Exact Results for a Quantum Many-Body Problem in One Dimension*

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(Received 12 April 1971)

We investigate exactly a system of either fermions or bosons interacting in one dimension by a two-body potential $V(r)=g/r^2$ with periodic boundary conditions. In addition to rederiving known results for correlation functions and thermodynamics in the thermodynamic limit, we present expressions for the one-particle density matrix at zero temperature and particular (nontrivial) values of the coupling constant g, as a determinant of order $N\times N$. These concise expressions allow a discussion of the momentum distribution in the thermodynamic limit. In particular, for a case of repulsive bosons, the determinant is evaluated explicitly, exhibiting a weak (logarithmic) singularity at zero momentum, and vanishing outside of a "Fermi" surface.

In a previous paper,[1] to be called I, we investigated the properties of a one-dimensional quantum N-body system interacting by the two-body potential $V(r)=g/r^2$. In order to take the thermodynamic limit, it was necessary to confine the system by a weak harmonic well. The ground-state wave function was then found to be intimately related to the probability distribution function for eigenvalues of matrices from a Gaussian ensemble.

The use of a weak harmonic well to confine the system, although it served the purpose, destroyed the translational invariance of the problem. In this paper we consider the same system with periodic boundary conditions. The resulting solution allows an even more complete description of the properties of this quantum fluid.

In order that the potential be periodic, and thus our Hamiltonian be periodic, we shall add the interaction of pairs of particles once, twice, etc. around the ring of circumference L. That is, we take our two-body potential to be

$$V(r)=g\sum_{n=-\infty}^{\infty}(r+nL)^{-2}=\frac{g\pi^2}{L^2}\left[\sin\left(\frac{\pi r}{L}\right)\right]^{-2}. \qquad (1)$$

For large systems, of course, the additional terms are unimportant. The mathematics, however, is considerably simplified by starting with periodic equations.

Thus, we wish to find the ground state of the Hamiltonian

$$H=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{g\pi^2}{L^2}\sum_{i>j}\left[\sin\frac{\pi(x_i-x_j)}{L}\right]^{-2} \qquad (2)$$

subject to

$$\Psi(x_1,\ldots,x_i+L,\ldots,x_N)=\Psi(x_1,\ldots,x_i,\ldots,x_N)\,. \qquad (3)$$

Peculiarities of g/r^2 potentials have been discussed at length in I, and will not be treated further here.

As before, we seek a wave function of product form, or a Bijl-Dingle-Jastrow[2] wave function:

$$\Psi=\prod_{i>j}[\psi(x_i-x_j)]^\lambda\,. \qquad (4)$$

This form is familiar as a trial wave function in the theory of quantum fluids.

Let us denote the logarithmic derivative of ψ by ϕ:

$$\phi=\psi'/\psi\,,\quad \phi'=\psi''/\psi-(\psi'/\psi)^2\,. \qquad (5)$$

Then the kinetic-energy operator acting on this wave function is

$$-\sum_i\frac{\partial^2\Psi}{\partial x_i^2}=\Big(-2\lambda\sum_{i>j}[\phi'(x_i-x_j)+\lambda\,\phi^2(x_i-x_j)] +2\lambda^2\sum_{\text{triples}}[\phi(x_1-x_2)\,\phi(x_2-x_3) + \text{cyclic permutations of } 123]\Big)\Psi\,. \qquad (6)$$

The last summation in Eq. (6) is over the $N(N-1)\times(N-2)/3!$ distinct triples ijk.

Apart from the particular Hamiltonian of Eq. (2), we can, for general information, ask when is the expression in braces on the right-hand side of Eq. (6) equal to a sum of two-body operators, and hence Ψ the wave function of some one-dimensional many-body problem. (Although this problem may in general not be very physical, at least the potential is of the two-body type.) We immediately see that it is sufficient that a solution of the three-body problem for some nonzero λ be given by Ψ of product form.

Let us write a typical triple term as

$$T\equiv\phi(x)\,\phi(y)+\phi(y)\,\phi(z)+\phi(z)\,\phi(x) =\phi(x)\,\phi(y)-\phi(x+y)\,[\phi(x)+\phi(y)]\,,\quad x+y+z=0\,. \qquad (7)$$

Therefore, we ask that there exist some even function $f(x)$ such that

$$T(x,y,z)=f(x)+f(y)+f(z)\,,\quad x+y+z=0\,. \qquad (8)$$

One may verify that Eq. (8) is satisfied for the known instances where a Hamiltonian has a Ψ of product form as a wave function[3]:

(i) $V(r) = -2c\delta(r)$, $\quad c \geq 0$;

(ii) $V(r) = \omega^2 r^2 + g/r^2$.

For our problem, with the particular Hamiltonian of Eq. (2), we try the product wave function

$$\Psi = \prod_{i>j} \left| \sin \frac{\pi(x_i - x_j)}{L} \right|^\lambda, \quad x_i > x_j \tag{9}$$

and thus

$$\phi(r) = (\pi/L)\cot(\pi r/L). \tag{10}$$

Recognizing that $\lambda = 1$ corresponds to the ground-state wave function for free fermions on a ring of circumference L, then by the considerations above, we know Ψ to be the solution of some problem.

First, comparing the addition formula for the cotangent function with Eq. (8), we find $T = \pi^2/L^2$. The complete expression for Eq. (6) then becomes

$$-\sum_i \frac{\partial^2 \Psi}{\partial x_i^2} = \left(\frac{\lambda^2 \pi^2 N(N^2-1)}{3L^2} - \frac{2\lambda(\lambda-1)\pi^2}{L^2} \times \sum_{i>j} \left[\sin \frac{\pi(x_i - x_j)}{L} \right]^{-2} \right) \Psi. \tag{11}$$

On comparison with the original Schrödinger equation, we find Ψ will be a solution if

$$2\lambda(\lambda-1) = g \quad \text{or} \quad \lambda = \tfrac{1}{2}[1 + (1+2g)^{1/2}]. \tag{12}$$

The energy eigenvalue is

$$E = \tfrac{1}{3}\lambda^2\pi^2 N(N^2-1)/L^2 \quad \text{or} \quad E/L \to \tfrac{1}{3}\lambda^2\pi^2 d^3. \tag{13}$$

This agrees with Eq. (24) of I. Other properties, of course, agree as well, for we are simply verifying that bulk properties are independent of boundary conditions; this is an assumption of the usual statistical mechanics.

Let us rewrite Ψ in terms of the variables $\theta_i = 2\pi x_i/L$ as

$$\Psi = \sqrt{C} \prod_{j>l} |e^{i\theta_j} - e^{i\theta_l}|^\lambda, \quad \theta_j > \theta_l \tag{14}$$

$$\Psi^2 = C \prod_{j>l} |e^{i\theta_j} - e^{i\theta_l}|^\beta, \quad \beta \equiv 2\lambda. \tag{15}$$

Then, as before in I, we recognize the expression for Ψ^2 to be identical with a probability distribution function from the theory of random matrices. However, this time the appropriate matrix ensembles are Dyson's circular ensembles,[4] instead of the Gaussian ensembles. Let us simply outline the relevant results from the literature on random matrices: (a) The normalization constant C is proven to be[5]

$$C^{-1} = (2\pi)^N \frac{\Gamma(1+\lambda N)}{[\Gamma(1+\lambda)]^N} \tag{16}$$

in contrast to the conjecture for the normalization constant of I. In terms of the x variables, the appropriate normalization constant is $(2\pi/L)^N C$. (b) The particle density d is constant and equal to N/L. (c) The pair correlations are the same as given in I. However, previously the case $\beta = 4$ rested on a conjecture, whereas we now have a proof.[6] In addition, Dyson in a recent paper[7] has given simple and elegant expressions for multiple particle correlation functions when $\beta = 1, 2, 4$. (d) And finally, the thermodynamics for this system has been given by the author in a previous paper.[8]

Although correlation functions and thermodynamics for this problem do not depend on the statistics of the particles, one should not conclude that all properties are independent of statistics. Now we derive concise expressions for the one-particle density matrix as a determinant of order $N \times N$, for the cases $\beta = 1, 2, 4$ and either fermions or bosons. Of course, we shall be especially interested in the macroscopic occupation of the zero-momentum state for bosons or a sharp Fermi surface for fermions.

The one-particle density matrix is defined as

$$\rho(x - x') = N \int_0^L dx_1 \cdots \int_0^L dx_{N-1} \Psi^*(x_1, \ldots, x_{N-1}, x') \times \Psi(x_1, \ldots, x_{N-1}, x). \tag{17}$$

This is normalized so that $\rho(0) = d$. The Fourier transform $n(k)$ is the distribution of particles with momentum k. It is normalized so that $\int n(k)\, dk = d$.

We shall write $\rho(x) \equiv L^{-1} R(2\pi x/L)$. Thus

$$R(\alpha) = N \frac{\Gamma(1+\lambda)\Gamma[1+\lambda(N-1)]}{\Gamma(1+\lambda N)} \times \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{N-1} \Psi_{N-1}^2 \prod_{i=1}^{N-1} f(\theta_i, \alpha), \tag{18}$$

where

$$f(\theta, \alpha) = 2^\lambda |\cos\theta - \cos(\tfrac{1}{2}\alpha)|^\lambda \times \begin{cases} 1, & \text{bosons} \\ \epsilon(\tfrac{1}{4}\alpha^2 - \theta^2), & \text{fermions}. \end{cases} \tag{19}$$

$\epsilon(x)$ is the sign function. Thus, for instance, if we have fermions with λ equal to an odd integer or bosons with λ equal to an even integer, then we may ignore the absolute sign in the expression for $f(\theta, \alpha)$ of Eq. (19) while dropping the final factor. Further, in these instances, since f is a polynomial in $\cos\frac{1}{2}\alpha$ of degree λ, $n(k)$ will vanish whenever $|k| > \pi\lambda d$. We now consider the following cases.

a. $\lambda = 1$. This case corresponds to free fermions or bosons with an infinite hard core. Free fermions are well understood, and we find

$$\rho(x) = \frac{\sin(\pi d x)}{\pi x}, \quad n(k) = \begin{cases} 1/2\pi, & |k| < \pi d \\ 0, & |k| > \pi d. \end{cases} \tag{20}$$

However the case of bosons, although at first glance simple, is actually quite complicated. It has been treated rather thoroughly in the literature.[9] Let us merely state that there is no Bose-Einstein condensation into a single momentum state. The indication is that $n(k)$ diverges as $|k|^{-1/2}$ at the origin; it may diverge less strongly.

b. $\lambda=\frac{1}{2}$. Since the expression Eq. (18) for $R(\alpha)$ is the expectation value with respect to a Ψ for $N-1$ particles, of a product of identical functions of single θ variables, we may use the Mehta-Gaudin method[10] to express $R(\alpha)$ as the determinant of a matrix. We find, assuming N is odd,

$$R(\alpha)=N\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}N+\frac{1}{2})}{\Gamma(\frac{1}{2}N+1)}\,\det[F]\,, \tag{21}$$

where

$$F_{p,q}=(p/4\pi)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\theta\,d\phi\,f(\theta,\alpha)\,f(\phi,\alpha)\,\epsilon(\theta-\phi)\times(\cos p\phi\,\sin q\theta-\cos p\theta\,\sin q\phi)\,,\quad p,q=\tfrac{1}{2},\tfrac{3}{2},\ldots,\tfrac{1}{2}N-1\,. \tag{22}$$

$f(\theta,\alpha)$ is given by Eq. (19) with $\lambda=\frac{1}{2}$. We see that F is an $\frac{1}{2}(N-1)\times\frac{1}{2}(N-1)$ matrix.

c. $\lambda=2$. Likewise, using the theorem relating the symplectic and orthogonal ensembles, we may again use the Mehta-Gaudin method to express $R(\alpha)$ as the determinant of a matrix:

$$R(\alpha)=N\,\frac{\Gamma(3)\,\Gamma(2N-1)}{\Gamma(2N+1)}\,\det[F]\,, \tag{23}$$

where

$$F_{p,q}=(1/2\pi)\int_{-\pi}^{\pi}d\phi\,f(\phi,q)\,[\sin q\phi\,\sin p\phi+(p/q)\cos q\phi\,\cos p\phi]\,,\quad p,q=\tfrac{1}{2},\tfrac{3}{2},\ldots,N-\tfrac{3}{2}\,. \tag{24}$$

$f(\phi,\alpha)$ is given by Eq. (19) with $\lambda=2$. F is an $(N-1)\times(N-1)$ matrix.

In a later paper, these matrix expressions for the one-particle density matrix will be investigated in detail. We simply note here, that in the case $\lambda=2$ bosons, the matrix F_{pq} has nonzero elements only for $|p-q|\le 2$. This simple form allows the explicit determination

$$\rho(x)=\frac{\mathrm{Si}(2\pi dx)}{2\pi x}\,,\qquad \mathrm{Si}(x)\equiv\int_0^x\frac{\sin y}{y}\,dy\,.$$

$$n(k)=\begin{cases}(4\pi)^{-1}\ln(2\pi d/|k|)\,, & |k|\le 2\pi d\\ 0\,, & |k|\ge 2\pi d\,.\end{cases} \tag{25}$$

It is really quite remarkable in what a diversity of situations the g/r^2 potential or the Ψ^2 distribution function appears. It brings to mind the similar versatility of Bethe's hypothesis as a nontrivial many-body wave function.

In this paper, we have related a one-dimensional quantum many-body problem with long-range forces to the theory of random matrices. Ψ^2 in turn is related to Dyson's one-dimensional Coulomb gas and a model for interacting Brownian particles. The g/r^2 potential arises as the dividing case between one-dimensional systems with and without phase transitions. Both the g/r^2 potential and a distribution similar to Ψ^2 occur in Anderson's treatment of the Kondo problem. Also, Ψ^2 is familiar in the theory of quantum fluids as that part of the ground-state wave function of a one-dimensional system due to zero-point motion. Needless to say, a deeper understanding of these interrelationships would be very welcome.

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*Work supported in part by National Science Foundation Grant No. GP 13632.

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