1)/2} u_0(r)\,. \tag{5.3.14b, c}$$

Thus, if u_n is an eigenstate of H belonging to the eigenvalue E_n, then $(Q_- u_n)$ is an eigenstate of H_1 belonging to the same eigenvalue E_n, except for $n=0$. (For $n=0$, the wavefunction $Q_- u_n$ itself vanishes.) Therefore, corresponding to the potential $\bar{V}(r)$ there exists another potential $\bar{V}_1(r)$, called the supersymmetric partner potential, which has the same energy spectrum as $\bar{V}(r)$ except for the GS. The $n=0$ state of $\bar{V}_1(r)$ would thus correspond to the $n=1$ state of $\bar{V}(r)$. One may notice that the supersymmetric partner potential may be regarded as being effectively in $(N+2)$ dimensions. Hence large-N expansions with $\bar{V}_1(r)$ are expected to give much better convergence than those with $\bar{V}(r)$. The method however requires for its applicability knowledge of the GS wavefunction to begin with. The convergence may be further improved by using the shifted $1/N$ expansion with $\bar{V}_1(r)$.

Imbo and Sukhatme [86] used this supersymmetric shifted $1/N$ expansion to obtain the energy of the first excited state of (i) the Hulthen potential, $-\delta\, \mathrm{e}^{-\delta r}/(1-\mathrm{e}^{-\delta r})$, $\delta = 0.05$, (ii) the logarithmically screened Coulomb potential, $-[(2-\delta)\delta^2 r - 2\delta]/2(1+\delta r)^2 \ln(1+\delta r)$, $\delta = 0.05$, and (iii) the quarkonium-type potential, $\frac{1}{2}(r/(2.34)^2 - 0.52/r)$, and compared their results with those obtained by the ordinary shifted $1/N$ expansion. For the first two potentials the GS wavefunctions are known analytically and therefore the supersymmetric partner potential can be obtained rather easily. In the

case of the potential (iii) for which the analytic solution is not available, the authors have obtained the GS wavefunction by numerically solving the Schrödinger equation. In all three cases it has turned out that the supersymmetry motivated shifted $1/N$ expansion provides still faster convergence than the ordinary shifted one. For instance, in the case of the quarkonium potential, the exact first excited state energy of which is 0.568 (in units $\hbar = m = 1$) [179], the shifted $1/N$ expansion with $\bar{V}_1(r)$ gives a value 0.569 just in the zeroth order calculation, while that with $\bar{V}(r)$ leads to the result 0.573 even when two higher order corrections are included. That just the leading-order term of the supersymmetric shifted $1/N$ expansion yields a remarkably high degree of accuracy makes this method extremely useful indeed.

6. Further applications in atomic and molecular physics

6.1. Hydrogen atom in a magnetic field

The study of atomic spectra in the presence of external magnetic fields of realizable intensities constitutes an interesting problem. Unfortunately however, this problem does not admit an exact solution even for the simplest atomic system. Various approximation schemes have therefore been proposed to study the effect of a uniform magnetic field on the hydrogen atom spectra. Numerical calculations are also available on this problem. For a review of the subject the reader is referred to the articles of Garstang [190] and Bayfield [191].

The idea of the $1/N$ expansion was first applied to this problem by Mlodinow and Papanicolaou [42] (MP) who used the pseudo-spin formulation. Their results are accurate for field strengths up to the intermediate region. Bender, Mlodinow and Papanicolaou [88] (BMP) later employed the algebraic recursion method and extended the calculation of MP to higher orders. They showed that a simple Shanks extrapolation of the $1/N$ expansion results provides commendable accuracy even for very strong fields.

The hamiltonian for a hydrogen atom in a uniform magnetic field $\boldsymbol{B}$ applied along the z-direction is given (in units $m = \hbar = 1$) by

$$H = H_0 + H_1 , \tag{6.1.1}$$

$$H_0 = \tfrac{1}{2}[(p_\perp^2 + \tfrac{1}{4}eB^2\rho^2) + p_z^2] - e^2/(\rho^2 + z^2) , \tag{6.1.2}$$

$$H_1 = \tfrac{1}{2}eBL_z , \tag{6.1.3}$$

where e is the electronic charge, L_z is the z-component of the angular momentum (the electronic spin being ignored), $\rho^2 = x^2 + y^2$ and $p_\perp^2 = p_x^2 + p_y^2$. Since H_1 is diagonal in the azimuthal angular momentum basis, it is sufficient to consider H_0. MP have addressed themselves to a more general problem

$$H_0 = \tfrac{1}{2}[(p_\perp^2 + \omega_0^2\rho^2) + p_z^2] + 2\nu e^2(\rho^2 + z^2)^\nu . \tag{6.1.4}$$

However at the end they set $\nu = -1/2$ and $\omega_0 = eB^2/4$ to go over to the original problem.

The original Fock space F can now be considered as the direct product of $\mathrm{F}_\perp$ and F_z, F_z involving the operators z and p_z and $\mathrm{F}_\perp$ the transverse components, for which one can make an analysis similar to that made in section 4.2.3., the only difference being that i appearing in (4.2.21), (4.2.22) and (4.2.23)

is now capable of taking only two values, 1 and 2. The Casimir invariant of the SO(2, 1) algebra associated with $F_\perp$ is

$$C = k'(k'-1)\,, \tag{6.1.5}$$

where

$$k' = \tfrac{1}{2}(|m|+1)\,, \quad |m| = 0, 1, 2, \ldots . \tag{6.1.6}$$

The space $F_\perp$ can now be decomposed into a direct sum of irreducible subspaces $\mathrm{F}_{|m|}$ and the dynamics can be effectively discussed within each subspace. Restricting H_0 to the subspace $\mathrm{F}_{|m|} \otimes \mathrm{F}_z$, MP then obtained

$$\begin{aligned} H_k = {} & \tfrac{1}{2}p_z^2 + \omega(1+\omega_0^2/\omega^2)(k'+\zeta^+\zeta) + \tfrac{1}{2}\omega(1-\omega_0^2/\omega^2)[(2k'+\zeta^*\zeta)^{1/2}\zeta + \zeta^+(2k'+\zeta^+\zeta)^{1/2}] \\ & + (2\nu e^2/\omega^\nu)[2k' + 2\zeta^+\zeta - (2k'+\zeta^+\zeta)^{1/2}\zeta - \zeta^+(2k'+\zeta^+\zeta)^{1/2} + \omega z^2]^\nu\,. \end{aligned} \tag{6.1.7}$$

The parameter ω was introduced in the definition of the creation and annihilation operators (4.2.22) associated with the space $\mathrm{F}_\perp$ and the operators ζ and ζ^+ enter as before through a Holstein–Primakoff representation (4.2.41) of the SO(2, 1) generators. Defining again a rescaled coupling constant g^2 by

$$e^2 = g^2(2k')^{1-\nu} = g^2\alpha^{2(\nu-1)}\,, \tag{6.1.8}$$

where

$$\alpha = 1/(2k')^{1/2} = 1/(|m|+1)^{1/2}\,, \tag{6.1.9}$$

and making the canonical transformation

$$\zeta = \xi + \sinh\phi/\alpha\,, \qquad [\xi, \xi^+] = 1\,, \tag{6.1.10}$$

the hamiltonian (6.1.7) can be written in the form

$$\begin{aligned} H_{k'} = {} & \frac{\mu}{2\alpha^2}\left(\frac{\nu+1}{\nu} + \frac{\nu-1}{\nu}\,\frac{\omega_0^2}{\mu^2}\right) + \mu(2\cosh^2\phi\;\xi^+\xi + \sinh^2\phi) \\ & + \tfrac{1}{2}[p_z^2 + (\mu^2-\omega_0^2)z^2] + \alpha H^{(1)} + \alpha^2 H^{(2)} + \cdots\,, \end{aligned} \tag{6.1.11}$$

where ω appearing in μ and ϕ are to be determined from the equations

$$\begin{aligned} & \mu^2 - \omega_0^2 = 4\nu^2 g^2 \mu^{1-\nu}\,, \\ & \cosh^2\phi = [1 + \tfrac{1}{2}(\nu-1)(\mu^2-\omega_0^2)/\mu^2]^{1/2}\,, \quad \mu \equiv \omega\, e^{2\phi}\,. \end{aligned} \tag{6.1.12}$$

Let us for the time being restrict our attention, following MP, to terms up to $O(\alpha^0)$ in $H_{k'}$. On introducing the creation and annihilation operators defined by

$$\xi_z = p_z/(2\mu\Omega_z)^{1/2} - \mathrm{i}(\tfrac{1}{2}\mu\Omega_z)^{1/2} z\,, \qquad \xi_z^+ = p_z/(2\mu\Omega_z)^{1/2} + \mathrm{i}(\tfrac{1}{2}\mu\Omega_z)^{1/2} z\,, \tag{6.1.13}$$

where

$$\Omega_z = (1 - \omega_0^2/\mu^2)^{1/2}\,, \tag{6.1.14}$$

the hamiltonian (6.1.11) reduces to

$$H_{k'} = \mu\{(1+\nu-\Omega^2)/\nu\alpha^2 + [(1+2\xi^+\xi)\Omega - 1] + \Omega_z(\xi_z^+\xi_z + \tfrac{1}{2}) + \cdots\} \tag{6.1.15}$$

which has the spectrum

$$\mathscr{E}_{k',n_1,n_2} = \mu\{(1+\nu-\Omega^2)/\nu\alpha^2 + [(1+2n_1)\Omega - 1] + \Omega_z(n_2 + \tfrac{1}{2}) + \cdots\} \tag{6.1.16}$$

where

$$\Omega = \cosh^2\phi\,. \tag{6.1.17}$$

For the problem of a hydrogen atom in a magnetic field ($\nu = -1/2$), (6.1.16) becomes

$$\mathscr{E}_{k',n_1,n_2} = \mu\{-(1-2\Omega^2)/\alpha^2 + [(1-2n_1)\Omega - 1] + \Omega_z(n_2 + \tfrac{1}{2}) + \cdots\}\,, \tag{6.1.18}$$

with

$$\Omega = \tfrac{1}{2}(1 + 3\omega_0^2/\mu^2)^{1/2}\,, \qquad \Omega_z = (1 - \omega_0^2/\mu^2)^{1/2}\,, \quad \omega_0^2 = \tfrac{1}{4}e^2B^2 \tag{6.1.19a, b, c}$$

and μ is determined from

$$\mu^2 - \omega_0^2 = g^2\mu^{3/2}\,, \tag{6.1.19d}$$

g^2 being given by

$$g^2 = e^2/(|m|+1)^{3/2}\,. \tag{6.1.19e}$$

MP have introduced the positive dimensionless parameter

$$\eta = (\omega_0/\mu)^{1/2}\,, \tag{6.1.20}$$

which satisfies the equation

$$\eta^4 + \lambda\eta = 1\,, \tag{6.1.21}$$

where

$$\lambda \equiv (g^4/\omega_0)^{1/4} = [2|e|^3/B(|m|+1)^3]^{1/2}\,. \tag{6.1.22}$$

Equation (6.1.18) then reads

$$\mathscr{E}_{k',n_1,n_2} = (|e|B/2\eta^2)\{-(1-3\eta^4)/2\alpha^2 + [(n_1+\tfrac{1}{2})(1+3\eta^4)^{1/2}-1] + (1-\eta^4)^{1/2}(n_2+\tfrac{1}{2}) + \cdots\}\,. \tag{6.1.23}$$

In the large magnetic field limit ($\lambda \to 0$), $\eta \to 1$ and (6.1.23) gives

$$\mathscr{E}_{k',n_1,n_2} = \tfrac{1}{2}|e|B(|m|+1+2n_1)\,. \tag{6.1.24}$$

When we add to this the energy corresponding to H_1 and the contribution from a similar term arising from the electronic spin (not considered here), we get the exact Landau spectrum. In the absence of the magnetic field ($\lambda \to \infty$), $\eta = 0$ and (6.1.18) reads

$$\mathscr{E}_{k',n_1,n_2} = [e^4/(|m|+1)^3][-\tfrac{1}{2}(|m|+1) + (n_1+n_2) + \cdots]\,, \tag{6.1.25}$$

which is just the large $(|m|+1)$ expansion of the exact energy of the hydrogen atom obtained by solving it in parabolic coordinates

$$\mathscr{E}_{n_1,n_2,|m|} = -e^4/2(n_1+n_2+|m|+1)^2\,. \tag{6.1.26}$$

It is indeed interesting to note that the leading-order term in the $1/(|m|+1)$ expansion yields the exact Coulomb spectrum specified by $n_1 = n_2 = 0$.

The next higher order correction to (6.1.23) can be calculated by straightforward perturbation theory. $H^{(1)}$ and $H^{(2)}$ may be written as

$$H^{(1)} = h^{(1)} + \delta h^{(1)}\,, \qquad H^{(2)} = h^{(2)} + \delta h^{(2)}\,, \tag{6.1.27a, b}$$

where $h^{(1)}$ and $h^{(2)}$ are given by (4.2.49) and (4.2.50), respectively, and $\delta h^{(1)}$ and $\delta h^{(2)}$ are given by

$$\delta h^{(1)} = \tfrac{1}{2}(\nu-1)(\mu^2-\omega_0^2)z^2V_1\,, \tag{6.1.28a}$$

$$\delta h^{(2)} = \tfrac{1}{4}(\nu-1)[(\mu^2-\omega_0^2)/\mu][\mu^2z^4 + 2\mu z^2V_2 + (\nu-2)\mu z^2V_1^2]\,, \tag{6.1.28b}$$

V_1 and V_2 being defined respectively by (4.2.50a) and (4.2.50b). Including the contributions of $\alpha H^{(1)}$ and $\alpha^2 H^{(2)}$ perturbatively, one obtains, for $n_1 = n_2 = 0$,

$$\mathscr{E}_{k',0} = (|e|B/2\eta^2)\{(3\eta^4-1)/2\alpha^2 + \tfrac{1}{2}[(1+3\eta^4)^{1/4} + (1-\eta^4)^{1/2} - 2] + \alpha^2E^{(1)} + \cdots\}\,, \tag{6.1.29}$$

with

$$E^{(1)} = -\frac{9}{32}\left(1 + \frac{\Omega_z^2}{\Omega^2}\,\frac{136\Omega^2-72\Omega+11}{108\Omega^2} + \frac{\Omega_z^2}{4\Omega^2}\,\frac{\Omega_z+3\Omega}{\Omega_z+\Omega} - \frac{\Omega_z}{3\Omega}\,\frac{2\Omega^2+4\Omega-1}{\Omega^2}\right). \tag{6.1.29a}$$

In the absence of the magnetic field (i.e. in the Coulomb limit $\eta = 0$) $E^{(1)} = 0$, as is expected, because for $n_1 = n_2 = 0$ the leading-order term coincides with the exact result. In the opposite limit

($B \to \infty$) the exact energy is known to be given by the Landau spectrum together with logarithmic corrections. This leads one to expect that $E^{(1)}$ should be again zero for $\eta = 1$. This is however, not the case. This anomaly crops up essentially because of truncating the $1/k'$ series at some finite order, and is indeed a signal that considering only a few terms of the $1/k'$ series may not be enough for very strong fields. The situation is, however, much less disturbing compared to that encountered in the Zeeman perturbative scheme.

For $n_1 = n_2 = 0$, the energy in each $|m|$ sector may be finally written as

$$E_{\pm|m|,0} = \pm |m|B + \mathscr{E}_{|m|,0}\,, \tag{6.1.30}$$

when $\mathscr{E}_{|m|,0}$ is given by (6.1.29). MP have compared their results [calculated from eq. (6.1.30)] with those of Praddaude [192] (see table 6.1). The agreement is satisfactory for fields up to the intermediate region.

BMP, by employing an algebraic recursive scheme, have added more higher order corrections to (6.1.29). They have considered the Schrödinger equation corresponding to H_0:

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right) + \tfrac{1}{8}B^2\rho^2 + \frac{m^2}{2\rho^2} - \frac{e^2}{(\rho^2+z^2)^{1/2}}\right]R = \mathscr{E}R\,. \tag{6.1.31}$$

On substituting $u = \rho^{1/2}R$, (6.1.31) reduces to

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{(k-1)(k-3)}{8\rho^2} + \tfrac{1}{8}B^2\rho^2 - \frac{e^2}{(\rho^2+z^2)^{1/2}}\right]u = \mathscr{E}u\,, \tag{6.1.32}$$

where

$$k = 2(|m|+1) = 4k'\,, \qquad m = 0, \pm1, \pm2, \ldots\,. \tag{6.1.33}$$

In terms of the rescaled variable q defined by

$$\rho = q\sqrt{k}\,, \tag{6.1.34}$$

(6.1.32) reads

$$\left[-\frac{1}{2k^2}\left(\frac{\partial^2}{\partial q^2} + k\frac{\partial^2}{\partial z^2}\right) + \frac{(1-1/k)(1-3/k)}{8q^2} + \tilde{V}(q,z)\right]u = \frac{\mathscr{E}}{k}u\,, \tag{6.1.35}$$

Table 6.1
Comparison of MP's results with those of Praddaude (PD) [192]. The unit for the magnetic field B is 2.35×10^9 G. The energies are in Rydberg [after Mlodinow and Papanicolaou [42], by courtesy of Ann. Phys. (NY)].

	$B = 0.1$		$B = 1$		$B = 3$	
	PD	MP	PD	MP	PD	MP
$m = 0$	−0.99505	−0.99505	−0.66233	−0.6714	0.67095	0.5992
$m = \pm1$	−0.10169	−0.10185	2.08682	2.0758	7.59297	7.5397
	−0.30169	−0.30185	0.08682	0.0758	1.59297	1.5397
$m = \pm2$	0.22432	0.22401	4.29389	4.2848	13.88551	13.843
	−0.17567	−0.17599	0.29389	0.2848	1.88551	1.843

where

$$\tilde{V}(q, z) = \tfrac{1}{8}B^2q^2 - \tilde{e}^2/(q^2 + z^2/k)^{3/2}\,, \tag{6.1.36}$$

$$\tilde{e} = e/k^{3/4}\,. \tag{6.1.36a}$$

In the limit $k \to \infty$, the energy to leading order is given by

$$\mathscr{E}_\infty = kV_{\text{eff}}(q_0)\,, \tag{6.1.37}$$

where q_0 is the value of q at which the large-k effective potential

$$V_{\text{eff}}(q) = 1/8q^2 + \tfrac{1}{8}B^2q^2 - \tilde{e}^2/u \tag{6.1.38}$$

is a minimum. Now introducing a new variable v through

$$v = \sqrt{k}(q - q_0)\,, \tag{6.1.39}$$

and writing, for the GS,

$$u = \exp[\phi(v)]\,, \tag{6.1.40}$$

BMP have cast eq. (6.1.35) into the form (3.2.84), to which they have then applied the algebraic recursion method. The first three terms in the $1/k$ series for the GS energy in each $|m|$ sector coincide with the pseudo-spin results of MP. They obtained an analytic expression for the next higher order term also. Still higher order corrections have been included numerically. A Shanks extrapolation is found to extend the applicability of the $1/k$ series into the domain of strong fields. For example, the first second-order Shanks extrapolant for the absolute GS energy ($m = 0$) for $B = 4.7 \times 10^{12}$ G yields (in atomic units) 990.919, the numerical value of Praddaude [192] being 990.695.

To our knowledge, extension of BMP's algebraic recursive scheme to excited states in each $|m|$ sector has not yet been possible.

6.2. *The Stark problem*

The Schrödinger equation for the hydrogen atom in a uniform electric field F is given by

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{(k-1)(k-3)}{8\rho^2} - \frac{e^2}{(\rho^2 + z^2)^{1/2}} + eFz\right]u = Eu\,, \tag{6.2.1}$$

where k is defined in (6.1.33). In terms of the rescaled variables q and v defined by

$$\rho = k^2q\,, \qquad z = k^2v\,, \tag{6.2.2}$$

eq. (6.2.1) reads

$$\left[-\frac{1}{2k^2}\left(\frac{\partial^2}{\partial q^2}+\frac{\partial^2}{\partial v^2}\right)+\frac{(1-1/k)(1-3/k)}{8q^2}-\frac{e^2}{(q^2+v^2)^{1/2}}+eF\tilde{v}\right]u=(Ek^2)u\,, \tag{6.2.3}$$

where $\tilde{F}=Fk^4$. In the limit $k\to\infty$, the particle sits at the bottom of the effective potential

$$V_{\text{eff}}(q,v)=1/8q^2-e^2/(q^2+v^2)^{1/2}+e\tilde{F}v\,, \tag{6.2.4}$$

and the GS energy is given by

$$E_\infty=(1/k^2)V_{\text{eff}}(q_0,v_0)\,, \tag{6.2.5}$$

where q_0 and v_0 are to be determined from

$$\partial V_{\text{eff}}/\partial q|_{q=q_0}=0=\partial V_{\text{eff}}/\partial v|_{v=v_0}\,, \tag{6.2.6}$$

which yields

$$v_0=-4e\tilde{F}q_0^4\,,\qquad 4e^2q_0=(1+16e^2\tilde{F}^2q_0^6)^{3/2}\,. \tag{6.2.7a, b}$$

For a very weak electric field F, (6.2.7b) yields distinct positive solutions q_{01} and q_{02}, q_{01} being smaller than q_{02}. Out of these two solutions only q_{01} gives a local minimum and therefore the large-k GS energy may be written as

$$E_\infty=(1/k^2)V_{\text{eff}}(q_{01},v_{01})\,, \tag{6.2.8}$$

where $v_{01}=4e\tilde{F}q_{01}^4$. It should however be mentioned that as F increases, the two solutions q_{01} and q_{02} come closer and closer to each other and finally become equal when $\tilde{F}$ reaches a value $\tilde{F}_0$ which gives the large-k quenching value of the electric field for an atom in its GS in a given $|m|$ sector. BMP have calculated its value, which is

$$F_0=1.07\times10^9/(|m|+1)^4\ \text{V/cm}\,. \tag{6.2.9}$$

For $F<F_0$, one can expand the potential around its minimum at $q=q_{01}$ and $v=v_{01}$, and follow a procedure essentially similar to that used by BMP in the Zeeman problem to generate the $1/k$ expansion. The pseudo-spin formulation may also be employed with some modification to study this problem.

6.3. *Hydrogen atom in a laser field*

The hydrogen atom interacting with an electromagnetic field is described by the time dependent Schrödinger equation

$$\{\tfrac{1}{2}[\hat{\boldsymbol{p}}+(e/c)\boldsymbol{A}(t)]^2-e^2/r\}\psi(\boldsymbol{r},t)=\mathrm{i}\,\partial\psi(\boldsymbol{r},t)/\partial t\,, \tag{6.3.1}$$

where $\boldsymbol{A}(t)=A(\hat{\boldsymbol{x}}\cos\omega t+\hat{\boldsymbol{y}}\sin\omega t)$ is the vector potential for a circularly polarized laser beam

propagating in the z-direction. Using the Kramers–Henneberger canonical transformation [193]

$$\psi(r, t) = e^{i\delta(t)\cdot p}\, e^{i\eta(t)}\, \phi(r, t)\,, \tag{6.3.2}$$

where

$$\delta(t) = -\frac{e}{c}\int^{t} dt'\, A(t')\,, \tag{6.3.3}$$

$$\eta(t) = -\frac{e^2}{2c^2}\int^{t} dt'\, A^2(t')\,, \tag{6.3.4}$$

eq. (6.3.1) can be transformed to [194]

$$(\tfrac{1}{2}\hat{p}^2 - e^2/|r-\delta|)\phi(r, t) = i\,\partial\phi(r, t)/\partial t \tag{6.3.5}$$

The laser-dressed binding potential $-e^2/|r-\delta|$ can be expanded as

$$-\frac{e^2}{|r-\delta|} = -\frac{e^2}{(r^2+a^2)^2}\left(1 + \frac{r\cdot\delta}{r^2+a^2} + \frac{1\cdot 3}{2}\,\frac{r\cdot\delta}{(r^2+a^2)^2} + \cdots\right), \tag{6.3.6}$$

where $|\delta| = a$. Since $[r\cdot\delta/(r^2+a^2)]^n$ will never exceed $\frac{1}{2}^n$, it appears to be a fairly good approximation to retain only the first term of the expansion (6.3.6). This zeroth order approximation, in which the effective potential for the problem becomes

$$V_{\text{eff}} = -e^2/(r^2+a^2)^{1/2}\,, \tag{6.3.7}$$

does not explicitly involve the strength of the laser field, so (6.3.7) is expected to describe both the weak and the strong field cases.

The potential (6.3.7) is not exactly solvable analytically. Lima and Miranda [194] used the variational method with hydrogenic trial functions to obtain the GS energy. They introduced a dimensionless quantity $\lambda = a/a_0$ (a_0 being the Bohr radius) as a measure of the field strength, which is connected with the frequency ω (s^{-1}) and the intensity I ($W\,cm^{-2}$) through the relation

$$\lambda = 6.5\times 10^{24}\,\omega^{-2} I^{1/2}\,. \tag{6.3.8}$$

They have shown that an intense laser field causes a weakening in the binding of the electron to the nucleus, in such a way that for large enough $\lambda (a \gg a_0)$ the GS energy scales down approximately as $1/\lambda$. To show that (6.3.7) is indeed a good model potential Lima and Miranda [195] estimated the first nonvanishing correction to the zeroth order energy, this comes out to be about 11% of the zeroth order result. For excited states, information about what is important when it comes to laser–atom interactions, the aforementioned simple variational procedure is not very useful, particularly for large λ. Analytic perturbation methods [196–199] are also not advisable when λ is large.

Dutt, Mukherjee and Varshni [73] examined the energy levels of the hydrogen atom in the presence of an intense laser field within the framework of the shifted $1/N$ expansion. The analytic expressions for the energy eigenvalues E_{nl} (n is principal quantum number) yield results which for a wide range of n, l and λ are in excellent agreement with the exact eigenvalues obtained by Singh et al. [200] by a high-precision numerical technique. It is observed that the degeneracy of the energy levels of the hydrogen atom disappears in the presence of an intense laser field and that the energy levels satisfy the ordering $E_{nl} > E_{nl'}$ for $l < l'$, which is consistent with the level ordering theorem of Grosse and Martin [201]. Furthermore the energy eigenvalues are found to decrease with increasing field strength and, for large enough λ, E_{nl} scales down approximately as λ^{-1}. For the GS the scaling sets in at $\lambda \gtrsim 100$, whereas for higher excited states it begins at a much higher value. For the GS the results gathered from the $1/N$ expansion are consistent with those obtained from the variational calculations of Choi et al. [202] and Lima and Mirande [198]. Finally, it may be mentioned that the approximation invoked in obtaining the truncated potential (6.3.7) is valid for intensities up to $I \sim 10^{-12}\omega^2$ (W cm^{-2}) and therefore calculations using (6.3.7) may be trusted for the available CO_2 laser ($\omega \simeq 1.8 \times 10^{14}$ s^{-1}) with intensity $I \approx 10^{10-11}$ (W cm^{-2}), which corresponds to λ in the range 20–60.

6.4. Helium atom

The study of the helium atom spectrum in the large-N approximation appears to have been initiated by Witten [2]. Subsequently, quite extensive investigations have been made in this field by Mlodinow and Papanicolaou [89], Van der Merwe [90–93] and Herschbach and collaborators [94–100].

The hamiltonian for a helium-like atom in N-dimensions is given by

$$H = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + \frac{\mathrm{d}^2}{\mathrm{d}y_i^2}\right) - Ze^2\frac{1}{|x|} + \frac{1}{|y|} + \frac{e^2}{|x-y|}, \tag{6.4.1}$$

where we put both the electronic mass and the Planck constant equal to unity. Here x and y denote the N-dimensional position vectors of the two electrons, the x_i and y_i are the corresponding cartesian coordinates, Z denotes the charge of the nucleus, and the mass of the nucleus is assumed to be infinitely large. In spherical polar coordinates the corresponding Schrödinger equation for the GS wavefunction, which is a function of the rotationally invariant quantities $x(=\sqrt{x^2})$, $y(=\sqrt{y^2})$ and θ, where x and y are the distances of the electrons from the nucleus and θ is the angle between x and y, reads

$$\left(-\frac{1}{2}\left\{\frac{1}{x^{N-1}}\frac{\partial}{\partial x}\left(x^{N-1}\frac{\partial}{\partial x}\right) + \frac{1}{y^{N-1}}\frac{\partial}{\partial y}\left(y^{N-1}\frac{\partial}{\partial y}\right) + \left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left[\frac{1}{\sin^{N-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{N-2}\theta\frac{\partial}{\partial\theta}\right)\right]\right\}\right.$$
$$\left. - Ze^2\left(\frac{1}{x}+\frac{1}{y}\right) + \frac{e^2}{(x^2+y^2-2xy\cos\theta)^{1/2}}\right)\psi = E\psi\,. \tag{6.4.2}$$

To eliminate the first derivatives we now substitute

$$\psi = (xy)^{-(N-1)/2}(\sin\theta)^{-(N-2)/2}\phi \tag{6.4.3}$$

in (6.4.2), which then becomes

$$\left\{-\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\right]+\frac{1}{8}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{(N-2)(N-4)}{\sin^2\theta}-1\right)\right.$$
$$\left.-Ze^2\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{e^2}{(x^2+y^2-2xy\cos\theta)^{1/2}}\right\}\phi=E\phi\,. \tag{6.4.4}$$

In the large-N limit, (6.4.4) reduces to

$$H_{\mathrm{eff}}\phi=(E/N^2)\phi\,, \tag{6.4.5}$$

where

$$H_{\mathrm{eff}}=-\frac{1}{2N^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\right]+V_{\mathrm{eff}}(x,y,\theta)\,, \tag{6.4.6a}$$

$$V_{\mathrm{eff}}(x,y,\theta)=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\frac{\lambda}{\sin^2\theta}-Z\tilde{e}^2\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{\tilde{e}^2}{(x^2+y^2-2xy\cos\theta)^{1/2}}\,, \tag{6.4.6b}$$

$$\tilde{e}=e/N\,,\qquad \lambda=1/8\,. \tag{6.4.6c, d}$$

The first term on the right hand side of the effective potential (6.4.6b) is of quantum mechanical origin. If $\hbar$ were retained throughout the analysis it would have appeared in this term. The parameter λ is thus an $\hbar$-dependent quantity which is given by (6.4.6d) when $\hbar$ and m are set equal to unity. $\lambda=0$ corresponds to the absence of fluctuations. In the limit $N\to\infty$, the kinetic energy terms are suppressed and the leading-order contribution to the GS energy is given by

$$E_\infty=N^2V_{\mathrm{eff}}(x_0,y_0,\theta_0)\,, \tag{6.4.7}$$

where x_0, y_0 and θ_0 are to be obtained from

$$\partial V_{\mathrm{eff}}/\partial x|_{x=x_0}=0\,,\qquad \partial V_{\mathrm{eff}}/\partial y|_{y=y_0}=0\,,\qquad \partial V_{\mathrm{eff}}/\partial\theta|_{\theta=\theta_0}=0\,. \tag{6.4.8}$$

It is now useful to make the transformations [90]

$$x=r\sin\alpha\,,\qquad y=r\cos\alpha\,. \tag{6.4.9}$$

The effective potential then becomes

$$V_{\mathrm{eff}}(r,\alpha,\theta)=\frac{4\lambda}{r^2\sin^2 2\alpha\sin^2\theta}-\frac{\tilde{g}}{r}\left(\frac{1}{\sin\alpha}+\frac{1}{\cos\alpha}\right)+\frac{\tilde{e}^2}{f(\alpha,\theta)r}\,, \tag{6.4.10}$$

where

$$\tilde{g}=Z\tilde{e}^2\,,\qquad f(\alpha,\theta)=(1-\sin 2\alpha\cos\theta)^{1/2}\,. \tag{6.4.11a, b}$$

Minimization of (6.4.10) with respect to r, α and θ yields

$$\frac{8\lambda}{r\sin^2 2\alpha \sin^2\theta} = \tilde{g}\left(\frac{1}{\sin\alpha} + \frac{1}{\cos\alpha}\right) - \frac{\tilde{e}^2}{f(\alpha,\theta)}\,, \tag{6.4.12}$$

$$\frac{16\cos 2\alpha}{r\sin^3 2\alpha \sin^2\theta} = \tilde{g}\left(\frac{\cos\alpha}{\sin^2\alpha} - \frac{\sin\alpha}{\cos^2\alpha}\right) + \frac{\tilde{e}^2}{f^3(\alpha,\theta)}\cos 2\alpha\cos\theta\,, \tag{6.4.13}$$

$$16\lambda\cos\theta\, f^3(\alpha,\theta) = -\tilde{e}^2\sin^3 2\alpha\sin^4\theta\,. \tag{6.4.14}$$

Equations (6.4.12) and (6.4.14) may be combined to yield

$$f^4(\alpha,\theta) = \cos^2 2\alpha - 4Zf^3(\alpha,\theta)\cos\theta(\cos\alpha + \sin\alpha)\,, \tag{6.4.15}$$

while equations (6.4.13) and (6.4.14) give

$$4Zf^3(\alpha,\theta)\cos\theta(\sin^3\alpha - \cos^3\alpha) = \sin^2 2\alpha\cos 2\alpha\,, \tag{6.4.16}$$

which is clearly satisfied by $\alpha = \pi/4$. One finally obtains

$$\cos\theta_0 = -[1 + (1 + 128Z^2)^{1/2}]/64Z^2\,, \tag{6.4.17}$$

$$x_0 = y_0 = (2\lambda/g)(1+\cos\theta_0)^{-2} = (2N^2/8Ze^2)(1+\cos\theta_0)^{-2}\,, \tag{6.4.18}$$

$$E_\infty = -e^4(2/N)^2 F\,, \tag{6.4.19}$$

where

$$F = Z^2(1+\cos\theta_0)^3/(1-\cos\theta)\,. \tag{6.4.19a}$$

It follows from eq. (6.4.17) that θ approaches 90° when the electron–electron repulsion becomes negligible ($z\to\infty$). For the helium atom ($Z=2$) in three dimensions ($N=3$) we get $\theta_0 = 95.28°$, $x_0 = y_0 = 1.3645$ au and $E_\infty = -1.2187$ au, the experimental value for the GS energy being -2.9037 au. Obviously the leading-order calculation does not provide a very accurate result for the energy, but the physical picture that emerges is quite interesting. For example, in the large-N limit the helium atom looks like a nonlinear molecule with well-defined bond lengths (x_0, y_0) and bond angle (θ_0). Since $x_0 = y_0$, the energy is equally shared between the electrons in the GS. This equipartition of energy and the result that the bond angle is greater than 90° show that the GS is a highly correlated state and thus provide a conceptual basis for describing the doubly excited states in the helium spectrum. The semiclassical large-N result for the correlation angle is in agreement with the variational calculation of Pekeris [203], which gives $\langle x\cdot y\rangle = -0.0647$ for the GS and shows that the correlation angle is indeed slightly greater than 90°. One can also make contact with other semiclassical models. For instance, eq. (6.4.18) indicates that when $\lambda = 0$ i.e. in the absence of fluctuations, θ is 180° i.e. the electrons are placed at the extremeties of a diameter. This corresponds to the planar model of Bohr, which has, however, been proved by Born [204] to be unstable.

MP have used the pseudo-spin formulation to obtain the first three terms of the $1/N$ expansion of the helium GS energy

$$E = -(2/N)[2.7377(N/2) + 3.0288 + 2.2155(2/N) + O(N^{-2})] \text{ au} . \tag{6.4.20}$$

Thus with the first three terms of the $1/N$ expansion, one gets for $N = 3$, $E = -2.5518$ au. MP have shown that their theory requires for its applicability $Z \geq Z_c = 1.228\ldots$. Thus apparently it seems that the method of MP cannot be transferred to the physically interesting case of the negative hydrogen ion H^-. Though this point is not completely understood, it appears that for $Z < Z_c$ the potential has two minima and the symmetric configuration (x_0, θ_0) is a saddle point between the two equivalent asymmetrical minima that differ by interchange of x_0 and y_0 [95].

It is known that the helium GS energy has a singularity at $N = 1$, so it seems more appropriate to use $1/(N-1)$ as the expansion parameter. The GS energy then reads [4]

$$E = -\{2.7377[2/(N-1)]^2 + 0.2911[2/(N-1)]^3 - 0.2744[2/(N-1)] + \cdots\} , \tag{6.4.21}$$

which for $N = 3$ gives $E = -2.7544$ au. Agreement with the experimental value is now quite satisfactory.

Van der Merwe [90] has analysed the large-N effective hamiltonian H_{eff} which may be written as

$$H_{\text{eff}} = (1/2M)[P^2 + Q^2 + (1/x^2 + 1/y^2)B^2] + V_{\text{eff}} , \tag{6.4.22}$$

where

$$P = -\mathrm{d}^2/\mathrm{d}x^2 , \qquad Q = -\mathrm{d}^2/\mathrm{d}y^2 , \qquad B = -\mathrm{d}^2/\mathrm{d}\theta^2 , \qquad M = N^2 , \tag{6.4.23}$$

in order to extract some interesting features of the independent particle model, the fragmentation regime and the collective motion, and the associated molecular spectra of the helium atom. Since in the large-N limit the kinetic energy is quenched because of the large effective mass ($M = N^2$) of the particle, fluctuations are not important and so Van der Merwe has used the Hamilton–Jacobi theory. The equations of motion are

$$\mathrm{d}x/\mathrm{d}t = \partial H_{\text{eff}}/\partial P = P/M , \qquad \mathrm{d}y/\mathrm{d}t = \partial H_{\text{eff}}/\partial Q = Q/M , \tag{6.4.24a, b}$$

$$\mathrm{d}\theta/\mathrm{d}t = \partial H_{\text{eff}}/\partial B = (1/x^2 + 1/y^2)B/M , \tag{6.4.24c}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\text{eff}}}{\partial x} = \frac{B^2}{Mx^3} + \frac{2\lambda}{x^3 \sin^2\theta} + \frac{\tilde{e}^2(x - y\cos\theta)}{(x^2 + y^2 - 2xy\cos\theta)^{3/2}} - \frac{\tilde{g}}{x^2} , \tag{6.4.24d}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -\frac{\partial H_{\text{eff}}}{\partial \theta} = \frac{2\lambda\cos\theta}{\sin^3\theta}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) + \frac{\tilde{e}^2 xy \sin\theta}{(x^2 + y^2 - 2xy\cos\theta)^{3/2}} . \tag{6.4.24e}$$

To get the discrete spectrum one should use together with the above semiclassical equations a suitable quantization condition [205, 206].

In the independent particle picture one takes $\tilde{g} = Z\tilde{e}^2 \gg \tilde{e}^2$. Although this model is not very realistic for the description of experimental two-electron spectra, it does offer a useful starting point to explore some of the interesting aspects of the large-N helium atom. In this independent particle limit pairing effects are ignored and eq. (6.4.24 e) has a solution $\theta = \frac{1}{2}\pi$. Integrating (6.4.24a) and (6.4.24d) leads to

$$\tfrac{1}{2}M(\mathrm{d}x/\mathrm{d}t)^2 = -\lambda/x^2 + \tilde{g}/x - C_1\,, \tag{6.4.25}$$

where C_1 is a constant of integration. For the variable y one has an analogous equation involving another constant C_2. The total energy E can be expressed in terms of C_1 and C_2. The quantization conditions used by Van der Merwe are

$$n_1\pi = \int_{x_{\min}}^{x_{\max}} P\,\mathrm{d}x = (2M)^{1/2}\int\left(-\frac{\lambda}{x^2} + \frac{\tilde{g}}{x} - C_1\right)^{1/2}\mathrm{d}x\,, \tag{6.4.26}$$

with a similar equation for y. The energy is finally given by

$$E_{\mathrm{ind}}(n_1, n_2) = -\frac{e^4}{2}\left(\frac{2}{N}\right)^2 Z^2\left(\frac{1}{[1+n_1(2/N)]^2} + \frac{1}{[1+n_2(2/N)]^2}\right), \quad n_1, n_2 = 0, 1, 2, \ldots. \tag{6.4.27}$$

Inclusion of higher order $1/N$ terms into (6.4.27) may be accomplished, as has been pointed out by Van der Merwe, by replacing N by $N-1$. The energy spectrum of the two-electron system is now given by the sum of two independent Rydberg series, in conformity with one's expectation. For the GS energy, (6.4.27) yields $E_{\mathrm{ind}}(0,0) = 4.003$ au. That this independent particle model result does not compare favourably with the experimental value shows that electron correlations are indeed important.

In the fragmentation regime the two electrons are placed at different distances from the nucleus, the one in the inner shell carrying the lion's share of the energy and at the same time screening the interaction of the outer shell electron with the ion core. The situation will be described by eqs. (6.4.24) with $x \gg y$. Again $\theta = \pi/2$, and following the same procedure as adopted for the analysis of the independent particle motion one gets

$$E_{\mathrm{frag}}(n_1, n_2) = -\frac{e^4}{2}\left(\frac{2}{N-1}\right)^2\left(\frac{(Z-1)^2}{[1+2n_1/(N-1)]^2} + \frac{Z^2}{[1+2n_2/(N-1)]^2}\right), \tag{6.4.28}$$

where $n_1, n_2 = 0, 1, 2, \ldots$, with the constraint $n_1 \gg n_2$. Comparing the results calculated from (6.4.28) with the experimental data [207, 208] Van der Merwe has shown that (6.4.28) describes the experimental situation fairly well if $n_1 > n_2$.

As has already been discussed, the GS of the helium atom is a highly correlated state in which the electrons are equidistant from the nucleus and the total energy is equipartitioned between them. The correlated electron motion is reflected by the doubly excited states which are manifestations of collective excitations where the pairing effects play the dominant role. The equations of motion (6.4.24) then reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}Mx^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \frac{4\lambda\cos\theta}{x^2\sin^3\theta} + \frac{b\sin\theta}{x(1-\cos\theta)^{3/2}}\,, \tag{6.4.29a}$$

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{1}{x^3}\left(\frac{B^2}{M} + \frac{2\lambda}{\sin^2\theta}\right) - \frac{\tilde{g}}{x^2}\left(1 - \frac{b}{\tilde{g}(1-\cos\theta)^{1/2}}\right), \tag{6.4.29b}$$

where $b = 2^{-3/2}\tilde{e}^2$. So far it has not been possible to solve the above set of coupled nonlinear equations

exactly. Van der Merwe has investigated the approximate solutions. He has shown that for intrashell states the angular variable may be regarded as frozen, as a first approximation, and then the evolution of the radial variable may be investigated. In the frozen angular variable configuration, eq. (6.4.29a) becomes

$$4\lambda C = -xb(1-C)^{1/2}(1+C)^2\,, \tag{6.4.30}$$

where $C = \cos\theta_e$, θ_e being the equilibrium value of θ.

The first integral of (6.4.29b) can be written as

$$\tfrac{1}{2}M\left(\frac{dx}{dt}\right)^2 = \frac{G_e}{x} - \frac{\lambda}{x^2\sin^2\theta} - D\,, \tag{6.4.31}$$

where

$$G_e = Z\tilde{e}^2 - b(1-C)^{-1/2}\,, \tag{6.4.32}$$

and D is proportional to the energy. The semiclassical quantization approach that we have already discussed in the context of the independent particle model leads to the discrete energy levels describing the doubly excited states. Van der Merwe has compared the leading-order $1/N$ results for the doubly excited s states to the experimental values [208]. The discrepancies that appear have been attributed by him to the uncertainties in the correlation angle (F is a sensitive function of the correlation angle). He has also calculated the correlation angle for different values of n. In the GS ($n = 0$) the correlation angle comes out to be the same as that obtained from (6.4.17), while as n increases, θ_e tends to π. Thus the situation is something like that of a nonlinear triatomic molecule evolving towards the XYX model proposed by Herrick and Kellman [209, 210].

If one now includes fluctuations in the analysis one would get vibrational and rotational levels accompanying the doubly excited states. Van der Merwe has investigated the vibrational states associated with the variations in the correlation angle.

The $1/N$ expansion thus supports the existence of molecular features in atomic systems. Because of the anharmonic contributions the spacing between the vibrational states however gradually increases, in contrast to what is known to happen in molecular spectra.

Van der Merwe has made further investigations on the electron correlations of the doubly excited states in his subsequent works [92, 93]. He has observed that his large-N results for the cosine of the correlation angle in the doubly excited s states are in substantial agreement with the corresponding results of Herrick, Kellman and Poliak [211] who have shown that the expectation value of the cosine of the angle between the position vectors of the two electrons in the lowest level of each intrashell state, with respect to the SO(4) basis of the hydrogenic orbital, is given by

$$\langle\cos\theta\rangle = -(1/2n_p^2)(2n_p^2 + 3n_p - 1)\,, \tag{6.4.33}$$

where n_p is the principal quantum number ($n_p = 1, 2, \ldots$). The accuracy of the leading-order $1/N$ result for the cosine of the correlation angle in GS (which is about -0.09) has furthermore been found to be comparable to that obtained from a one-parameter variational calculation, the accurate variational result [203] being -0.075.

The N-dimensional helium atom problem has been investigated in detail by Herschbach [95] and Herschbach and collaborators [96–100]. By scaling coordinates the helium atom in the $N = 1$ limit can be mapped on to a δ-function model for which an exact solution is possible [212]. On the other hand, the $N \to \infty$ limit also (as we have already discussed) leads to an exactly solvable problem in which the electrons find themselves in fixed classical configuration. Herschbach [95] has calculated the GS energy of the helium atom by interpolating between exact solutions for the limiting cases of $N = 1$ and $N = \infty$. He has shown that the GS energy can be expressed in the form

$$E_N = -E_\infty[1 + C_1/N + C_2/N^2 + F(N, Z)/N^3], \tag{6.4.34}$$

where the first three terms have been obtained by a large-N perturbative expansion and $F(N, Z)$, called the interpolation function, has been approximated as a geometric series, obtained by fitting the accurately known $N = 1$ and $N = 5$ results. This dimensional interpolation method yields an accuracy of 2 parts in 10^5 or better for $N = 3$ and $Z \geq 2$. Again, for each N the interpolation function is found to vanish for a critical nuclear charge Z_c close to 12, and then the first three terms of the $1/N$ expansion give the exact energy.

Loeser and Herschbach [96] have extended the Pekeris formulation [213] for s states of the helium atom to N dimensions with a view to evaluating more higher-order terms in the $1/N$ expansion so as to be able to test its convergence and the nature of the interpolation function. It turns out that the expansion is not convergent for $N = 3$, but terms from third order to sixth order do approximately form a geometric series. In a subsequent paper [97] Loeser and Herschbach have extended the Hartree–Fock method for the two-electron atoms to N dimensions. The problem admits exact solution for $N \to 1$ and $N \to \infty$ for any Z. The correlation energy turns out to be more or less independent of Z for any N. Moreover, for $Z \geq 2$, it is found to be nearly linear in $1/N$. Therefore, the dimensional interpolation between the exactly known $N \to 1$ and $N \to \infty$ limits yields even better results for the correlation energy than for the total energy. This indicates the bright prospects of the Hartree–Fock method in the context of using the dimensional interpolation for the many-electron systems.

Doren and Herschbach [99] have studied the interdimensional degeneracies present in the two-electron systems. The knowledge of such degeneracies is useful, because it is then possible to obtain energies of some excited states in three dimensions from the simpler GS energy calculations in other dimensions which may be otherwise unphysical dimensions.

Goodson and Herschbach [100], generalizing the method of Ader [45], originally suggested for spherically symmetric potential problems, have developed an efficient algorithm for the $1/N$ expansion calculation of the GS energy of the helium atom which uses recursion relations for the moments of the coordinate operators. The difficulty with the $1/N$ expansion is that it is a divergent asymptotic expansion. The helium GS energy exhibits a singularity at $N = 1$, where it has a second-order pole and a confluent first-order pole. On removing the effect of these poles the asymptotic convergence of the first few terms is improved but ultimately the series still diverges. The authors have finally used Padé summation, which yields accurate results. The [5/5] Padé approximant gives an energy which agrees with the highly accurate result of Pekeris to 0.005%. The accuracy of their result is comparable to that of configuration interaction calculations [214, 215]. The GS of the helium atom in five dimensions and the doubly excited $(2p)^2$ 3P state of it in three dimensions are known to have identical energies [216, 217]. Goodson and Herschbach have calculated the [5/5] Padé approximant at $N = 5$; this gives an energy equal to 2.842 001 96 au, which differs from the exact result [218, 219] only by 0.000 04%.

6.5. Rotational–vibrational levels of the H_2^+ molecule

The Schrödinger equation for the H_2^+ molecule is given by

$$\left[-\frac{1}{2m}\nabla_r^2 - \frac{1}{2M}\sum_{j=1}^{\infty}\nabla_{R_j}^2 + \left(\frac{1}{R} - \frac{1}{r_1} - \frac{1}{r_2}\right)\right]\psi = E\psi\,, \tag{6.5.1}$$

where we have put $\hbar = e = 1$ and $|r - R_1| = r_1$, $|r - R_2| = r_2$, $|R_1 - R_2| = R$, r being the position of the electron and R_1 and R_2 being the positions of the two protons. Here m is the electronic mass and M is that of each proton. In the Born–Oppenheimer (BO) approximation one can treat the electronic and the nuclear motions separately. The first step is to consider electronic motion in the configuration of momentarily frozen nuclei.

$$H\phi_{k,R}(r) \equiv \left[-\frac{1}{2m}\nabla_r^2 - \left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{R}\right]\phi_{k,R}(r) = E_e(R)\phi_{k,R}(r)\,. \tag{6.5.2}$$

Here the electronic wavefunction ϕ and the energy E_e contain R only as a parameter. The total wavefunction is now written as

$$\psi = \phi_{k,R}(r)\chi_{k,n}(R_j)\,, \tag{6.5.3}$$

where $\chi_{k,n}(R_j)$ denotes the nuclear wavefunction corresponding to the electronic state k and satisfies the Schrödinger equation

$$\left(-\frac{1}{2M}\sum_{j=1}^{2}\nabla_{R_j}^2 + E_e(R)\right)\chi_{k,n}(R_j) = E_{k,n}\chi_{k,n}(R_j)\,. \tag{6.5.4}$$

Since the potential function E_e depends only on the relative coordinate R of the nuclei, one can separate in a standard fashion the centre-of-mass motion and reduce the two-body problem described by (6.5.4) to the problem of a single particle with a reduced mass. Of course, the potential function $E_e(R)$ has to be first obtained from (6.5.2), which however does not admit an exact solution. Equation (6.5.2) has been solved variationally with a trial function [101]

$$\phi = a\phi_1 + b\phi_2\,, \tag{6.5.5}$$

$$\phi_1 = (\gamma^{3/2}/\sqrt{\pi})\,e^{-\gamma r_1}\,, \qquad \phi_2 = (\gamma^{3/2}/\sqrt{\pi})\,e^{-\gamma r_2}\,. \tag{6.5.6}$$

Minimization of $\langle\phi|H|\phi\rangle$ with respect to a and b gives, for the symmetric case (which corresponds to a lower energy),

$$a = b = [2(1+S)]^{-1/2}\,, \tag{6.5.7}$$

$$E_e(y) = f(y)\gamma^2 + g(y)\gamma\,, \tag{6.5.8}$$

with

$$y = \gamma R\,, \tag{6.5.9}$$

$$S = \langle \phi_1 | \phi_2 \rangle = (1 + y + \tfrac{1}{3}y^2)\,\mathrm{e}^{-y}\,, \tag{6.5.10}$$

$$f(y) = -\tfrac{1}{2} + [1/(1+S)][1 + H_1(y)]\,, \tag{6.5.11}$$

$$g(y) = 1/y - [1/(1+S)][1 + 2H_1(y) + H_2(y)]\,, \tag{6.5.12}$$

where

$$H_1(y) = (1+y)\,\mathrm{e}^{-y}\,, \tag{6.5.13}$$

$$H_2(y) = (1/y)[1 - (1+y)\,\mathrm{e}^{-2y}]\,. \tag{6.5.14}$$

Minimization of (6.5.8) with respect to γ finally yields

$$\gamma = -g(y)/2f(y)\,, \tag{6.5.15}$$

$$E_e = -g^2(y)/4f(y)\,. \tag{6.5.16}$$

The BO variational energy expression (6.5.16) should now serve as the effective potential for the nuclear motion. However, because of its complexity, this potential has not evoked much interest. Instead, it has become customary to use the empirically derived Morse potential [180].

Atag [101] has recently obtained the rotational–vibrational levels of the H_2^+ molecule using the BO variational effective potential (6.5.16) in the framework of the shifted $1/N$ expansion of Sukhatme and Imbo [67]. He has calculated the first three terms (E^0, E^1 and E^2) of the shifted expansion. His results compare favourably with those obtained from the Morse potential. Furthermore, agreement is found to be better for vibrational states $l = 0$; $n = 0, 1, 2, 3$. Atag has also compared his results with the values obtained by numerically solving the Schrödinger equation with (6.5.16). This comparison shows that the $1/N$ expansion results are highly accurate, particularly for small values of n. The inadequacy present in the $1/N$ expansion results has therefore been attributed by Atag to the variational procedure and the BO approximation. It thus appears that if the variational procedure leading to the effective potential (6.5.16) is sufficiently reliable, then the shifted $1/N$ expansion is a useful approach to obtain the rotational–vibrational states of similar molecules with the effective potential (6.5.8).

7. Large-N expansions in relativistic quantum mechanics

The large-N approximation was first invoked in relativistic quantum mechanics by Miramontes and Pajares [102] (MIP) who have shown that for a class of potentials $V(r) \sim r^n$ (where $-2 < n < 0$) the relativistic and spin corrections are nonleading in $1/N$. An iterative $1/N$ expansion technique has recently been suggested by the present author [103] to obtain the energy spectrum of a scalar particle moving in any spherically symmetric potential. For the Coulomb potential, in particular, the method looks quite attractive, for it yields the exact energy series. This method has very recently been employed by Atag [104] to solve the Dirac equation.

7.1. The Klein–Gordon equation

7.1.1. The large-N limit

The time-independent N-dimensional Klein–Gordon equation (in units $c = \hbar = 1$) for a scalar particle of mass m moving in a Lorentz vector potential whose only surviving component is the fourth component $V(r)$ is given by

$$\{\nabla_N^2 + [E - V(r)]^2 - m^2\}\psi(r) = 0\,, \tag{7.1.1}$$

where ∇_N^2 is given by (3.0.14) and E is the total relativistic energy of the particle. On using (3.0.26), (3.0.28), (3.0.14) and (3.0.25), eq. (7.1.1) reduces to

$$(-\mathrm{d}^2/\mathrm{d}r^2 + (k-1)(k-3)/4r^2 - \{[E - V(r)]^2 - m^2\})u(r) = 0\,, \tag{7.1.2}$$

where $k = N + 2l$, l being the angular momentum quantum number. In the limit $N \to \infty$ the leading-order contribution to the energy is given by

$$E_\infty = V(r_0) + m(1 + k^2/4m^2r_0^2)^{1/2}\,, \tag{7.1.3}$$

where r_0 is to be obtained from

$$r_0^3 V'(r_0)(1 + k^2/4m^2r_0^2)^{1/2} = k^2/4m\,. \tag{7.1.4}$$

At this point, MIP have made an interesting observation. They have considered the case

$$r_0 \gg N/2m\,. \tag{7.1.5}$$

Then expression (7.1.3) reduces to

$$E_\infty = m + [V(r_0) + N^2/8mr_0^2]\,, \tag{7.1.6}$$

where k has been approximated by N in the limit $N \to \infty$. Correspondingly, eq. (7.1.4) becomes

$$r_0^3 V'(r_0) = N^2/4m\,. \tag{7.1.7}$$

Thus it is clear that if condition (7.1.5) is satisfied, the large-N behaviour of the Klein–Gordon equation becomes identical to that of the nonrelativistic Schrödinger equation.

To find the range of the potential $V(r)$ for which the condition (7.1.5) may be satisfied, MIP have considered the power-law potential

$$V(r) = r^n\,. \tag{7.1.8}$$

Equation (7.1.4) then reads

$$nr_0^{n+2}(1 + N^2/4m^2r_0^2)^{1/2} = N^2/4m\,,$$

or

$$(nr_0^2/m)^2\{[(r_0/N)2m]^4 + [(r_0/N)2m]^2\} = 1\,. \tag{7.1.9}$$

If condition (7.1.5) is obeyed, then (7.1.9) may be approximated by

$$(nr_0^n/m)[(r_0/N)2m]^4 \approx 1\,, \tag{7.1.10}$$

which yields

$$r_0 \approx N^{2/(n+2)}\,. \tag{7.1.11}$$

Compatibility of (7.1.11) with (7.1.5) now fixes n within the following range:

$$-2 < n < 0\,. \tag{7.1.12}$$

Thus for a class of potentials $V(r) = r^n$ $(-2 < n < 0)$ (MIP criterion) the relativistic corrections are nonleading in $1/N$. As a specific example, MIP have considered the Coulomb potential. For a spin-zero particle the exact energy for the Coulomb potential $V(r) = -\beta/r$ obtained by solving the N-dimensional Klein–Gordon equation is [220]

$$E = m\{1 + 4\beta^2[2n - 2l + 1 + \sqrt{(k-2)^2 - 4\beta^2}]^{-2}\}^{1/2}\,, \tag{7.1.13}$$

which when expanded in powers of $1/k$ reads for the GS

$$E = m\left(1 - \sum_{n=1}^{\infty} C_n \frac{2\beta^{2n}}{k^{2n}}\right), \tag{7.1.14}$$

where

$$C_1 = 1 + 2/k + 3/k^2 + 4/k^3 + \cdots\,, \tag{7.1.15a}$$

$$C_2 = 1 + 8/k + 34/k^2 + 108/k^3 + \cdots\,, \tag{7.1.15b}$$

$$C_3 = 2(1 + 12/k + 81/k^2 + 400/k^3 + \cdots)\,, \tag{7.1.15c}$$

etc. In (7.1.14) the first term denotes the rest energy, the $n = 1$ term represents the nonrelativistic energy and the $n = 2$ term gives the lowest-order relativistic correction. It is clear that the relativistic corrections are indeed nonleading in $1/N$ (or more precisely in $1/k$).

7.1.2. An iterative procedure for finite-N corrections

One may notice that the usual methods of $1/N$ expansion cannot be directly applied to (7.1.2), for in this equation the energy E and the potential $V(r)$ are coupled. We shall describe here an iterative $1/N$ expansion technique [103]. Let us rewrite (7.1.2) in the following form:

$$(-\mathrm{d}^2/\mathrm{d}r^2 + (k-1)(k-3)/4r^2 - \{[E_a - V(r)]^2 - m^2\} + 2(E - E_a)[V(r) - V(r_0)])u(r)$$
$$= \{[E - V(r_0)]^2 - [E_a - V(r_0)]^2\}u(r)\,, \tag{7.1.16}$$

where the artificially introduced quantity E_a may be chosen arbitrarily. Let us first choose $E_a = E_\infty$. So far we have made no approximation. The simplest approximation now consists in neglecting the term $2(E - E_\infty)\,[V(r) - V(r_0)]$, which is obviously nonleading in $1/k$. Then denoting the energy in this approximation by $E^{(1)}$ and the wave function by $u^{(1)}(r)$, we get from (7.1.16)

$$(-\mathrm{d}^2/\mathrm{d}r^2 + (k-1)(k-3)/4r^2 - \{[E_\infty - V(r)]^2 - m^2\})u^{(1)}(r) = \mathscr{E}_1 u^{(1)}(r)\,, \tag{7.1.17}$$

where

$$\mathscr{E}_1 = [E^{(1)} - V(r_0)]^2 - [E_\infty - V(r_0)]^2\,. \tag{7.1.18}$$

Equation (7.1.17) now looks like a Schrödinger equation to which the usual $1/N$ expansion techniques can be applied. After a $1/N$ expansion solution for $E^{(1)}$ is obtained, we set (in the next improved approximation) $E_a = E^{(1)}$ in (7.1.16) and drop the term $2(E - E^{(1)})\,[V(r) - V(r_0)]$. This is obviously a better approximation. Let us denote the energy and the wave function in this approximation by $E^{(2)}$ and $u^{(2)}(r)$ respectively. We then obtain, after some reorganization of the terms,

$$(-\mathrm{d}^2/\mathrm{d}r^2 + (k-1)(k-3)/4r^2 - \{[E_\infty - V(r)]^2 - m^2\}$$
$$+ 2(E^{(1)} - E_\infty)[V(r) - V(r_0)])u^{(2)}(r) = \mathscr{E}_2 u^{(2)}(r)\,, \tag{7.1.19}$$

where

$$\mathscr{E}_2 = [E^{(2)} - V(r_0)]^2 - [E_\infty - V(r_0)]^2\,. \tag{7.1.20}$$

Equation (7.1.19) can again be solved by any of the standard $1/N$ expansion methods. After $E^{(2)}$ is determined, E_a should be replaced by $E^{(2)}$ in (7.1.16) and then, ignoring the term $2(E - E^{(2)})$ $\times[V(r) - V(r_0)]$, the resulting equation can be solved to obtain a more improved approximate solution $E^{(3)}$. This iterative procedure can be continued to achieve any desired order of accuracy. For example, the nth order iterative result $E^{(n)}$ is to be obtained from

$$(-\mathrm{d}^2/\mathrm{d}r^2 + (k-1)(k-3)/4r^2 - \{[E_\infty - V(r)]^2 - m^2\} + 2(E^{(n-1)} - E_\infty)[V(r) - V(r_0)])u^{(n)}(r)$$
$$= \mathscr{E}_n u^{(n)}(r)\,, \quad n = 1, 2, 3, \ldots\,, \tag{7.1.21}$$

where

$$\mathscr{E}_n = [E^{(n)} - V(r_0)]^2 - [E_\infty - V(r_0)]^2\,, \tag{7.1.22}$$

$$E^{(0)} \equiv E_\infty\,. \tag{7.1.23}$$

The method outlined above is applicable to any spherically symmetric potential. For the Coulomb potential $V(r) = -\beta/r$, the results of the first few iterations for the GS energy are

$$E^{(1)} = m[1 - (2\beta^2/k^2)(1 + 2/k + 3/k^2 + \cdots) - (2\beta^4/k^4)(1 + 8/k + 50/k^2 + \cdots) - \cdots], \tag{7.1.24a}$$

$$E^{(2)} = m[1 - (2\beta^2/k^2)(1 + 2/k + 3/k^2 + \cdots) - (2\beta^4/k^4)(1 + 8/k + 34/k^2 + \cdots) - \cdots], \tag{7.1.24b}$$

$$E^{(3)} = m[1 - (2\beta^2/k^2)(1 + 2/k + 3/k^2 + \cdots) - (2\beta^4/k^4)(1 + 8/k + 34/k^2 + \cdots) - \cdots]. \tag{7.1.24c}$$

Comparison of (7.1.24) with (7.1.14) shows that in the case of the Coulomb potential one may stop at $E^{(2)}$ to get the exact series for the lowest-order relativistic correction. For higher-order relativistic corrections however, one will have to make further iterations.

7.2. The Dirac equation

By following the standard factorization method [221, 222] the radial equations for a spin-1/2 fermion moving in an N-dimensional spherically symmetric potential $V(r)$ may be obtained as a pair of two first-order differential equations

$$dF/dr = \{[K - \tfrac{1}{2}(N-3)]/r\}F - [E - m - V(r)]G, \tag{7.2.1a}$$

$$dG/dr = \{[K + \tfrac{1}{2}(N-3)]/r\}G + [E + m - V(r)]F, \tag{7.2.1b}$$

where

$$K = -(\tfrac{1}{2}k_j - 1), \qquad k_j = N + 2j, \qquad j = l + \tfrac{1}{2}. \tag{7.2.2}$$

Equations (7.2.1a) and (7.2.1b) can be transformed to a single second-order differential equation [104]

$$\frac{d^2g}{dr^2} + \frac{V'(r)}{A}\frac{dg}{dr} + \left(-\frac{K(K+1)}{r^2} + \frac{V'(r)}{A}\frac{K}{r} + AB\right)g = 0, \tag{7.2.3}$$

$$A = E + m - V(r), \qquad B = E - m - V(r), \tag{7.2.4}$$

which on substituting

$$u = \sqrt{A}g, \tag{7.2.5}$$

reduces to

$$-\frac{d^2u}{dr^2} + \left(\frac{V''(r)}{2A} + \frac{3V'(r)^2}{4A^2} + \frac{V'(r)(k_j-2)}{2Ar} + \frac{(k_j-2)(k_j-4)}{4r^2} - \{[E - V(r)]^2 - m^2\}\right)u(r) = 0. \tag{7.2.6}$$

The energy in the large-N limit is given by

$$E_\infty = V(r_0) + m(1 + k_j^2/4m^2r_0^2)^{1/2}\,, \tag{7.2.7}$$

where r_0 is to be determined from

$$r_0^3V'(r_0)(1 + k_j^2/4m^2r_0^2) = k_j^2/4m\,. \tag{7.2.8}$$

Equations (7.2.7) and (7.2.8) are the same as the corresponding ones obtained from the Klein–Gordon equation except for the spin terms in k_j. Therefore we can make the same analysis as was made in subsection 7.1.1 to conclude that for potentials $V(r) = r^n$ $(-2 < n < 0)$, the relativistic and spin corrections are nonleading in $1/N$ [102].

Approximating (7.2.6) by

$$-\frac{d^2u}{dr^2} + \left(\frac{V''(r)}{2A_\infty} + \frac{3V'(r)^2}{4A_\infty^2} + \frac{V'(r)(k_j-2)}{2A_\infty r} + \frac{(k_j-2)(k_j-4)}{4r^2} - \{[E - V(r)]^2 - m^2\}\right)u(r) = 0\,, \tag{7.2.9}$$

where

$$A_\infty = E_\infty - m + V(r_0)\,, \tag{7.2.10}$$

Atag [104] has applied the iterative procedure described in section 7.1.2 to solve for its energy spectrum. For the Coulomb potential $(-\beta/r)$ the first two iterations yield for the GS energy,

$$E^{(1)} = m[1 - (2\beta^2/k_j^2)(1 + 4/k_j + 12/k_j^2 + \cdots) - (2\beta^4/k_j^4)(1 + 8/k_j + 88/k_j^2 + \cdots) - \cdots]\,, \tag{7.2.11a}$$

$$E^{(2)} = m[1 - (2\beta^2/k_j^2)(1 + 4/k_j + 12/k_j^2 + \cdots) - (2\beta^4/k_j^4)(1 + 8/k_j + 40/k_j^2 + \cdots) - \cdots]\,. \tag{7.2.11b}$$

The Dirac equation for the Coulomb potential is, however, exactly solvable in N dimensions and the GS energy reads

$$E = m\left(1 - \sum_{n=1} D_n \frac{2\beta^{2n}}{k_j^{2n}}\right)\,, \tag{7.2.12}$$

$$D_1 = 1 + 4/k_j + 12/k_j^2 + \cdots\,, \tag{7.2.13a}$$

$$D_2 = 1 + 8/k_j + 40/k_j^2 + \cdots\,, \tag{7.2.13b}$$

etc. Thus the large-N iterative procedure appears to yield the exact energy series for the Coulomb potential.

Finally we note that in the Coulomb problem the actual expansion parameter is either β/k (in the Klein–Gordon case) or β/k_j (in the Dirac case). These expansions are therefore useful for heavier atoms for which the usual perturbative expansions are not very productive.

8. Large-N expansion for potential scattering

Though in bound state problems the large-N methodology has found immense application, very little effort has so far been directed towards applying it to scattering problems. Sinha-Roy, Gangopadhyay and Dutta-Roy (SGD) [105] extended for the first time in 1984 the idea of the large-dimension expansion to low energy potential scattering by providing an intuitive approach for the estimation of scattering lengths for spherically symmetric potentials.

The radial Schrödinger equation for a particle moving in an N-dimensional potential $V_N(r)$ can be written (in units $\hbar = 2m = 1$) for s wave ($l = 0$) states as

$$[\mathrm{d}^2/\mathrm{d}r^2 + K^2 - (N-3)(N-1)/4r^2 - V_N(r)]u(r) = 0\,, \tag{8.1}$$

where K^2 is the energy of the particle. For potentials less singular than $1/r^2$ at the origin, the wavefunction has the form

$$u(r) \sim r^{(N-1)/2}, \tag{8.2}$$

for small r. Thus if the potential $V_N(r)$ is short-ranged, then

$$V_N(r)u(r) = 0\,, \quad \text{at } r = 0\,, \qquad V_N(r)u(r) = 0\,, \quad \text{at large } r\,, \tag{8.3}$$

i.e. the product $V_N(r)u(r)$ appearing in (8.1) will in general possess a maximum, which in the limit $N \to \infty$ and with a suitable scaling of the potential can be replaced by an approximate effective potential [105]

$$V_{\mathrm{eff}}(r) \approx -\lambda\delta(r-a)\,. \tag{8.4}$$

The location a of the δ-function is obtained by extremizing the quantity

$$f(r) = \ln V_N(r) + \tfrac{1}{2}(N-1)\ln r\,, \tag{8.5}$$

at $r = a$, while the strength λ is given by

$$\lambda = \mathrm{e}^{f(a)}(2\pi/f''(a))^{1/2}\,, \quad f''(a) = \mathrm{d}^2 f/\mathrm{d}r^2|_{r=a}\,. \tag{8.6}$$

Following SGD, let us consider for an illustration the case of a Yukawa potential $V(r) = -V_0\,\mathrm{e}^{-\mu r}/\mu r$, the N-dimensional version of which may be prescribed as

$$V_N(r) = -V_0\,\mathrm{e}^{-\tilde{\mu}Nr}/\tilde{\mu}Nr\,, \tag{8.7}$$

where $\tilde{\mu} = \mu/N$. Accordingly we have

$$f(r) = -\tilde{\mu}Nr + \tfrac{1}{2}(N-3)\ln r\,, \tag{8.8}$$

which on extremization yields

$$a = (N-3)/2\tilde{\mu}N \approx 1/2\tilde{\mu}\,. \tag{8.9}$$

The effective potential V_{eff} is now given by

$$V_{\text{eff}}(r) = -(2V_0/N\tilde{\mu})\sqrt{\pi/N}\,\mathrm{e}^{-N/2}\,\delta(r - 1/2\tilde{\mu})\,, \tag{8.10}$$

which in three dimensions reads

$$V_{\text{eff}}(r) = -(2V_0/\mu)\sqrt{\pi/N}\,\mathrm{e}^{-3/2}\,\delta(r - 3/2\mu)\,. \tag{8.11}$$

The partial-wave phase shift for the δ-function potential is known,

$$\tan\delta_l = K\lambda a^2[j_l(Ka)]^2/[1 + Ka^2 j_l(Ka)\eta_l(Ka)]\,. \tag{8.12}$$

For the s-wave scattering we get

$$\tan\delta_0 = \lambda\sin^2(Ka)/[K - \lambda\sin(2Ka)/2]\,, \tag{8.13}$$

and the corresponding scattering length is given by

$$a_{\text{sc}} = -a^2/(1 - \lambda a)\,. \tag{8.14}$$

Thus once the pseudopotential (8.4) is obtained the scattering length can be calculated by using (8.14). SGD have employed this technique to calculate the scattering lengths of various potentials as a function of the depth of the potential and have compared their results with the exact values and also with the Born approximation results (see fig. 8.1). Clearly the simple-minded large-N approximation is quite adequate to exhibit the broad qualitative features of the scattering length, but the quantitative agreement with the exact values is rather poor particularly when the strength of the interaction approaches the optimal strength necessary to produce a zero-energy bound state. Actually, for all the potentials considered by these authors, the divergence in the scattering length appears at a depth much lower than that which is necessary to support a bound state. Furthermore, the accuracy of the phase shifts calculated within their scheme is far from encouraging.

Gangopadhyay, Dutt and Varshni (GDV) [106] have attributed the discrepancies of SGD's results to the fact that in their scheme the location of the effective pseudopotential has no dependence on the strength of the actual potential. GDV have demonstrated that the required strength-dependence of the position of the effective δ-function potential can be generated by resorting to the shifted $1/N$ expansion. They have considered, for an illustration, the Gaussian potential, for which the SGD prescription leads to the poorest results. In this modified approach the position a of the effective δ-function potential contains the shift parameter, which GDV have chosen so as to reproduce the

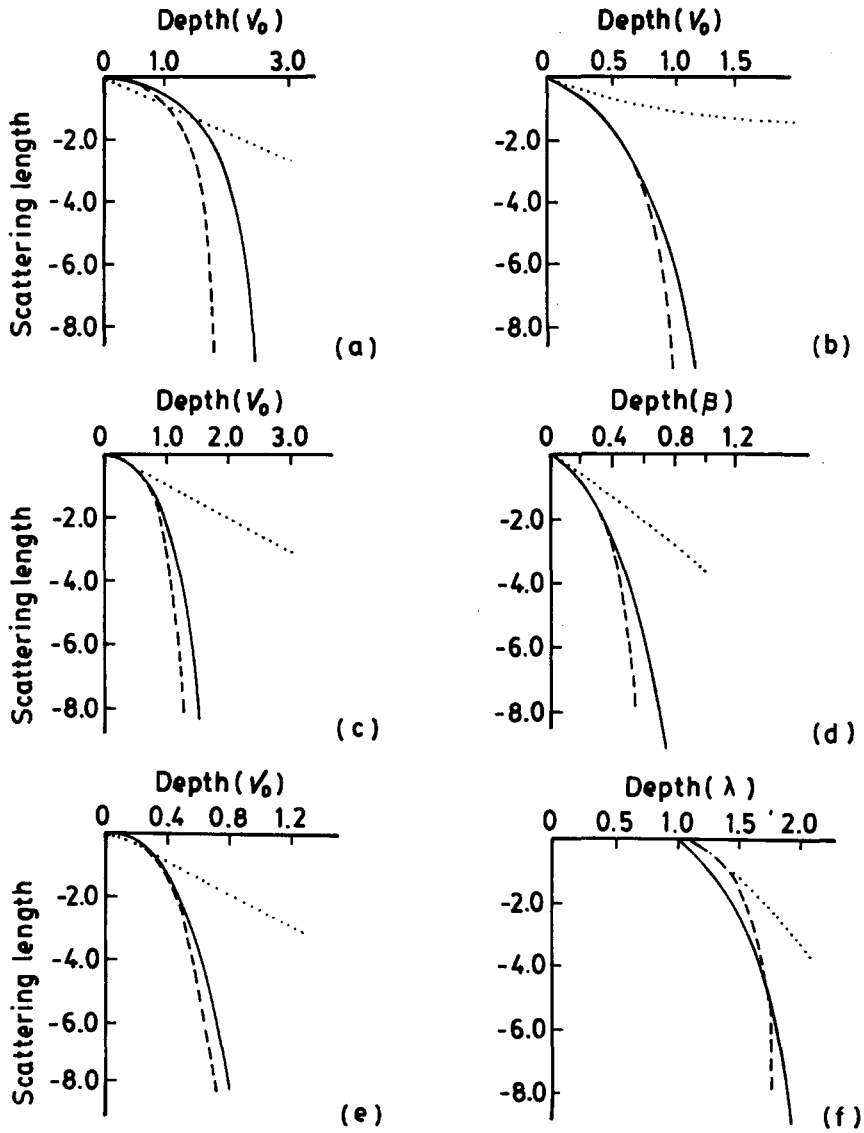

Fig. 8.1. Scattering length, in units where $\mu = 1$ as a function of potential depth for various potentials. Exact values are displayed by solid curves, while the Born and large-N approximation results are shown by dotted and dashed curves respectively. (a) Gaussian, $V(r) = -V_0\, e^{-\mu r^2}$; (b) exponential, $V(r) = -V_0\, e^{-\mu r}$; (c) Yukawa, $V(r) = -(V_0/\mu)\, e^{-\mu r} r^{-1}$; (d) Bargmann, $V(r) = -2\beta\mu^2\, e^{-\mu r}\,(1 + e^{-\mu r})^{-2}$; (e) Hulthen, $V(r) = -V_0 \times e^{-\mu r}(1 - e^{-\mu r})^{-1}$; (f) Poschl Teller, $V(r) = -\mu^2\lambda(\lambda - 1)\ \mathrm{sech}^2\mu r$ (after Sinha-Roy et al. [105], by courtesy of J. Phys.).

position of the first divergence of the exact scattering length as a function of the strength of the actual potential. Their results for the scattering lengths show significant improvement over the unshifted results (see fig. 1 of ref. [106]). Also the accuracy of the phase shifts calculated by the shifted $1/N$ expansion is quite satisfactory, at least in the low-energy region (see fig. 2 of ref. [106]). Of course the accuracy deteriorates with increasing strength. Furthermore, the results deviate more and more from the exact ones as the energy increases. GDV have conjectured that this discrepancy may possibly be reduced if the shift parameter can be made to depend also on the energy.

Further progress has been reported in this field by Sukhatme, Lauer and Imbo [107]. They have developed a systematic $1/N$ expansion for the phase shifts with the δ-function results as the leading-order term. Using this formalism they have calculated scattering lengths of various potentials. Agreement with the exact results is found to be excellent (see fig. 2 of ref. [107]). The reliability of the $1/N$ expansion however decreases with increasing energy. Thus the $1/N$ expansion turns out to be complementary to the Born approximation (in the context of scattering phenomena).

9. Concluding remarks

We have reviewed here some selected methods of $1/N$ expansion and their application to quantum mechanics, atomic and molecular physics and some $O(N)$ invariant models of general interest. We have also discussed how the concept of the large-N expansion originated and was applied in the area of critical phenomena where it served a novel purpose. This idea was introduced in the field of quantum mechanics by Ferrel and Scalapino [35] in 1974 and since then it has continued to evoke interest in this field.

Several $1/N$ expansion schemes in general are now available in the literature. We have included in our review only those which have proved particularly useful in quantum mechanics and atomic physics. For simple potential problems, the Riccati equation method [37, 39] and the perturbed oscillator methods [2, 45] have been most extensively used. One useful aspect of the Riccati equation methods is that here one has to ultimately solve a set of algebraic recurrence relations which are easier to handle (compared to the differential equations). Moreover, in these methods one can go to very high order in $1/N$ in a quite systematic and straightforward manner. But the disadvantage with these methods is that the calculations become highly cumbersome even for the first excited state and perhaps intractable for states with large radial quantum numbers. The perturbed oscillator method (discussed in the initial part of 3.2.1) [2], on the other hand, provides the entire energy spectrum but since the method uses the RSPT it is difficult to go beyond the first few low-order terms. The shortcomings of these two methods are overcome in the hypervirial $1/N$ expansion [65] in which one can obtain the entire energy spectrum and can also calculate the higher-order corrections in a systematic way, although the method is applicable only to a limited class of potentials.

The collective field method and the pseudospin formulation are [41, 42] two powerful methods of $1/N$ expansion and they display the semiclassical nature of the large-N approximation in a quite transparent way. These methods are extremely general in the sense that they find applications in a wide variety of fields. The collective field method has been successfully applied to problems such as the high density Bose plasma [57], N identical harmonic oscillators [57], the $SU(N)$ matrix model [57], the linear and nonlinear σ models [59] and the Yang–Mills gauge theories [140, 142]. This method is best-suited for investigating the collective excitations of a system. For the high density Bose plasma it yields the random phase approximation (RPA) results of Bohm and Pines [173]. A similar equivalence with the RPA is found also in the Lipkin, Glick and Meshkov model [32], as we have mentioned in section 1. It emerges in general that the large-N collective field equations are obtainable from the original classical equations subject to some special constraints. A similar conclusion has been drawn by Bardakci [108]. The assertion of Chatterjee [63] is that the large-N quantum energy can be obtained from the classical energy expression if one imposes the quantum constraint of minimum uncertainty.

The pseudospin formulation is a group theoretic approach which exploits the underlying symmetry of a problem. This method has been applied to the BCS superconductivity [41], the quartic anharmonic

oscillator [41], a wide class of spherically symmetric potentials [42], ϕ^4 theory on a lattice [116], the Zeeman problem [42], the helium atom [89], and some other problems in condensed matter physics [137] and field theory [115]. The method shows [41] that N^{-1} plays the same role as $\hbar$ in the path integral representation of the transition amplitude and hence that the limit $N \to \infty$ is in effect equivalent to the stationary phase approximation. The pseudospin equations obtained in this approximation are then given by original classical equations of motion subject to some algebraic constraint, called the quantum Casimir constraint. The pseudospin formulation also shows [42] that if a problem has an $O(N)$ symmetry then $1/N$ is indeed a natural expansion parameter. Through the works of various authors it has thus become well-established that the large-N limit is some kind of a classical limit. The generalized coherent state formalism of Yaffe [3] is devoted to the explication of this point. Another interesting point is that even for the most unfavourable case of $N = 3$ and $l = 0$, just the first few terms of the $1/N$ expansion often lead to accurate results, implying that $1/3$ is a sufficiently small number, a point that is still not well-understood.

$1/N$ expansions are often found to have poor convergence, particularly for higher excited states. To improve the situation various modified approaches have been suggested, such as the shifted $1/N$ expansion [67], the expansion about the singular points in the energy as a function of the dimensionality [84], and the supersymmetric $1/N$ expansion [86]. These modified $1/N$ expansions have been applied to a number of potentials and they have proved extremely useful in general. In the shifted $1/N$ expansion, or in its supersymmetric version, one can choose the shift parameter as one wishes. Sukhatme and Imbo's prescription is physically motivated and leads to an order-independent shift, whereas the prescription of Maluendes et al. [80, 81] leads to an order-dependent shift. Papp [82, 83] has advocated yet another choice. It would be however more desirable, in our opinion, to attach to the choice of the shift parameter some rigorous mathematical basis, variational or otherwise.

Recent years have witnessed a flurry of investigations in large-N atomic physics. The idea of $1/N$ expansion has been used to study the effect of a uniform magnetic field on the hydrogen atom spectra, as early as 1980, by Mlodinow and Papanicolaou [42], who have applied their pseudospin formulation to obtain results which are accurate for field strength up to the intermediate region. Bender, Mlodinow and Papanicolaou [88] (BMP), using the algebraic recursion method, have extended MP's calculation to higher orders. A simple Shanks transformation extends the applicability of the $1/N$ expansion well into the domain of strong fields.

Considerable effort has lately gone into understanding the helium spectrum by $1/N$ expansions. Quite an interesting physical picture emerges even in the leading-order calculation. For example, in the large-N limit the helium atom behaves like a nonlinear molecule with well-defined bond lengths and bond angle (which comes out to be slightly greater than 90°) and the energy is equally partitioned between the two electrons in the GS. This indicates that there is a high degree of electron correlation in the helium GS, and thus gives a conceptual basis for the doubly excited states in the helium spectrum. Furthermore, since the bond lengths and the bond angle will in reality fluctuate around their well-defined equilibrium values, one also expects vibrational and rotational levels accompanying the doubly excited states in the helium atom spectrum. Highly accurate results for the GS energy have been obtained by Herschbach and collaborators [94–100] by using the idea of dimensional interpolation. Extending the Hartree–Fock method for two-electron atoms to N dimensions Loeser and Herschbach [97] have shown that the dimensional interpolation between the exactly known $N \to 1$ and $N \to \infty$ limits yields even better results for the correlation energy than for the total energy. The Hartree–Fock method is thus expected to prove promising in the context of using the dimensional interpolation for many-electron systems. Also in molecular physics, the large-N expansion has emerged as a useful approximation.

In the field of scattering, only a very few investigations have been made so far and the subject is still in its infancy. The results that have been reported are however sufficiently encouraging to motivate further investigations. Attention should be given particularly to developing a systematic and rigorous $1/N$ expansion scheme for computing scattering cross-sections. In view of the nonperturbative character of the $1/N$ expansion, such a scheme is expected to be useful for solving strong potential scattering problems.

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