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Study of Exactly Soluble One-Dimensional N-Body Problems

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In this paper it is shown that several cases of one-dimensional N-body problems are exactly soluble. The first case describes the motion of three one-dimensional particles of arbitrary mass which interact with one another via infinite-strength, repulsive delta-function potentials. It is found in this case that the stationary-state solution of the scattering of the three particles is analogous to an electromagnetic diffraction problem which has already been solved. The solution to this analogous electromagnetic problem is interpreted in terms of particles. Next it is shown that the problem of three particles of equal mass interacting with each other via finite- but equal-strength delta-function potentials is exactly soluble. This example exhibits rearrangement and bound-state effects, but no inelastic processes occur. Finally it is shown that the problem of N particles of equal mass all interacting with one another via finite- but equal-strength delta functions is exactly soluble. Again no inelastic processes occur, but various types of rearrangements and an N-particle bound state do occur. These rearrangements and the N-particle bound state are illustrated by means of a series of sample calculations.

I. INTRODUCTION

SINCE the advent of quantum theory, physicists have relied on exactly soluble problems to describe some of the strange effects which have quantum mechanical origin. The way in which the potential enters the Schrödinger wave equation makes this equation soluble only for a very limited class of potentials, and with the exception of the Coulomb potential and the harmonic-oscillator potential, the exactly soluble problems are not particularly good imitations of the interactions which exist in the physical world. On the other hand, these exactly soluble problems illustrate a broad range of effects which are present in the physical world, and therefore at least allow us a qualitative description of the processes which can occur, and perhaps an insight into perturbation and approximation methods which can be used in more physical problems.

One would hope that exact solutions of N-body problems would be of help in producing similar insights into qualitative effects and possible approximation methods for problems of this type. There are, however, additional mathematical difficulties introduced by the presence of more than two particles which have made the exact solution of an N-body problem a more elusive goal.[1,2] These mathematical difficulties are related to the broad range of physical effects which are possible due to the presence of more than two particles.

Let us discuss the kinds of effects which may occur in N-body problems. We know that the many-particle wavefunction will contain all of the information about two-particle interactions because we may isolate two particles by putting the other particles so far away that their influence on the remaining two is negligible. Under these circumstances we will recover the two-particle wavefunction. More complicated effects arise when the N particles are close together in space and time.

Our task is to discuss those effects which arise from the proximity of the N particles, so let us focus our attention on the simplest problem which contains these effects, the three-body problem. Even here we expect a large number of physical effects. We expect finite probabilities for any two particles with an attractive potential between them to be bound in the final state, even though all of the particles are free in the initial state. Also, there will be finite probabilities for the particles to be scattered from a free state to another free state with a different distribution of energy among the particles. If a free particle is incident on a bound state we would expect that this free particle could ionize, excite, or perhaps replace a bound particle. In general we would expect finite matrix elements between any initial and final state which have the same energy.

In view of the many effects which exist in problems of this type it is not surprising that exact solutions or even reliable approximation methods are difficult to find. In order to construct exactly soluble problems we are going to be forced to make many simplifying assumptions. We are going to deal with a three-body problem where all of the particles move in only one dimension and interact with one

[1] E. Lieb and H. Koppe, Phys. Rev. **116**, 367 (1959).
[2] R. Jost, Commun. Math. Helv. **28**, 173 (1954).

another through delta-function potentials. We will later argue that this does not *a priori* restrict the number of physical effects which can occur, except for the fact that two particles which interact through an attractive delta-function potential have only a single bound state, and therefore an incident third particle cannot excite to another bound state, but only to the continuum.

II. FORMULATION OF THE PROBLEM

We consider the Hamiltonian

$$H = -\frac{\hbar^2}{2}\left(\frac{1}{M_1}\frac{d^2}{dx_1^2} + \frac{1}{M_2}\frac{d^2}{dx_2^2} + \frac{1}{M_3}\frac{d^2}{dx_3^2}\right) + A\,\delta(x_1 - x_2) + B\,\delta(x_2 - x_3) + C\,\delta(x_1 - x_3),$$

which arises when three particles of mass M_1, M_2, M_3 at positions x_1, x_2, x_3 interact with one another via delta-function potentials with strengths A, B, C which depend on coordinate differences between particles.

If we make the change of variables

$$z = (M_1 + M_2 + M_3)^{-\frac{1}{2}}(M_1x_1 + M_2x_2 + M_3x_3),$$

$$y = \frac{[M_3(M_1 + M_2)]^{\frac{1}{2}}}{(M_1 + M_2 + M_3)^{\frac{1}{2}}}\left(\frac{M_1x_1 + M_2x_2}{M_1 + M_2} - x_3\right),$$

$$x = (M_1M_2)^{\frac{1}{2}}/(M_1 + M_2)^{\frac{1}{2}}(x_1 - x_2),$$

the resulting Hamiltonian will be

$$H = -\frac{\hbar^2}{2}\left(\frac{d^2}{dz^2} + \frac{d^2}{dy^2} + \frac{d^2}{dx^2}\right) + A\,\delta\!\left(\frac{1}{\mu_{12}}\,x\right) + B\,\delta\!\left(\frac{1}{\mu_{23}}\,[x\cos\alpha + y\sin\alpha]\right) + C\,\delta\!\left(\frac{1}{\mu_{13}}\,[x\cos\beta - y\sin\beta]\right),$$

$$\frac{1}{\mu_{ij}} = \left[\frac{1}{M_i} + \frac{1}{M_j}\right]^{\frac{1}{2}},$$

$$\tan\alpha = [(M_1 + M_2 + M_3)M_2/M_1M_3]^{\frac{1}{2}},$$

$$\tan\beta = [(M_1 + M_2 + M_3)M_1/M_2M_3]^{\frac{1}{2}}.$$

Transformations of this type are discussed in the Appendix. Formulas are derived which are valid for N particles and not restricted to one dimension.

If we remove the center-of-mass motion of all three particles and eliminate the time from Schrödinger's equation, the stationary-state equation for the internal motion of the three particles will be

$$\left[-\frac{\hbar^2}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + A\,\delta\!\left(\frac{1}{\mu_{12}}\,x\right) + B\,\delta\!\left(\frac{1}{\mu_{23}}\,[x\cos\alpha + y\sin\alpha]\right) + C\,\delta\!\left(\frac{1}{\mu_{13}}\,[x\cos\beta - y\sin\beta]\right) - E\right]\psi = 0.$$

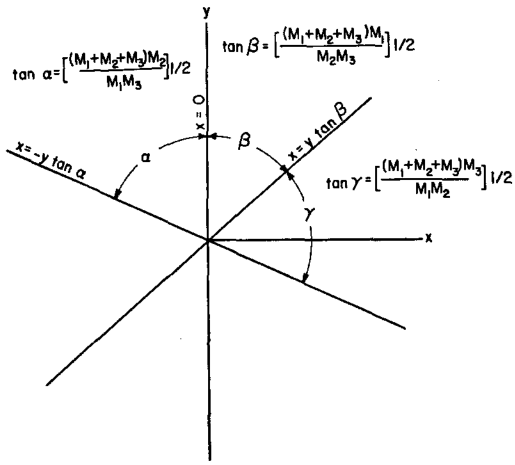

FIG. 1. Potential diagram for three particles interacting in one dimension.

This differential equation may be interpreted as describing the motion of a single particle in a two-dimensional space. Interpreting the differential equation this way may seem awkward since it tends to obscure the true nature of the physical problem, but we will find that the interpretation in terms of particles is not difficult once we have the solution to this mathematically equivalent problem.

The potential in which the single two-dimensional particle moves is zero everywhere except on the lines $x = 0$, $x = y\tan\beta$, $x = -y\tan\alpha$, as shown in Fig. 1. We see that we must solve $(\nabla^2 + k^2)\psi = 0$ everywhere except on the boundaries provided by the "line" delta functions. It is well known that the boundary condition on such a delta-function line is that there is a discontinuity in the normal derivative of the wavefunction which is equal to the strength of the delta function times the value of the wavefunction on the boundary.

Even this problem is an extremely difficult one mathematically and only limited progress has been made toward its solution. In Sec. IV we will solve a special case of this problem where the masses of the particles and the strengths of the delta functions are chosen in a particular way.

Another special case of interest has already been solved for us. If we take the strength of the delta-function interactions to be infinite, then the wavefunction must approach zero on the delta-function boundaries. This problem is analogous to the diffraction of electromagnetic waves from wedges and corners made of conducting materials, and is soluble for arbitrary angles between the delta-function lines and hence for arbitrary masses of the interacting

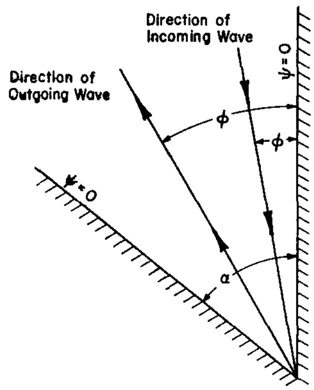

FIG. 2. Coordinate system for the electromagnetic diffraction problem.

particles. We propose to make a quantum mechanical interpretation of the solution to this analogous electromagnetic problem.

III. INFINITE-STRENGTH DELTA FUNCTIONS

A. Discussion of the Solution

The analysis of the infinite-delta-function problem is simplified by the fact that the wavefunction is confined to one of the wedges of Fig. 1. We interpret this as meaning that the particles stay in a particular order in the one dimension. The particles cannot transmit through one another because they cannot penetrate the infinite-strength delta-function wall.

Most readers will recall that certain wedge problems may easily be solved by the method of images, which is equivalent to tracing rays through a wedge until the wave vector for the ray is pointed in such a direction that it will not hit one of the sides of the wedge again. For a wedge of arbitrary angular opening, there will be two such rays emerging from the wedge corresponding to the bifurcation of the incident wave by the two sides of the wedge. If, however, the angular opening of the wedge is π/n, the reflected rays will emerge in parallel with one another and fill all of the space within the wedge. In this case the entire solution to the problem requires only the sum of the incident plus the reflected waves.

For a wedge of arbitrary angular opening, the outgoing waves will emerge in different directions and either overlap or not fill all of the space within the wedge, and thus something must be added to the solution to fit the continuity conditions along the so-called "boundaries of geometric optics" which are the terminators of the regions filled by the outgoing waves.

Diffraction problems of this type have received extensive treatment in the literature beginning with Sommerfeld's paper in 1896.[3] The interested reader can trace the literature from Oberhettinger[4] (our principal source), who has given a particularly convenient treatment for quantum mechanical interpretation.

The solution, as we have argued that it should, consists of all of the reflections in the various regions of space plus a diffraction term which fits the continuity conditions along the boundaries of geometric optics. For the scattering solution we are interested only in the far-field part of this diffraction term. For a discussion of the general boundary conditions and form of the solution of problems of this type, see Ref. [5]. We write the far-field solution as

$$\psi = \psi_{\text{incident}} + \text{two-body reflections} + f(\varphi, \varphi', \alpha)(e^{ikr}/r^{\frac{1}{2}}) + O(r^{-\frac{3}{2}}),$$

where φ and φ' are, respectively, the angles of the incoming and outgoing k vectors, and α is the angular opening of the wedge. The coordinate system is shown in Fig. 2.

Oberhettinger has provided an expansion from which we may calculate f,

$$f(\varphi, \varphi'; \alpha) = \frac{\lambda^{\frac{1}{2}}}{4} \frac{\sin(\pi^2/\alpha)}{\alpha} \times \left[\frac{\sin\dfrac{\pi\varphi}{\alpha} \sin\dfrac{\pi\varphi'}{\alpha}}{\left(\sin\dfrac{\pi\varphi}{\alpha}\sin\dfrac{\pi\varphi'}{\alpha}\right)^2 - \left(\cos\dfrac{\pi\varphi}{\alpha}\cos\dfrac{\pi\varphi'}{\alpha} - \cos\dfrac{\pi^2}{\alpha}\right)^2} \right].$$

Notice that $f(\varphi, \varphi'; \pi/n) = 0$, so that, as asserted earlier, there is no diffraction when the wedge is of angular opening π/n. The singularity of f along the boundaries of geometric optics, where $\pi \pm (\varphi \pm \varphi') = 2n\alpha$, is required to fit continuity conditions.

Before going further let us analyze a particular example to practice interpreting this solution in terms of particles.

B. Analysis of a Particular Example

As an example we will take the interaction between two particles of equal mass and a third particle of infinite mass. The two light particles interact with each other, but we will assume that only one of the light particles interacts with the massive particle. All interactions are infinite-strength, repulsive delta functions.

The potential diagram for this problem is shown in Fig. 3. We must solve $(\nabla^2 + k^2)\psi = 0$ with $\psi = 0$ on the lines $x = 0$ and $x = y$. The interpretation of the solution of this problem is simplified

[3] A. Sommerfeld, Math. Ann. **47**, 317 (1896).

[4] F. Oberhettinger, J. Res. Natl. Bur. Std. **61**, 343 (1958).

[5] E. Gerjuoy, Phys. Rev. **109**, 1806 (1958).

by the fact that the x coordinate of the potential diagram is the position of the x particle relative to the massive particle (the x particle is the one which interacts with the massive particle), and the y coordinate is the coordinate of the y particle relative to the massive particle.

If the incoming wave is in Region I no three-body reactions occur since Region I is a wedge of π/n. The total solution in this region is a succession of the two-body problems. The experimental arrangement corresponding to Region I would have the y particle starting to the left of the x particle as shown in Fig. 3.

In Region II the situation is more interesting because the angular opening is $\frac{3}{4}\pi$. The experimental arrangement which corresponds to this region has the x particle starting to the left of the y particle, again as shown in Fig. 3.

We will assume that the incoming beam is "collimated" in the sense that the x and y particles in the incoming beam are adjusted so as to be at the origin at about the same time. This introduces a correlated distribution in the incoming state, that is the probability of finding an x particle per unit length depends upon where the y particle is. The collimation is necessary because otherwise the probability per unit volume to find the x particle at x_0 and the y particle at y_0 would depend on interference terms between the incident wave, the two-particle reflected waves and the true three-body waves. By introducing a collimation we have allowed the possibility of positioning the detector outside the beam and the two-particle reflections where the true three-particle effects are directly observable without interference.

As is usual in problems of this type, the probability per second to be scattered into some "solid" angle is proportional to the incident flux. This flux is neither the x-particle flux nor the y-particle flux, but the magnitude of the vector flux in the two-dimensional space. We will assume that the incident beam is normalized such that

$$\psi_{\text{incident}} = (\rho_x\rho_y)^{\frac{1}{2}} e^{ik_x x} e^{ik_y y},$$

where ρ_x and ρ_y are, respectively, the number of x particles per unit length and the number of y particles per unit length. Under these circumstances the magnitude of the flux[6] is $\rho_x\rho_y(v_x^2 + v_y^2)^{\frac{1}{2}}$ and

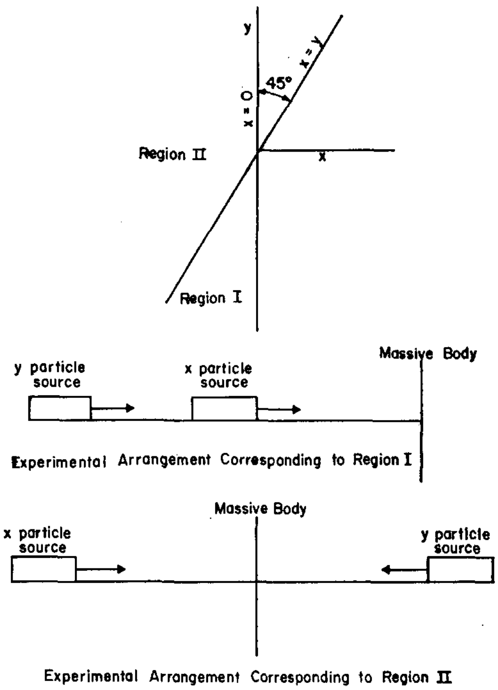

Fig. 3. Potential diagram and corresponding experimental arrangements for the particular example.

the reaction rate is

$$w = \rho_x\rho_y(v_x^2 + v_y^2)^{\frac{1}{2}} \, |f(\varphi)|^2$$

per second per unit solid angle.

A possible experiment would be to place an x-particle momentum detector to the left of the massive particle and measure the outgoing momentum distribution of x particles. This distribution would have two high peaks corresponding to the geometrical reflections of the incoming beam, but it would also have x particles of every possible momentum from zero up to the maximum possible consistent with the conservation of energy. The height of the peaks of the distribution would be proportional to the total number of the incoming particles, whereas the distributed portion of the spectrum would be proportional to the beam flux.

A second type of experiment would be to count coincidences of particle x situated between x_0 and $x_0 + \Delta x_0$ and particle y situated between y_0 and $y_0 + \Delta y_0$. The coincidence rate for this experiment is computed from the reaction rate given above where $\tan \varphi' = x_0/y_0$. For fixed $\Delta x_0 \, \Delta y_0$, the coincidence rate would decrease as

$$(x_0^2 + y_0^2)^{-\frac{1}{2}}.$$

C. Summary of the Infinite-Delta-Function Results

The goals of analyzing the infinite-strength delta-function problem were limited. Of the three-body

[6] It can be verified that this quantity is the scale factor for the reaction rate by calculating the reaction rate from the "golden rule" or by analyzing what happens to each Fourier component of a situation where a packet of x particles is incident from the left and a packet of y particles is incident from the right.

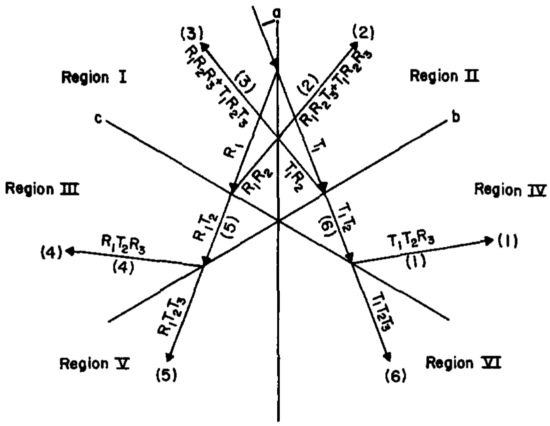

FIG. 4. Ray diagram which applies when the incoming (6) hits potential (a) first.

physical effects outlined in the Indrotuction the only one we expected to see was the redistribution of energy among the three particles, and this we have seen in our particular example. We could not have expected to see any of the other effects, because by using infinite-repulsive delta-functions for all of the interactions we have guaranteed that there will be no bound states to be ionized or rearranged.

We have, however, learned a good deal about the structure of solutions of problems of this type, and we could at least make a guess as to the form the solutions might take if the delta-function walls were transmissive. We would guess that the wavefunction would consist of all of the transmitted and reflected waves in the various regions of space, plus diffracted waves which fit the continuity conditions along the boundaries of geometric optics. In spite of these insights no one has yet been able to construct a general solution to the problem where the interpotential angles are arbitrary.

Suppose we could construct a problem which bears the same relations to the finite-strength delta-function case as does the π/n wedge to the infinite-delta-function case, that is a problem in which there is no diffraction. If such a problem exists, all of the angles between the potential walls must be π/n because the transmission from wedge to wedge would assure that there would be some problability to get into a diffracting (non π/n) wedge. Since the three interpotential angles of Fig. 1 must add up to 180° there are only three possibilities for mass ratios where all angles are π/n. These possibilities are:

(1) The masses of two like particles are infinitesimal compared with that of a third particle. The interpotential angles in this case are 45°, 45°, and 90°.

(2) Particle 1 is of infinite mass compared to particle 2 which in turn is three times the mass of particle 3 (90°, 60°, 30°).

(3) All three particles have equal mass (all angles are 60°).

We have examined all three cases and it turns out that the first two possibilities will always diffract if the strengths of all of the potentials involved are finite, but as we will now proceed to show, the third possibility will not diffract if the strengths of all the potentials are the same.

IV. EXPLICIT SOLUTION TO A PROBLEM WITH FINITE-STRENGTH DELTA-FUNCTION POTENTIALS

A. Free-Particle Solution

The Hamiltonian for the internal motion of three-one dimensional particles of equal mass interacting with one another by equal-strength delta-function potentials is

$$H = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) - g\,\delta(x) - g\,\delta\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) - g\,\delta\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right).$$

We have chosen units so that $\hbar = M = 1$. If c is the "true" strength of the delta-function potentials, the equivalent strength is

$$-g = \sqrt{2}\,c.$$

The potential diagram for this problem is three-line delta functions intersecting at 60° angles. The method of solution will be to trace rays through this complex of delta functions and verify that there are no boundaries of geometric optics and hence no diffraction.

We again wish to take literally the mathematical equivalence of this Hamiltonian to a single particle in a two-dimensional space, and return to the interpretation of one-dimensional particles after we have solved the problem.

The potential diagram and the rays which result are shown in Figs. 4 and 5. Any sequence of reflections of the incident ray result in one of six rays as shown in Fig. 6. As is indicated in Fig. 6, there are three possible angles of incidence for these rays to strike a potential. These angles are φ, $60° + \varphi$, and $60° - \varphi$.

The rays transmit or reflect with an amplitude which is dependent only on the component of momentum perpendicular to the potential surface, that is, the sine of the angle of incidence. As is usual in problems of this type we do not have to consider path-length effects because two nearby pieces of the phase front travel the same distance between incoming and outgoing wave.

For a delta-function potential of strength g the transmission coefficient may easily be shown to be[7]

$$T = \frac{2ik/g}{2ik/g + 1} = \frac{S}{S+1}, \qquad S = \frac{2ik}{g}.$$

Similarly, the reflection coefficient is

$$R = \frac{-1}{2ik/g + 1} \frac{-1}{S+1},$$

where k is the component of the wave vector perpendicular to the delta-function surface.

We denote the six possible plane waves ψ_1 through ψ_6. Their momentum vectors are shown in Fig. 6. For convenience we have labeled the potentials a, b, and c and numbered the Regions I through VI.

In Fig. 4 we consider the incoming wave to be of Type 6 in Region I. The incoming wave may strike potential a first. If it does so it has an amplitude T_1 to be transmitted into Region II and an amplitude R_1 to be reflected into a Type-5 wave and stay in Region I. If the plane wave is transmitted through potential a, it will then hit potential b and again be transmitted or reflected, and so on. Notice that each ray interacts only three times, once at each angle of incidence, before becoming an outgoing ray. This is a consequence of equal-massed particles. Figure 5 illustrates the sequence of reflections we would obtain if the incoming Type-6 plane wave in Region I struck potential b first.

By following rays through the potential complex

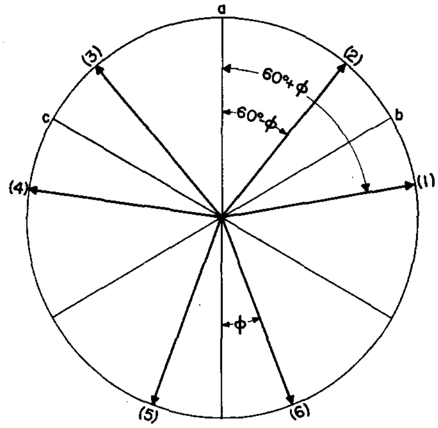

FIG. 6. Representation of the six plane waves which may be generated by reflections in the potential complex.

it is possible to evaluate the amplitude for each type of wave (i.e., Types 1 through 6) to be present in each region. As we have seen in the case of the infinite-strength delta functions, the incoming wave is bifurcated depending on which potential wall it hits first. As we saw in the infinite-delta-function case, the two halves of the plane wave must reunite to form a complete plane wave or diffraction will result. In this problem the two halves of the plane wave must be parallel and fill all of space and be equal in magnitude and phase.

It would seem at first that this problem would contain diffraction because the outgoing 2 in Region II is made up of the sum of two amplitudes from the potential a side and only one amplitude from the b side. From the a side we have

$$T_3R_2R_1 + R_3R_2T_1 = (s_3 + s_1)/(s_1 + 1)(s_2 + 1)(s_3 + 1),$$

but

$$s_2 - s_3 = (2ik/g)[\sin(60° + \varphi) - \sin(60° - \varphi)] = (2ik/g)\sin\varphi = s_1.$$

Thus

$$s_1 + s_3 = s_2$$

$$T_3R_2R_1 + R_3R_2T_1 = s_2/(s_3 + 1)(s_2 + 1)(s_1 + 1) = R_1T_2R_3,$$

which is exactly equal to the contribution from the b side; thus there is no diffraction. A similar situation occurs in Region III and the same relationship between s's again shows that there is no diffraction. In all of the other wedges it is clear that the two halves have the same magnitude and phase because the amplitudes of the two halves are made up of

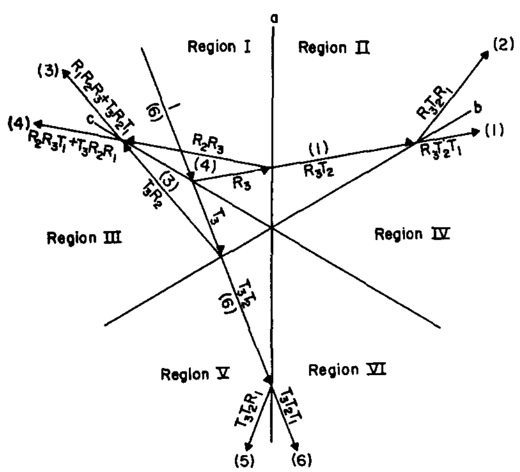

FIG. 5. Ray diagram which applies when the incoming (6) hits potential (c) first.

[7] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 1644.

TABLE I. Amplitude of plane waves in the various regions for the free wave solution.

Wave type	Region I	Region II	Region III	Region IV	Region V	Region VI
6	1	T_1	T_3	T_1T_3	T_3T_2	$T_1T_2T_3$
5	R_1		R_1T_2		$T_3T_2R_1$	
4	R_2R_3		$T_3R_2R_1 + R_3R_2T_1 = R_1T_2R_3$			
3	$R_1R_2R_3 + T_3R_2T_1$	T_1R_2	T_3R_2			
2	R_1R_2	$T_1R_2R_3 + R_1R_2T_3 = R_3T_2R_1$				
1	R_3	R_3T_2		$T_1T_2R_3$		

the product of three complex numbers which are the same for both halves.

Since there is no diffraction, the solution to the problem may be specified by giving the amplitude for each of the six plane waves in each of the six regions. These amplitudes are given in Table I.

B. Interpretation of the Solution in Terms of Particles

Let us now interpret our solution in terms of particles. If we have the three particles oriented along a line, say in the order 1 2 3 from left to right, and we are interested in the scattering of these three particles off one another, we see first that particle 1 must be traveling to the right faster than particle 2, which in turn must be traveling to the right faster than particle 3. This is because we are interested in a scattering problem and the initial state of a scattering problem must be such that if the state is projected backwards in time there are no collisions. If particle 1 were traveling slower than particle 2 and we projected this state backwards in time, there would be a collision between 1 and 2 at some time in the past. This "no collision in the past" condition is the condition that the incoming plane wave be aimed into a wedge in such a way that the tail of the k vector not intersect any of the delta-function walls.

In the potential diagram we recall that the incoming plane wave was bifurcated by the two walls bounding the Region-I wedge. In terms of particles this means that there are two possible first interactions among the three particles, viz., particle 1 may hit particle 2 or particle 2 may hit particle 3.

When two particles of equal mass collide in one dimension, the amplitude to reflect is the amplitude that the particles retain their original order along the one-dimensional line and the amplitude to transmit is the amplitude that they exchange positions along the line. Each of the six wedges of the potential diagram represent a given order of the particles along the line. If Region I is the order 1 2 3 from left to right, then Region II must be the order 2 1 3 because to get from Region I to Region II, particle 1 must transmit through particle 2 since a is the potential between particles 1 and 2.

In two-body collisions between particles of equal mass no new velocities are generated. That is to say that the particles in the incoming state have the same velocities as the particles in the outgoing state, although the particles may switch velocities during the collision.

What we have demonstrated in our problem is that there are no new velocities generated even though there are three particles present. If we make any small change in our problem, such as letting the strength of one of the delta functions change or one of the masses be slightly different from the other two, the character of the solution will change radically due to the presence of diffraction, and there will be an infinity of new velocities brought into the problem.

We now see that we have calculated the scattering, or S matrix for this problem, which is simply a 6×6 matrix, the elements of which tell how each of the six possible initial permutations of the particles on a line couple to each of the final six permutations of the particles. To specify this 6×6 matrix entirely, it is sufficient to write down how one permutation, say 1 2 3, propagates into the six possible outgoing permutations. This is shown in Table II. The remaining elements of the 6×6 matrix may be derived by a relabeling of the particles in the initial state.

TABLE II. Elements of the S matrix.

Wavefunction	Amplitude	Region
$\exp i(k_1x_1 + k_2x_2 + k_3x_3)$	1 (Incoming Wave)	Region I: $x_1 < x_2 < x_3$
$\exp i(k_3x_1 + k_2x_2 + k_1x_3)$	$\dfrac{-1 - s_1s_3}{(s_1+1)(s_2+1)(s_3+1)}$	Region I: $x_1 < x_2 < x_3$
$\exp i(k_2x_1 + k_3x_2 + k_1x_3)$	$\dfrac{s_2}{(s_1+1)(s_2+1)(s_3+1)}$	Region II: $x_2 < x_1 < x_3$
$\exp i(k_3x_1 + k_1x_2 + k_2x_3)$	$\dfrac{s_2}{(s_1+1)(s_2+1)(s_3+1)}$	Region III: $x_1 < x_3 < x_2$
$\exp i(k_1x_1 + k_3x_2 + k_2x_3)$	$\dfrac{-s_1s_2}{(s_1+1)(s_2+1)(s_3+1)}$	Region IV: $x_2 < x_3 < x_1$
$\exp i(k_2x_1 + k_1x_2 + k_3x_3)$	$\dfrac{-s_2s_3}{(s_1+1)(s_2+1)(s_3+1)}$	Region V: $x_3 < x_1 < x_2$
$\exp i(k_1x_1 + k_2x_2 + k_3x_3)$	$\dfrac{s_1s_2s_3}{(s_1+1)(s_2+1)(s_3+1)}$	Region VI: $x_3 < x_2 < x_1$

C. Rearranged Solutions

Although we have seen that there is no diffraction in the free wave solution there is still the possibility that a particle which is bound in the initial state may be free in the final state.

Suppose we choose particles 1 and 2 to be bound in the initial state. The boundary conditions at infinity requires that there be no incoming waves in particles 1 and 2. The only way this can happen is to make both T_1 and R_1 infinite by choosing $s_1 = -1$ so that the ratio of the amplitudes of the incoming to the outgoing waves in particles 1 and 2 is zero. In the limit as s_1 approaches -1, the ratio of T_1 to R_1 is unity.

Let us evaluate the entries in Table I in this limit, that is, we set $T_1 = R_1 = 1$ and any product of amplitudes which does not contain T_1 or R_1 is set equal to zero. This result is given in Table III. The amplitudes given in Table III may be verified to constitute a solution.

In order to interpret the entries in Table III we return to Fig. 4. The incoming 6 in Region I now is at an imaginary angle of incidence with respect to potential a. Its amplitudes to either transmit or reflect are infinite and equal. This transmission and reflection together represent an incoming wave, bound in potential a. This incoming wave is a decaying exponential in both the positive and negative x direction and a propagating exponential in the y direction.

The outgoing 2 in Region II makes the same imaginary angle to potential b as does the 5 in Region I to potential a. This outgoing 2 together with the outgoing 1 in Region IV form a bound-state wavefunction in the direction perpendicular to b which is propagating in the direction parallel to b. Notice that all of the intermediate states, such as the two in Region I tend to zero exponentially at infinity in the region in which they exist.

There are three outgoing waves, a bound state in potential b propagating up to the right parallel to potential b, a bound state in potential c propagating up to the left parallel to c, and a bound state in potential a propagating down parallel to potential a.

The interpretation in terms of particles may be made without difficulty since we know that potential b, for example, is the potential between particles 1 and 3 and if the outgoing wave is bound in potential b, particles 1 and 3 must be bound together

TABLE III. Amplitudes of waves in various regions for the rearranged solution.

Wave Type	Region I	Region II	Region III
6		1	
5	1		T_2
4			$R_2T_3 + R_3R_2$
3	$R_2R_3 + R_2T_3$	R_2	
2	R_2	$R_2R_3 + R_2T_3$	
1			
Wave Type	**Region IV**	**Region V**	**Region VI**
6	T_2		T_3T_2
5		T_3T_2	
4			
3			
2			
1	T_2R_3		

and particle 2 free. This would represent a rearrangement of our initial state which had particles 1 and 2 bound and particle 3 free.

We follow this line of reasoning and conclude that the amplitude to leave the vertex bound in potential a is the amplitude for no rearrangement to occur. This amplitude is given from Table III,

$$T_3T_2 = \left(\frac{s_3}{s_3+1}\right)\left(\frac{s_2}{s_2+1}\right) = \frac{s_3-1}{s_3+1}.$$

The amplitude to go up to the right along potential b and the amplitude to go up to the left along potential c are interpreted as the amplitudes for the particles 1 and 2, respectively, to have been replaced by the incoming particle 3. They are

$$T_2R_3 = R_2R_3 + R_2T_3 = -s_2/(s_3+1)(s_2+1),$$

since $s_1 = -1$;

$$\tfrac{1}{4}[2i(k_v^2 - \tfrac{1}{4}g^2)^{\frac{1}{2}}]\sin\varphi = -1,$$

$$\sin\varphi = \frac{+ig}{2(k_v^2 - \frac{1}{4}g^2)^{\frac{1}{2}}},$$

$$s_2 = \frac{2ik\sin(60° + \varphi)}{g} = \frac{\sqrt{3}\,ik_v}{g} - \frac{1}{2},$$

$$s_3 = \frac{2ik\sin(60° - \varphi)}{g} = \frac{\sqrt{3}\,ik_v}{g} + \frac{1}{2}.$$

The corresponding probabilities are:

Probability that 3 replaces 1 = Probability that 3 replaces 2,

$$\left(\frac{3ky^2}{g^2} + \frac{9}{4}\right)^{-1}.$$

$$\text{Probability of no rearrangement} = \frac{(3k_v^2/g^2 + \frac{1}{4})}{(3k_v^2/g^2 + \frac{9}{4})}.$$

Note the following results of this rearrangement solution:

(1) There is no ionization, that is there is no amplitude for the final state of the system to be three free particles. This is intimately connected with the lack of diffraction in the free wave solutions.

(2) There is no reflection. If some particle is incident from the left in the initial state, some particle will be moving to the right in the final state with the same velocity as the initially incident particle.

(3) Even if the incident particle is moving toward the bound pair with an infinitesimal velocity, it has a probability of $\frac{1}{9}$ to transmit through the bound pair.

D. The Bound State of Three Particles

In addition to the free wave and rearranged solutions there is one totally bound state of the three particles. The condition for this state is that there be no incoming waves in any of the particles. This is achieved by imposing the further condition that $s_3 = -1$ on the rearranged solution. All of the outgoing waves then have equal amplitude. Their k vectors are pure imaginary and are pointed along the bisectors of the angles of the six wedges. Apart from the normalization factor, this wavefunction may be written as

$$\psi = n\exp\{-\sqrt{2}\,g[|x_1 - x_2| + |x_2 - x_3| + |x_1 - x_3|]\},$$

where x_1, x_2, x_3 are the positions of the three particles along the one-dimensional line. This wavefunction is totally symmetric to the interchange of any pair of particles and its energy is $E = -\frac{1}{2}g^2$.

All of the properties of the outgoing wavefunction for the rearranged state and the bound state may be deduced directly from the S matrix by simply considering the behavior of the S matrix at the values of k for which it has a pole when it is analytically continued to complex or imaginary k. The discussion in this section was carried out in terms of the wavefunction for purposes of clarity, henceforth we shall discuss rearrangements and bound states from the analytically continued S matrix.[8]

V. N-PARTICLE SOLUTION

In this section we will show that the corresponding N-particle problem is exactly soluble, that is, the problem of an arbitrary number of particles of equal mass all interacting with one another via equal-strength delta-function potentials.

The Hamiltonian is

$$H = -\frac{\hbar^2}{2M}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + C\sum_{i>j=1}\sum^{N}\delta(x_i - x_j).$$

We will continue to use the units

$$\hbar = M = 1, \qquad \sqrt{2}\,C = -g.$$

[8] We should note here that there is a universal peculiarity bred into this problem which is retained in all of the problems we shall discuss subsequently. This peculiarity is that there are no bound-state solutions where any two particles are moving with zero relative velocity. From an inspection of the S matrix it would appear that the condition $s_1 = -1$, $s_3 = 0$, $s_2 = -1$ is also a state which has no incoming waves. If one applies this condition and looks at the wavefunction, one finds that it does not satisfy the boundary conditions on the delta-function surfaces. One can construct a wavefunction which does satisfy the boundary conditions on the delta-function surfaces by a careful limiting process, but this wavefunction increases exponentially at infinity in certain domains.

It is apparently quite impossible to continue to think of the N-particle problem as an equivalent one-particle problem in a multidimensional space, for the dimensionality of this equivalent space is $N - 1$. An alternative point of view exists, however, in which the difficulty of increasing dimensionality may be avoided.

Suppose we consider the space–time plot of a two-particle problem. The particles enter at some momentum which dictates the slope of the line in space–time. When the two particles collide they either transmit or reflect, but since no new velocities are generated, the space–time plot looks as shown in Fig. 7. If particle 1 started on the left and particle 2 on the right, then the transmission coefficient is the amplitude for particle 1 to come out on the right and particle 2 to come out on the left. The reflection coefficient is the amplitude for particle 1 to come out on the left and particle 2 to come out on the right.

We should remark here that there is no intention of changing our formulation of the many-particle problem from the stationary-state type to that of space–time. We intend only to argue that by interpreting the space–time plots we may derive all of the information which would be available in a ray tracing argument such as we used in Sec. IV.

Now let us consider the three-particle problem. There are two possible topologically different three-particle space–time diagrams which are again shown in Fig. 7. These two diagrams correspond exactly to the bifurcation of the incoming plane wave with which we dealt in the previous section. If the particles are ordered 1 2 3 from left to right, then the diagram on the left is the diagram which occurs when particle 1 strikes particle 2 first and the diagram on the right is the diagram which occurs when particle 2 strikes particle 3 first. It is now obvious that there is one collision at each of the three possible relative velocities and that there are exactly three collisions between incoming and outgoing waves. As a matter of fact even the "miraculous" property that $s_1 + s_3 = s_2$ is now evident because

$$s_1 = \sqrt{2}\, i(k_1 - k_2)/g,$$
$$s_3 = \sqrt{2}\, i(k_2 - k_3)/g;$$

thus

$$s_1 + s_3 = \sqrt{2}\, i(k_1 - k_3)/g = s_2.$$

What we have shown in the previous section is that, as far as the outgoing waves are concerned, it does not matter which of the two possible diagrams is used, for both give exactly the same result.

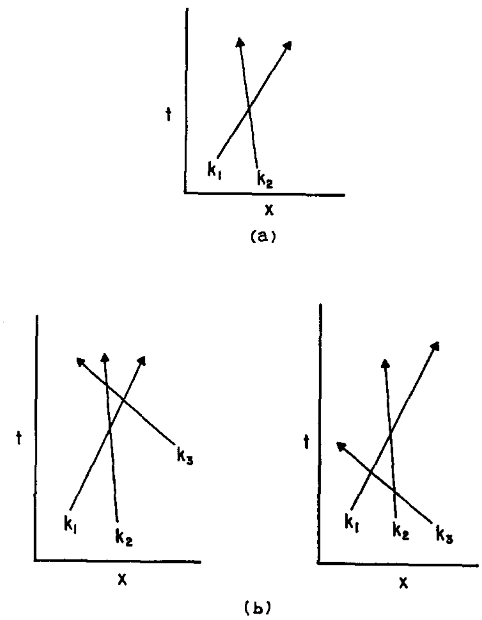

FIG. 7. Space–time plots for (a) two- and (b) three-particle problems.

If we were to change one of the particle masses or one of the delta-function strengths, the two diagrams would not give the same result and diffraction would occur.

In order to calculate the amplitude of the outgoing waves, let us invent two operators T_{ij}^{ll+1} and R_{ij}^{ll+1} which are to operate on some permutation of particles along the line. The indices $ll + 1$ label the position of an adjacent pair of particles which are interacting, and i and j label the k vectors with which the particles are interacting. The operator T interchanges the particle in the lth slot and the particle in the $l + 1$ slot with the amplitude t_{ij} where

$$t_{ij} = \frac{\sqrt{2}\, i(k_i - k_j)/g}{\sqrt{2}\, i(k_i - k_j)/g + 1} = \frac{s_{ij}}{s_{ij} + 1}.$$

The operator R leaves the same particles in the l and $l + 1$ slot, with the amplitude r_{ij} where

$$r_{ij} = -1/(s_{ij} + 1).$$

We denote the order of the particles by (1 3 2) meaning that particle 1 is in the first slot (that is, it is to the left of all of the other particles), particle 3 is in the second slot, and particle 2 is in the last slot. Thus, for example

$$T_{12}^{12}(132) = [s_{12}/(s_{12} + 1)](312),$$
$$R_{12}^{12}(132) = [-1/(s_{12} + 1)](132).$$

We use the three-particle diagrams of Fig. 7

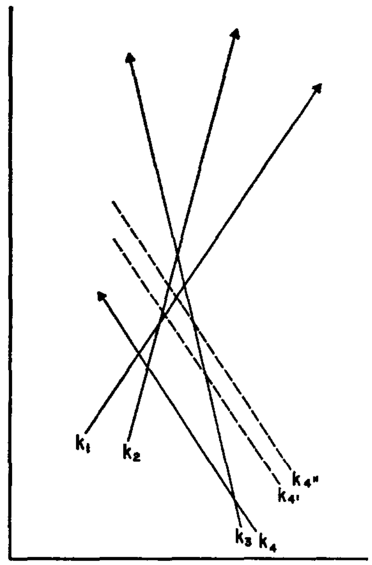

FIG. 8. Four-particle space–time plot.

to tell us in what order these operators work. For example the three-particle diagram on the left in Fig. 7 implies the order

$$(T_{23}^{12}+R_{23}^{12})(T_{13}^{23}+R_{13}^{23})(T_{12}^{12}+R_{12}^{12}),$$

which operates on some linear combination of the initial permutations of the three particles.

The three-particle diagram on the right in Fig. 7 implies the order

$$(T_{12}^{23}+R_{12}^{23})(T_{13}^{12}+R_{13}^{12})(T_{23}^{23}+R_{23}^{23}).$$

It is easily verified that these operators on any permutation of the particles give exactly the same result.

If we go on to four particles it will require fourteen diagrams to fill all space, and there will be six operator products going from incoming to outgoing states. We will now show that the outgoing waves are the same from each of the possible diagrams.

In order to do this let us start with a typical four-particle diagram such as the one shown in Fig. 8. This diagram implies the sequence of six operators

$$O_{24}^{23}\,O_{13}^{34}\,O_{12}^{23}\,O_{14}^{12}\,O_{24}^{23}\,O_{34}^{34},$$

where

$$O_{ij}^{l\,l+1}=T_{ij}^{l\,l+1}+R_{ij}^{l\,l+1}.$$

Suppose we now imagine moving the bottom line, that is, the k_4 line, up the page. We generate a new sequence when this line passes the collision between k_1 and k_2 indicated by the dotted line in Fig. 8. This second sequence of collisions gives exactly the same result as the first because all we have done is change the order of operators and the diagram in exactly the way we changed them in the three-body problem. If we continue to move the k_4 line up the page, another new sequence will be generated when the k_4 line reaches the position indicated by the second dotted line. This sequence involves an interchange of the operators O_{12}^{12} and O_{34}^{34} which commute because they have no slot in common. Thus this diagram gives exactly the same result as do the first two.

A continuation of this argument will show that every possible diagram contributes exactly the same outgoing waves. The argument does not depend on the number of particles, for all that is ever required is to move lines across intersections or to move intersections past commuting intersections. In order to show there is no diffraction, one must show that the amplitudes for every intermediate state which may be reached by more than one route are equal. The argument for these states proceeds in exactly the same way, and requires no more than the operators discussed above.

VI. N-PARTICLE CALCULATIONS

A. The Three-Particle Problem

We are now in a position to calculate amplitudes for various N-particle processes with relative ease. We could now draw some convenient sequence diagram with N k lines, write the appropriate operator sequence, and generate the S matrix by the operator rules of the preceding section. We know that we may pick any sequence whatever to generate the S matrix, for all sequences yield the same result. If it is desired to study scatterings in which particles are bound to one another we simply take ratios of elements of the S matrix where the elements are evaluated at the poles which correspond to the desired bound state.

In order to make the method more clear, let us redo the three-particle problem by the methods of Sec. V.

Let us evaluate the three-particle S matrix by using the space–time sequence diagram on the left in Fig. 7. We will assume that the incoming wave has the particles in the order (123) from left to right. Now

$$\begin{aligned}S(123)&=(T_{23}^{12}+R_{23}^{12})(T_{13}^{23}+R_{13}^{23})(T_{12}^{12}+R_{12}^{12})(123)\\&=(T_{23}^{12}+R_{23}^{12})(T_{13}^{23}+R_{13}^{23})[t_{12}(213)+r_{12}(123)]\\&=(T_{23}^{12}+R_{23}^{12})[t_{13}t_{12}(231)+r_{13}t_{12}(213)\end{aligned}$$

$$+ t_{13}r_{12}(132) + r_{13}r_{12}(123)]$$
$$= (t_{23}r_{13}t_{12} + r_{12}r_{13}r_{12})(123)$$
$$+ (r_{23}r_{13}t_{12} + t_{23}r_{13}r_{12})(213) + (r_{23}t_{13}r_{12})(132)$$
$$+ r_{23}t_{13}t_{12}(213) + t_{23}t_{13}r_{12}(312) + t_{23}t_{13}t_{12}(321),$$

if we let

$$s_{12} = s_1, \qquad s_{23} = s_3, \qquad s_{13} = s_2.$$

The elements above are exactly the same as the elements of the S matrix as given in Table II.

Of course, the other sequence diagram of Fig. 7 gives the same result. As we have seen previously, all of the scattering amplitudes for all possible processes as well as the bound-state energies may be calculated from analytic continuation of the S matrix.

The evaluation of S matrices for more particles is a straightforward but tedious process. We will consider here some processes whose amplitudes may be calculated without calculating the entire S matrix.

B. Four-Particle Processes

Suppose we consider the scattering of a pair of bound particles incident on a second bound pair of particles. We will denote the incoming state as

$$|(12)(34)\rangle$$

where the (12) indicates that particles one and two are bound and the order inside the "ket" indicates the order along the one dimension from left to right.

Let us first calculate the amplitude for the bound aggregates to pass through one another, that is, for the outgoing state to be

$$\langle(34)(12)|.$$

We know that the wavefunction is symmetric to the interchange of particles 1 and 2, for the single bound-state wavefunction of two particles bound by a delta-function potential is symmetric. If one of the particles has momentum k_1 and the other has momentum k_2 we know that

$$\sqrt{2}\, i(k_1 - k_2)/g = -1,$$

for this is the condition that the two particles be bound. Similarly the condition

$$\sqrt{2}\, i(k_3 - k_4)/g = -1$$

is the condition that particles 3 and 4 be bound. Let

$$\sqrt{2}\, i(k_2 - k_3)/g = s.$$

If the bound aggregate (12) is to pass through (34), clearly both particles 1 and 2 must transmit through particles 3 and 4. The amplitude for particle 2 to transmit through particle 3 is

$$\langle 32 \mid 23\rangle = \frac{\sqrt{2}\, i(k_2 - k_3)/g}{\sqrt{2}\, i(k_2 - k_3)/g + 1} = \frac{s}{s+1}.$$

The amplitude for 2 to transmit through 4 is

$$\langle 42 \mid 24\rangle = (s-1)/s.$$

Similarly,

$$\langle 31 \mid 13\rangle = (s-1)/s,$$
$$\langle 41 \mid 14\rangle = (s-2)/(s-1).$$

The amplitude for all four of these events is the product of their respective amplitudes and is the amplitude for the aggregate (12) to pass through the aggregate (34). Thus,

$$\langle(34)(12) \mid (12)(34)\rangle = (s-1)(s-2)/s(s+1).$$

Let us write this amplitude in terms of the energy in the center-of-mass system. By use of the formula in the Appendix which relates the energy to the s variables we can write

$$-4E_{\text{tot}}/g^2 = [s-1]^2 + 1.$$

The factor $+1$ on the right-hand side is just the binding energy of the two pairs of particles, and in this problem it is more convenient to represent the solution in terms of the kinetic energy of incidence of the bound aggregates in their center of mass; thus if we remove the binding energy term

$$-4E/g^2 = [s-1]^2,$$
$$s = 1 + 2iE^{\frac{1}{2}}/g.$$

The amplitude for transmission is then

$$\langle(34)(12) \mid (12)(34)\rangle = \frac{(2iE^{\frac{1}{2}}/g)(2iE^{\frac{1}{2}}/g - 1)}{(2iE^{\frac{1}{2}}/g + 1)(2 + 2iE^{\frac{1}{2}}/g)},$$

and the probability of transmission is

$$|\langle(34)(12) \mid (12)(34)\rangle|^2 = \frac{E/g^2}{E/g^2 + 1}.$$

So the probability for the (12) aggregate to pass through the (34) aggregate is zero at zero energy and monotonically increasing to unity at infinite energy.

We have several other possibilities for outgoing states. It is possible for one of the particles of the bound aggregate (12) to switch places with one of the bound particles of the aggregate (34). One way this could happen is for the first interaction to be a reflection and for the last three interactions to be transmissions. Thus we have the amplitude

$$\langle(13)(24) \mid (12)(34)\rangle = (s-1)(s-2)/s^2(s+1),$$

which yields the probability

$$|\langle(13)(24) \mid (12)(34)\rangle|^2 = \frac{4E/g^2}{(4E/g^2+1)(4E/g^2+4)}.$$

This probability is zero both at zero and at infinite energy. It is a "resonance" probability having a maximum at $E = \frac{1}{2}g^2$ where the probability of the production of the $\langle(13)(24) \mid$ state is $\frac{1}{9}$.

There is no easy way to calculate the amplitude for the reflection of the (12) aggragate off the (34) aggregate; one must return to the sequence diagrams and analyze the S matrix at the appropriate poles. We will simply state the result

$$\langle(12)(34) \mid (12)(34)\rangle = 2(s-2)/s^2(s+1),$$

$$|\langle(12)(34) \mid (12)(34)\rangle|^2 = \frac{4}{(4E/g^2+1)(4E/g^2+4)}.$$

C. The Many-Particle Bound State

From the four-particle amplitudes worked out above, we can see that there is a four-particle bound state. Every element of the S matrix is proportional to $1/[s(s+1)]$; thus if we let $s = -1$ there are only outgoing waves, and we will have a bound state. It would also appear that $s = 0$ would give a bound state, but as we have seen in the three-particle case there are no solutions where two particles have zero relative velocity.

The condition for the four-particle bound state is

$$\frac{\sqrt{2}\, i(k_1 - k_2)}{g} = \frac{\sqrt{2}\, i(k_2 - k_3)}{g} = \frac{\sqrt{2}\, i(k_3 - k_4)}{g} = -1.$$

One may show using the many-particle S matrix that the condition for an N-particle bound state is

$$\sqrt{2}\, i(k_j - k_{j+1})/g = -1 = s_j$$

for all j.

Using this condition we may evaluate the energy of the N-particle bound state using the formula derived in the Appendix which relates $k_i - k_{i+1}$ to the internal energy,

$$E = \frac{g^2}{4}\sum_{n=1}^{N-1}\frac{1}{n(n+1)}\left[\sum_{l=1}^{n} l s_l\right]^2$$
$$= -\tfrac{1}{48}(g^2)N(N^2-1).$$

There is no saturation; the energy decreases as N^3. The wavefunction is symmetric to the interchange of any pair of particles and the average density of particles in the vicinity of the center of mass is of the order of Ng.

D. Scattering of One-Particle by $N - 1$ Bound Particles

As a final example of an N-body calculation let us consider the scattering of one free particle by $N - 1$ bound particles. By the usual method the relation between s and kinetic energy in the center of mass is found to be

$$s = \frac{2iN^{\frac{1}{2}}E^{\frac{1}{2}}}{g(N-1)^{\frac{1}{2}}} + \frac{N-2}{2} = \frac{\sqrt{2}\, i(k_1 - k_2)}{g}.$$

The amplitude for the incident particle to pass through the bound aggregate of $N - 1$ particles is the product of its amplitude to pass through each particle individually, and is

$$\langle(2 \cdots N)(1) \mid (1)(2 \cdots N)\rangle$$
$$= \left(\frac{s}{s+1}\right)\left(\frac{s-1}{s}\right)\left(\frac{s-2}{s-1}\right) \cdots \frac{[s-(N-2)]}{s-(N-3)}$$
$$= \frac{s-(N-2)}{s+1} = \frac{2i(NE)^{\frac{1}{2}}/g(N-1)^{\frac{1}{2}} - \frac{1}{2}(N-2)}{2i(NE)^{\frac{1}{2}}/g(N-1)^{\frac{1}{2}} + \frac{1}{2}N}.$$

The amplitude for the incident particle to replace one of the bound particles is the same for all bound particles, and is

$$\langle(13 \cdots N)(2) \mid (1)(2 \cdots N)\rangle$$
$$= \langle(124 \cdots N)(3) \mid (1)(2 \cdots N)\rangle$$
$$= \left(\frac{-1}{s+1}\right)\left(\frac{s-1}{s}\right) \cdots \frac{s-(N-2)}{s-(N-3)}$$
$$= \left[\frac{-1}{\dfrac{2i(NE)^{\frac{1}{2}}}{g(N-1)^{\frac{1}{2}}} + \frac{1}{2}N}\right]\left[\frac{\dfrac{2i(NE)^{\frac{1}{2}}}{g(N-1)^{\frac{1}{2}}} - \frac{1}{2}(N-2)}{\dfrac{2i(NE)^{\frac{1}{2}}}{g(N-1)} + \frac{1}{2}(N-2)}\right].$$

We noted previously that one of the peculiar things about the three-particle solution was that the amplitude for the incident particle to transmit through the bound aggregate was nonzero even at zero energy. Here we see that this transmission amplitude is always nonzero for one particle incident on $N - 1$ particles, and in fact the amplitude to transmit approaches 1 for infinite N. Thus in the limit of large N nothing happens to the incident particle; it simply passes through this extremely dense bound aggregate as though it were not there.

VII. SUMMARY

In the Introduction we stated that one of our objectives in studying exactly soluble N-body problems was the illustration of physical effects. We have succeeded in this objective to some degree, for we have seen a number of the possible effects outlined in the Introduction. In the infinite-strength delta-function case we have seen particles redistribute their energy among the particles in a way which cannot be understood by a sequence of the two-body interactions. For infinite-strength delta functions we have illustrated the possibility of rearrangement of particles and the existence of N-particle bound states.

On the other hand we have not been able to illustrate any inelastic processes which involve disassociation or recombination of particles out of or into bound states. We have, however, learned to associate these processes with some generalization of diffraction processes in a multidimensional space. This multidimensional diffraction must be dealt with in any successful approximation method so at least in this sense we have provided some insight into the approximation methods which might be used in more physical problems. Moreover, we now have an exactly soluble problem involving a rearrangement of particles which can be used to check the existing approximation methods, and perhaps lead to a better understanding of the lack of convergence which seems to be implicit in problems of this type.[9]

Finally one must wonder about the statistical mechanics of a one-dimensional system of particles of equal mass which interact through equal-strength delta-function potentials. If the particles were bosons or distinguishable particles and the potentials attractive, the problem would make no sense, for the system would collapse into the N-particle bound state independent of temperature. The case of repulsive bosons has recently been worked out by Lieb[10] who, independent of this work, constructed the totally symmetric wavefunction for an arbitrary number of particles of equal mass interacting via finite-strength, repulsive delta-function potentials. The situation with attractive or repulsive fermions remains open and should prove to be an interesting area of further research.

ACKNOWLEDGMENTS

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APPENDIX. DISCUSSION OF COORDINATE SYSTEMS FOR N-BODY PROBLEMS

Consider an N-body Hamiltonian of the form

$$H = -\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{1}{M_i}\frac{\partial^2}{\partial x_i^2} + \sum_{j>i}^{N}\sum v_{ij}(x_i - x_j).$$

We will take the x_i to be one-dimensional variables, but the results we derive will be independent of dimensionality and may be extended to more dimensions by simply substituting the vector quantities ∇_i^2 for $\partial^2/\partial x_i^2$ and $\mathbf{x}_i$ for x_i.

We wish to make a change of variables which will allow us to separate out the center of mass of the entire system. In addition we will find that a more unified view of N-body problems is attained if we pick a "rationalized" coordinate system so that the second derivative terms in the new variables all have the same coefficient. The following has been shown to be such a coordinate system[11,12]:

$$z_1 = \frac{(M_1M_2)^{\frac{1}{2}}}{(M_1+M_2)^{\frac{1}{2}}}(x_1 - x_2),$$

$$z_2 = \frac{M_3^{\frac{1}{2}}(M_1+M_2)^{\frac{1}{2}}}{(M_1+M_2+M_3)^{\frac{1}{2}}}\left(\frac{M_1x_1+M_2x_2}{M_1+M_2} - x_3\right),$$

$$z_n = \frac{M_{n+1}^{\frac{1}{2}}\left(\sum_{i=1}^{n} M_i\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n+1} M_i\right)^{\frac{1}{2}}}\left[\frac{\sum_{i=1}^{n} M_ix_i}{\sum_{i=1}^{n} M_i} - x_{n+1}\right], \quad n < N,$$

$$z_N = \sum_{i=1}^{N} M_ix_i \Big/ \left(\sum_{i=1}^{N} M_i\right)^{\frac{1}{2}}.$$

Pick the first new coordinate to be the relative coordinate between any two particles multiplied by the square root of the reduced mass of those two particles. Pick the second new coordinate to be the coordinate of a third particle relative to the center of mass of the first two multiplied by the square root of the reduced mass of particle three and the sum of the masses of the first two, etc. The last coordinate is the center of mass of the whole system multiplied by the square root of the sum of the masses.

We can verify that this transformation has the property that the coefficients of the new second-derivative terms are equal by observing that the transformation between z and x can be written as an orthogonal matrix times a diagonal matrix, where

[9] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

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the elements of the diagonal matrix are the square roots of the masses of the particles. That is,

$$z = UM^{\frac{1}{2}}x,$$

where

$$M^{\frac{1}{2}} = M_i^{\frac{1}{2}}\delta_{ij},$$

and

$$U_{ni} = (M_{n+1}^{\frac{1}{2}})(M_i^{\frac{1}{2}}) \Big/ \left(\sum_{j=1}^{n+1} M_j\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} M_j\right)^{\frac{1}{2}}, \quad N > n \geq i,$$

$$U_{nn+1} = -\left[\left(\sum_{j=1}^{n} M_j\right)^{\frac{1}{2}} \Big/ \left(\sum_{j=1}^{n+1} M_j\right)^{\frac{1}{2}}, \qquad N > n,\right.$$

$$U_{Ni} = M_i^{\frac{1}{2}} \Big/ \left(\sum_{j=1}^{N} M_j\right)^{\frac{1}{2}},$$

$$U_{ni} = 0, \qquad i > n + 1.$$

The operator $\partial/\partial x$ transforms by the rule

$$\partial/\partial x = (M^{\frac{1}{2}}U')\,\partial/\partial z.$$

Thus the quadratic form

$$\left(\frac{\partial}{\partial x}\right)'(M)^{-1}\frac{\partial}{\partial x} = \sum_{i=1}^{N}\frac{1}{M_i}\frac{\partial^2}{\partial x_i^2} = \left(\frac{\partial}{\partial z}\right)' UM^{\frac{1}{2}}(M)^{-1}M^{\frac{1}{2}}U'\frac{\partial}{\partial z} = \sum_{i=1}^{N}\frac{\partial^2}{\partial z_i^2}.$$

To find the arguments of the potentials we invert the transformation between x and z for form $x_i - x_j$. The result is

$$x_i - x_j = \left(\frac{M_i + M_j}{M_iM_j}\right)^{\frac{1}{2}} \times \left[\sum_{n=i}^{j-2}\frac{(M_iM_j)^{\frac{1}{2}}}{(M_i + M_j)^{\frac{1}{2}}}\frac{M_{n+1}^{\frac{1}{2}}}{\left(\sum_{l=1}^{n} M_l\right)^{\frac{1}{2}}\left(\sum_{l=1}^{n+1} M_l\right)^{\frac{1}{2}}} z_n + \frac{\left(\sum_{l=1}^{j} M_l\right)^{\frac{1}{2}} M_i^{\frac{1}{2}}}{(M_i + M_j)^{\frac{1}{2}}\left(\sum_{l=1}^{j-1} M_l\right)^{\frac{1}{2}}} z_{j-1} - \frac{M_j^{\frac{1}{2}}\left(\sum_{l=1}^{i-1} M_l\right)^{\frac{1}{2}}}{(M_i + M_j)^{\frac{1}{2}}\left(\sum_{l=1}^{i} M_l\right)^{\frac{1}{2}}} z_{i-1}\right].$$

The factor of the square root of the reduced mass is introduced so that the sum of the squares of the coefficients of the z's add up to unity. These coefficients may be looked at as the "direction cosines" of the potential in the multidimensional space.

Another result which we shall find useful is the transformation law between momenta in the two systems. This transformation is

$$\partial/\partial z = [U(M^{\frac{1}{2}})^{-1}]\,\partial/\partial x,$$

$$P_z = U(M^{\frac{1}{2}})^{-1}P_x,$$

which leads to

$$P_{z_n} = \frac{M_{n+1}^{\frac{1}{2}}}{\left(\sum_{i=1}^{n+1} M_i\right)^{\frac{1}{2}}\left(\sum_{n=1}^{n} M_i\right)^{\frac{1}{2}}}\sum_{i=1}^{n}(P_{x_i} - P_{x_{n+1}}), \qquad n < N.$$

The total energy of internal motion is

$$E = \frac{\hbar^2}{2}\sum_{n=1}^{N-1} P_{z_n}^2.$$

We shall need to calculate this energy for the case when all of the masses are equal. For convenience we set $\hbar = 1$ and $M = 1$. This leads to

$$E = \frac{1}{2}\sum_{n=1}^{N-1}\left[\frac{1}{n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}}\sum_{i=1}^{n}(P_{x_i} - P_{x_{n+1}})\right]^2,$$

which can be written

$$E = \frac{1}{2}\sum_{n=1}^{N-1}\frac{1}{n(n+1)}\left[\sum_{i=1}^{n} ik_i\right]^2,$$

where

$$k_i = P_{x_i} - P_{x_{i+1}}.$$