Space Quantization in a Gyrating Magnetic Field

I. I. RABI
Columbia University, New York, N. Y.
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The nonadiabatic transitions which a system with angular momentum J makes in a magnetic field which is rotating about an axis inclined with respect to the field are calculated. It is shown that the effects depend on the sign of the magnetic moment of the system. We therefore have an absolute method for measuring the sign and magnitude of the moment of any system. Applications to the magnetic moment of the neutron, the rotational moment of molecules, and the nuclear moment of atoms with no extra-nuclear angular momentum are discussed.

IN a previous paper[1] the effect of a rapidly varying magnetic field on an oriented atom possessing nuclear spin and extra-nuclear angular momentum. It appeared that it was possible to deduce the sign of the magnetic moment of the nucleus from the nature of the nonadiabatic transitions which occur if the field rotates an appreciable amount in the time of a Larmor rotation. This effect was applied experimentally[2] with the method of atomic beams to measure the sign of the proton, deuteron, K^{39}, etc. The evaluation of the sign was possible because the experiment decided whether the h.f.s. level was normal or inverted. Since the sign of the electronic moment is known to be negative a normal level meant positive nuclear moment and an inverted level negative moment.

Clearly it is desirable to find another effect which will make it possible to find the sign of the nuclear moment in cases where the normal state of the atom is one in which there is no electronic angular momentum as in the alkaline earths. Spectroscopic methods where applicable will yield this information, but there are numerous important instances in which molecular and atomic beam methods are the only ones available. For example, it would be very desirable to measure the sign of the moment of the neutron directly. Although it would be very difficult to apply atomic beam methods to the neutron, the polarization effect of magnetized iron suggested by Bloch may possibly be useful in this connection as a device for measuring the degree of depolarization caused by the nonadiabatic transitions to be described below. Another example is the sign of the moment arising from molecular rotation which results in a positive contribution from the motion of the nuclei about the centroid and a negative contribution from the electrons.

The following considerations should make it possible to make the same sort of observations with simple systems as are made in the Einstein-de Haas and Barnett experiments: namely, the magnitude and sign of the gyromagnetic ratio.

Consider a simple system such as a neutron with magnetic moment $\mu=-g\mu_0 J$, where g is the Landé g factor, J is the total angular moment due to all causes. If g is positive the total moment is negative as in the spinning electron. If g is negative the moment is positive. In a magnetic field H the system precesses with the Larmor frequency $\nu=g\mu_0 H/h$. If g is positive the precession is in the positive direction and if negative in the negative direction. We shall now consider our system initially quantized with magnetic quantum number m in a field H which is constant in magnitude but rotates with a frequency $\omega/2\pi$ about some direction which is at an angle ϑ with respect to the direction of the field.

This problem was solved by Güttinger[3] for the particular case when the angle is $\pi/2$. He found that transitions will occur to other magnetic levels with quantum number m' when $\omega/2\pi$ is of the order of magnitude of ν. The transition probabilities in this case do not depend on the direction of the field. It will be shown that in the more general case the direction of rotation introduces an asymmetry into the problem and

[1] Rabi, Phys. Rev. **49**, 324 (1936).
[2] Kellogg, Rabi and Zacharias, Phys. Rev. **50**, 472 (1936); Torrey and Rabi, Phys. Rev. **51**, 379A (1937) Millman and Zacharias, Phys. Rev. **51**, 380A (1937).
[3] P. Güttinger, Zeits. f. Physik **73**, 169 (1931); E. Majorana, Nuovo Cimento **9**, 43 (1932).

as a consequence with the same $|\nu|$ and $|\omega|$ the transition probabilities will be different depending on whether g is positive or negative. The Majorana and Güttinger arrangements do not possess this property.

It will suffice to consider only the case where $J=\frac{1}{2}$ since the solution of more general problems depends only on the solution of this simple case. The Schrödinger equation for this case is

$$i\hbar\dot{\psi}=\mathcal{H}\psi,\qquad \mathcal{H}=g(\mu_0/2)(\sigma_1H_1+\sigma_2H_2+\sigma_3H_3). \tag{1}$$

If we set

$$\psi=C_{\frac{1}{2}}\psi_{\frac{1}{2}}+C_{-\frac{1}{2}}\psi_{-\frac{1}{2}} \tag{2}$$

we obtain from (1)

$$\begin{aligned} dC_{\frac{1}{2}}/dt&=(-ig\mu_0/2\hbar)[H_3C_{\frac{1}{2}}+(H_1-iH_2)C_{-\frac{1}{2}}],\\ dC_{-\frac{1}{2}}/dt&=(-ig\mu_0/2\hbar)[-H_3C_{-\frac{1}{2}}+(H_1+iH_2)C_{\frac{1}{2}}]. \end{aligned} \tag{3}$$

With the substitutions

$$\begin{aligned} &H_1=H\sin\vartheta\cos\varphi,\quad H_2=H\sin\vartheta\sin\varphi,\quad H_3=H\cos\vartheta,\\ &g\frac{u_0}{2}\frac{H}{\hbar}\cos\vartheta=a,\quad g\frac{u_0}{2\hbar}H\sin\vartheta=b,\quad \varphi=\omega t, \end{aligned} \tag{4}$$

which represents a field H constant in magnitude and precessing about the z direction with angular velocity ω, we have

$$\begin{aligned} dC_{\frac{1}{2}}/dt&=-iaC_{\frac{1}{2}}-ibe^{-i\omega t}C_{-\frac{1}{2}},\\ dC_{-\frac{1}{2}}/dt&=iaC_{-\frac{1}{2}}-ibe^{i\omega t}C_{\frac{1}{2}}. \end{aligned} \tag{5}$$

Therefore

$$d^2C_{\frac{1}{2}}/dt^2+i\omega\frac{d}{dt}C_{\frac{1}{2}}+(a^2+b^2-\omega a)C_{\frac{1}{2}}=0. \tag{6}$$

The solution of this familiar equation is

$$\begin{aligned} C_{\frac{1}{2}}&=Ae^{ip_1t}+Be^{ip_2t},\\ C_{-\frac{1}{2}}&=-e^{i\omega t}\left[\frac{a+p_1}{b}Ae^{ip_1t}+\frac{a+p_2}{b}e^{ip_2t}\right],\\ p_1&=-(\omega/2)+\tfrac{1}{2}(\omega^2+4a^2+4b^2-4\omega a)^{\frac{1}{2}},\\ p_2&=-(\omega/2)-\tfrac{1}{2}(\omega^2+4a^2+4b^2-4\omega a)^{\frac{1}{2}}. \end{aligned} \tag{7}$$

The quantities A and B are determined from the initial conditions and the normalization condition $|C_{\frac{1}{2}}|^2+|C_{-\frac{1}{2}}|^2=1$. If ψ_α and ψ_β are the vectors which correspond to $m=+\frac{1}{2}, -\frac{1}{2}$, respectively, in the direction of H we obtain[4]

$$\begin{aligned} \psi_\alpha&=(2\alpha)^{-\frac{1}{2}}(\beta e^{-i\omega t}\psi_{\frac{1}{2}}+\alpha\psi_{-\frac{1}{2}}),\\ \psi_\beta&=(2\gamma)^{-\frac{1}{2}}(-\beta e^{-i\omega t}\psi_{\frac{1}{2}}+\gamma\psi_{-\frac{1}{2}}),\\ \beta&=\sin\vartheta,\quad \alpha=1-\cos\vartheta,\quad \gamma=1+\cos\vartheta,\\ \alpha^2&+\beta^2=2\alpha,\quad \gamma^2+\beta^2=2\gamma. \end{aligned} \tag{8}$$

If $\psi(0)=\psi_\alpha(0)$ at $t=0$ we start with the system quantized in the direction of H with $m=\frac{1}{2}$, and

$$\begin{aligned} &A+B=\beta/(2\alpha)^{\frac{1}{2}},\\ &-\left(\frac{a+p_1}{b}\right)A-\left(\frac{a+p_2}{b}\right)B=\frac{\alpha}{(2\alpha)^{\frac{1}{2}}}. \end{aligned} \tag{9}$$

Utilizing these values of A and B and setting $n=(\omega^2+4a^2+4b^2-4\omega h)^{\frac{1}{2}}$ we obtain for the probability amplitudes

$$\begin{aligned} C_{\frac{1}{2}}&=\frac{1}{(2\alpha)^{\frac{1}{2}}}e^{-(i\omega t/2)}\left[\beta\cos\frac{n}{2}t-i\frac{(2\alpha b+2\beta a-\beta\omega)}{n}\sin\frac{n}{2}t\right],\\ C_{-\frac{1}{2}}&=\frac{1}{(2\alpha)^{\frac{1}{2}}}e^{i\omega t/2}\left[\alpha\cos\frac{n}{2}t-i\frac{(2\beta b-2\alpha a+\alpha\omega)}{n}\sin\frac{n}{2}t\right]. \end{aligned} \tag{10}$$

The probability of finding the system in a state with $m=-\frac{1}{2}$ with respect to H is therefore

$$P_{(\frac{1}{2},-\frac{1}{2})}=|\psi_\beta{}^*\psi|^2=\frac{1}{(4\alpha\gamma)^{\frac{1}{2}}}|-\beta e^{i\omega t}C_{\frac{1}{2}}+\gamma C_{-\frac{1}{2}}|^2 \tag{11}$$

with the values given in Eq. (12)

$$P_{(\frac{1}{2},-\frac{1}{2})}=\frac{\beta^2\omega^2}{n^2}\sin^2\frac{n}{2}t. \tag{12}$$

In terms of the Larmor frequency $\nu=g\mu_0H/h$, the

[4] Dirac, *Quantum Mechanics*, p. 70.

angle ϑ and the frequency of rotation $r=\omega/2\pi$

$$P_{(\frac{1}{2},\,-\frac{1}{2})}=\frac{\sin^2\vartheta r^2}{\nu^2+\nu^2-2\nu r\cos\vartheta}\sin^2\pi t \times(\nu^2+\nu^2-2\nu r\cos\vartheta)^{\frac{1}{2}}. \quad (13)$$

This result reduces to Güttinger's formula when $\vartheta=\pi/2$. For other values of ϑ it is apparent that the transition probability for given ϑ, H and ω will be quite different depending on whether g is positive or negative since ν appears linearly in the result. Expressed in another way we may say that $P_{(\frac{1}{2},\,-\frac{1}{2})}$ depends on the direction of rotation of the field for a given sign and magnitude of g.

Since rotating fields are usually realized by allowing the system to pass through a field which changes in direction from point to point, the total change in direction is fixed. If we set $\varphi=2\pi rt$ we obtain from (15) setting $\nu/r=q$

$$P_{(\frac{1}{2},\,-\frac{1}{2})}=\frac{\sin^2\vartheta}{1+q^2-2q\cos\vartheta}\sin^2\frac{\varphi}{2} \times(1+q^2-2q\cos\vartheta)^{\frac{1}{2}}. \quad (14)$$

If we set $q=\cos\vartheta$ which can be arranged by suitably varying the magnitude and direction of the field,

$$P_{(\frac{1}{2},\,-\frac{1}{2})}=\sin^2(\tfrac{1}{2}\varphi\sin\vartheta), \quad (15)$$

which can be made as close to unity as one pleases by arranging experimental conditions so that $2\pi>\varphi>\pi$. If one were then to leave everything unchanged but reversed the direction of rotation we would get

$$P_{(\frac{1}{2},\,-\frac{1}{2})}=\frac{\sin^2\vartheta}{1+3\cos^2\vartheta}\sin^2\left[\frac{\varphi}{2}(1+3\cos^2\vartheta)^{\frac{1}{2}}\right], \quad (16)$$

a very much smaller quantity.

It is clear therefore that from this qualitative difference and a knowledge of the sense of rotation one can infer the sign of the moment. The magnitude may be inferred from the absolute value of the fields and their direction and the angular velocity of its rotation.

It is unnecessary to consider the details of the realization of these rotating fields and the detection of these transitions since similar conditions have already been obtained in the experiments cited above.

To generalize these results we apply the general result of Majorana[2, 3] for any value of J

$$P_{(\alpha,\,m,\,m')}=(\cos\tfrac{1}{2}\alpha)^{4J}(J+m)!(J+m')!\times(J-m)!(J-m')! \times\left[\sum_{\nu=0}^{2J}\frac{(-1)^\nu(\tan\tfrac{1}{2}\alpha)^{2\nu-m+m'}}{\nu!(\nu-m+m')!(J-m-\nu)!(J-m'-\nu)!}\right]^2, \quad (17)$$

where the value of the parameter α is given by

$$\sin^2\tfrac{1}{2}\alpha=P_{(\frac{1}{2},\,-\frac{1}{2})} \quad (18)$$

and depends only on g and not on m or J.

It may be of interest to note that in cases where we have a coupled system such as a nuclear spin coupled to molecular rotation, in which the coupling is weak and g small, the field required for these transitions can be such that the component systems are completely decoupled. As was shown by Motz and Rose,[5] each system would then make these transitions independently. In particular, if the moments of the two systems are opposite in sign it should be possible to arrange conditions so that only one of these two systems makes these transitions. In this case it should be possible by means of the focusing methods developed in this laboratory to measure directly one moment in the presence of another (rotational and nuclear) with only slight interference.

[5] Motz and Rose, Phys. Rev. **50**, 348 (1936).