NOTE ON THE FORCED AND DAMPED OSCILLATOR IN QUANTUM MECHANICS[1]

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ABSTRACT

The wave equations for the forced, damped, and forced and damped oscillators are solved in closed form for an arbitrary forcing function; the solutions produced being in one-to-one correspondence with the stationary states of the unforced, undamped Hamiltonian H_0. The quantal motion is closely connected with the classical: for the forced oscillator the probability density is that of H_0 but moves as a whole with the classical motion; for the forced and damped oscillator this motion is accompanied by a contraction progressing eventually into a delta function at the classical position. Transition probabilities between states of H_0 are computed in the case of forced motion and depend solely on the classically acquired energy of the oscillator at any time. The transition probability vanishes strictly only when this energy has a value falling at the roots of a Laguerre polynomial associated with the transition. The classical dipole radiation emitted by a disturbed oscillator is, when the damping force is identified with the force of radiation damping, that of the classical oscillator: a shifted and broadened line.

INTRODUCTION

The customary development of perturbation theory for time-varying perturbations ordinarily leaves unanswered the question of the long-time behavior of the perturbed system. Even supposing a perturbation calculation to be carried out to any desired order of accuracy, there remains the serious problem of the convergence of the calculation, about which little seems to be understood. We should like here to sketch the rigorous solution of the quantum motion of an oscillator exposed to an arbitrary time-varying external force. Special though the oscillator is, one may hope from a full knowledge of a simple problem to learn something of the nature of the answers to the more general questions posed above, or perhaps to replace these questions by more relevant ones. It will appear below, for example, that the effect of a perturbing force is intimately related to the corresponding classical motion, and it is not at all clear that discussing the perturbation in terms of transition probabilities between unperturbed states is in general very illuminating.

Additionally it will be pointed out how the motion of the damped and simultaneously forced and damped oscillator may be solved. We shall have altogether a little isomorphism of the simple classical motions that are so instructive and their equally simple quantal counterparts. Our object will be principally mathematical for the present, namely to demonstrate techniques for managing the wave equation; in another report it is planned to work out illustrative examples and physical consequences in greater detail.

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FORCED OSCILLATOR

For the Newtonian motion of an oscillator subject to an external force $F(t)$ we have

$$m\ddot{x}_0 + kx_0 = F(t), \tag{1}$$

and for the quantum motion,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + [\tfrac{1}{2}kx^2 - xF(t)]\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{2}$$

We may construct solutions to (2) that are in one-to-one correspondence with the unforced stationary solutions ($F = 0$) as follows. Write $\psi = \chi\exp[xg(t)]$ and then $\chi = \phi(x-u(t), t)$, where g and u are to be determined. The Schrödinger equation transforms* into

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + \left(i\hbar\dot{u} - \frac{\hbar^2}{m}g\right)\frac{\partial\phi}{\partial\xi}$$
$$+\left\{\tfrac{1}{2}k\xi^2 + \xi(ku - F - i\hbar\dot{g}) + \left(\tfrac{1}{2}ku^2 - Fu - i\hbar u\dot{g} - \frac{\hbar^2}{2m}g^2\right)\right\}\phi = i\hbar\frac{\partial\phi}{\partial t},$$

where $\xi = x - u(t)$. Choosing now g and u so that the coefficients of $\partial\phi/\partial\xi$ and $\xi\phi$ vanish,

$$-m\dot{u} = i\hbar g,$$

$$ku - F - i\hbar\dot{g} = 0 \quad \text{or} \quad m\ddot{u} + ku = F, \tag{3}$$

one is left with

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + [\tfrac{1}{2}k\xi^2 + \delta(t)]\phi = i\hbar\frac{\partial\phi}{\partial t}, \tag{4}$$

$$\delta(t) = \tfrac{1}{2}ku^2 - Fu - i\hbar u\dot{g} - \frac{\hbar^2}{2m}g^2$$
$$= \tfrac{1}{2}m\dot{u}^2 - \tfrac{1}{2}ku^2.$$

Thence, ξ and t variables being separable,

$$\phi = N_n\exp\left\{-\frac{i}{\hbar}\int[\delta(t) + E_n]dt\right\}\exp(-\tfrac{1}{2}\alpha^2\xi^2)H_n(\alpha\xi),$$

$$\alpha^4 = mk/\hbar^2, \qquad E_n = (n+\tfrac{1}{2})\hbar\omega_0 \quad (\omega_0 = [k/m]^{\frac{1}{2}}),$$
$$H_n = n\text{th Hermite polynomial}, \qquad N_n^2 = \alpha/\pi^{\frac{1}{2}}2^n n!$$

In words, a class of solutions of equation (2) is that formed from the stationary states of the unforced problem, with x replaced by ξ and with a phase $\exp(-i/\hbar)(\int\delta dt + xg)$; by equation (3) ξ is nothing other than $x - x_0(t)$, and by (4) δ is the classical Lagrangian for the unforced motion written as a function in time of classical forced position and velocity.

*An alternative route for transformation is via contact transformation of the classical Hamiltonian.

Conventionally we may suppose an initial state $\psi(x, 0)$ that coincides with one of the eigenfunctions $v_m(x)$ of the unforced Hamiltonian. Then if the classical motion (1) be restricted to the initial conditions $x_0(0) = 0 = \dot{x}_0(0)$, $\psi(x, t)$ is

$$(5) \quad \psi_m(x, t) = N_m \exp\left(\frac{i}{\hbar} p_0(t) x\right) \exp\left(-\frac{i}{\hbar}\int_0^t (\delta + E_n)dt\right) \times \exp\{-\tfrac{1}{2}\alpha^2[x - x_0(t)]^2\} H_m(\alpha[x - x_0(t)]),$$

p_0 denoting classical momentum $m\dot{x}_0(t)$. Plainly these eigenfunctions form a complete set at any time t and a superposition of them can represent an arbitrarily specified wave function at any time. Though $i\hbar\dot{\psi} = H(t)\psi$ can possess no "true" stationary states, it is clear that in some suitable $\xi(x, t)$-space it may do so.

The meaning of the ξ-stationary states is at once visible: $\bar{x}(t)$ is just $x_0(t)$: the probability density

$$|\psi_m|^2 = N_m^2 \exp\{-\alpha^2[x - x_0(t)]^2\} H_m^2(\alpha[x - x_0(t)])$$

dances a classical dance centered at the instantaneous classical position $x_0(t)$, moving *in toto* classically. This generalizes the well-known "oscillating wave-packet" solution of the unforced problem.

The computation of the transition probability

$$P_{mn} = \left| \int_{-\infty}^{\infty} \psi_m(x, t)\, v_n^*(x) dx \right|^2,$$

giving the probability that at time t the oscillator is in the unperturbed state v_n if initially it was in the state v_m, is readily performed using the generating function for the Hermite polynomials. The result is

$$P_{mn} = \frac{\mu!}{\nu!} e^{-\epsilon_0} \epsilon_0^{\nu-\mu} |L_\nu^{\nu-\mu}(\epsilon_0)|^2,$$

where

$$\epsilon_0(t) = \frac{\text{classical energy, } \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}kx_0^2}{\text{quantum energy, } \hbar\omega_0}$$

and ν is the greater, μ the lesser of m, n; $L_\nu^{\nu-\mu}$ denotes the associated Laguerre polynomial. There follows the rigorous selection rule that the transition probability vanishes only when the classically accumulated energy of the oscillator in units of $\hbar\omega_0$ falls at a zero of the appropriate Laguerre polynomial. The discussion of several interesting illustrative examples will be taken up at a later time.

DAMPING

For the linearly damped motion of a particle in a field of force $V(\mathbf{r})$ the Newtonian equation

$$m\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} = -\nabla V(\mathbf{r})$$

may be cast into Lagrangian form by means of

$$L = \exp(\lambda t)(\tfrac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r})), \qquad \lambda \equiv \gamma/m.$$

A brief quantum-mechanical discussion of the damped oscillator was given first by Kanai (1948). Havas (1956, 1957) has considered quite generally the theory of multipliers, such as $\exp(\lambda t)$ above, that allow a Lagrangian formulation of a broad class of Newtonian problems not otherwise fitting into the Lagrange–Hamilton scheme of mechanics; and he has emphasized the impossibility or else ambiguity (due to the multiplicity of possible integrating factors) of proceeding to quantization by the usual rules.* Although the meaning of doing so is not altogether clear at this point, we shall in the conventional way quantize the damped motion described by L and show that physically reasonable results are consequent.

By the customary route, the Hamiltonian implied by L is

$$H = \frac{\mathbf{p}^2}{2m}\exp(-\lambda t)+V(\mathbf{r})\exp(\lambda t), \qquad \mathbf{p} = m\dot{\mathbf{r}}\exp(\lambda t), \tag{6}$$

and is neither the energy nor any constant of the motion, but is just the generator of motion. To place H in a more familiar light, make the contact transformation

$$\mathbf{r} = \exp(-\tfrac{1}{2}\lambda t)\mathbf{R}, \qquad \mathbf{p} = \exp(\tfrac{1}{2}\lambda t)\mathbf{P},$$

giving the new Hamiltonian

$$G = \frac{\mathbf{P}^2}{2m}+\tfrac{1}{2}\lambda\mathbf{R}\cdot\mathbf{P}+\exp(\lambda t)\,V(\exp(-\tfrac{1}{2}\lambda t)\mathbf{R}),$$

in which the whole burden of time dependence is in the potential-energy term. When V is homogeneous of the second degree, the V term is $V(\mathbf{R})$, and G, being free of t, is a constant of the motion and is an energy though not *the* energy. For the one-dimensional oscillator, for example,

$$G = \frac{P^2}{2m}+\tfrac{1}{2}\lambda RP+\tfrac{1}{2}kR^2. \tag{7}$$

The further transformation in this case generated by the generating function

$$F(R,\Pi) = \frac{\lambda}{4m\omega^2}\Pi^2+\frac{\omega_0}{\omega}R\Pi$$

gives a new Hamiltonian

$$\Gamma = \frac{\Pi^2}{2m}+\tfrac{1}{2}KX^2 \tag{8}$$

in the new momentum $\Pi = (\omega/\omega_0)P$ and conjugate coordinate $X = (\lambda/2m\omega\omega_0)P+(\omega_0/\omega)R$, where ω_0 is the undamped frequency $(k/m)^{\frac{1}{2}}$, ω the damped frequency $(K/m)^{\frac{1}{2}} = (k/m-\lambda^2/4)^{\frac{1}{2}}$.

It is interesting to see here that the Gibbs statistical mechanics is applicable to ensembles of weakly coupled damped oscillators, just as for the undamped ones; for in the phase space (Π_i, X_i) Liouville's theorem holds and Γ is conserved. Hence, for example, in a microcanonical ensemble mean values for

*I am indebted to Dr. Havas for correspondence on this matter.

any quantity of interest may be found, in a parallel with the usual computations; and in a canonical ensemble, with a "heat" bath of damped oscillators, there is a modulus of the distribution playing a role analogous to temperature and telling the preferred direction of flow of Γ between ensembles with different moduli. There is, in short, a kind of thermodynamics of macroscopic systems damped in their microscopic coordinates.

Turning now to quantization we have that, as a fundamental proposition stemming from the commutation rules, $\mathbf{p}$ may be represented as $-i\hbar\nabla$, and that, because only of the meaning of the Hamiltonian as generator of the motion, the Schrödinger wave function is controlled by $i\hbar\dot{\psi} = H\psi$, with H as in (6). That ψ retains its meaning of positional probability amplitude is indicated by maintenance of probability conservation (since H is Hermitian) and of the validity of Ehrenfest's theorem, to the effect that the quantal mean position is Newtonian.

For the oscillator, either by direct transformation of the wave equation or by use of the transformed Hamiltonians (7) or (8), the states corresponding to the undamped ones come out to be

$$\psi_n = \exp\left\{\left(-\frac{i}{\hbar}\epsilon_n+\tfrac{1}{4}\lambda\right)t-\left(\frac{im\lambda}{4\hbar}+\tfrac{1}{2}\beta^2\right)x^2\exp(\lambda t)\right\}H_n(\beta x\exp(\tfrac{1}{2}\lambda t)),$$

where the ϵ_n are eigenvalues of Γ, $\hbar\omega(n+\frac{1}{2})$, and $\beta^4 = mK/\hbar^2$ $(K > 0)$. The factor $\exp(\frac{1}{4}\lambda t)$ guarantees the time-independent normalization of ψ_n. These are stationary states in R-space: a measurement of G for them is certain to give the value ϵ_n. So to speak the moving states are stationary with respect to the moving Hamiltonian G. The position–momentum uncertainty product is $\Delta x\Delta p = \hbar(\omega_0/\omega)(n+\frac{1}{2})$ while the mean energy $(\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2)$ becomes $\hbar\omega_0(\omega_0/\omega)(n+\frac{1}{2})\exp(-\lambda t)$. Altogether the quantum motion, like the classical, winds down (or up, for increasing t), $|\psi_n|^2$ shrinking eventually into a delta function. So long as quantum mechanics is linked to classical mechanics via Ehrenfest's theorem such a result is inevitable; the ground state of the *undamped* oscillator can not in the course of time be the favored one, as it might at first sight be expected to be.

SIMULTANEOUS FORCING AND DAMPING

The forced and damped oscillator, controlled classically by

$$m\ddot{x}_0+\gamma\dot{x}_0+kx_0 = F(t), \tag{9}$$

has the Hamiltonian

$$H = \frac{p^2}{2m}\exp(-\lambda t)+(\tfrac{1}{2}kx^2-xF)\exp(\lambda t)$$

with Hamiltonian equations equivalent to (9).

The Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\exp(-\lambda t)\frac{\partial^2\psi}{\partial x^2}+(\tfrac{1}{2}kx^2-xF)\exp(\lambda t)\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

First put $\zeta = x\exp(\frac{1}{2}\lambda t)$ and $\psi = U(\zeta, t)$; then with $U = V\exp[-(im/4\hbar)\zeta^2]$, the transformed wave equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2 V}{\partial\zeta^2}+[\tfrac{1}{2}K\zeta^2+\tfrac{1}{4}i\hbar\lambda-F\exp(\tfrac{1}{2}\lambda t)\zeta]V = i\hbar\frac{\partial V}{\partial t},$$

with K as previously defined. From this point on, with the problem reduced to an equivalent forced and undamped oscillator problem in ζ-space, we proceed as before, placing $V = W\exp[\zeta f(t)]$ and $W = S(\zeta-w(t), t)$ to give (η being $\zeta-w(t)$)

$$-\frac{\hbar^2}{2m}\frac{\partial^2 S}{\partial\eta^2}+[\tfrac{1}{2}K\eta^2+\Delta(t)]S = i\hbar\frac{\partial S}{\partial t},$$

provided f, w satisfy

$$-m\dot{w} = i\hbar f,$$

(10) $$Kw-F\exp(\tfrac{1}{2}\lambda t)-i\hbar\dot{f} = 0 \quad \text{or} \quad m\ddot{w}+Kw = F\exp(\tfrac{1}{2}\lambda t),$$

and Δ is

$$\Delta = \tfrac{1}{4}i\hbar\lambda+\tfrac{1}{2}m\dot{w}^2-\tfrac{1}{2}Kw^2.$$

Finally, for $K > 0$,

$$S = \exp\left(-\frac{i}{\hbar}\int(\Delta+\epsilon_n)dt\right)\exp(-\tfrac{1}{2}\beta^2\eta^2)H_n(\beta\eta).$$

For $K < 0$ (no oscillation at all classically in the damped but unforced motion) S is of continuum type. By (9) and (10) we identify

$$w(t) = x_0(t)\exp(\tfrac{1}{2}\lambda t),$$

$$\Delta(t) = L_0+\tfrac{1}{2}m\lambda x_0\dot{x}_0\exp(\lambda t)+\tfrac{1}{4}i\hbar\lambda,$$

where L_0 represents the classical Lagrangian for the damped but unforced motion as a function in time of the damped and forced position and velocity.

Altogether, for the ψ_n that are in correspondence with the v_n, we get

$$\psi_n = \exp\left(-\frac{i}{\hbar}\{\textstyle\int\Delta dt+\epsilon_n t+\frac{1}{4}m\lambda x^2\exp(\lambda t)-x[p_0+\frac{1}{2}\lambda m x_0\exp(\lambda t)]\}\right)$$
$$\times\exp\{-\tfrac{1}{2}\beta^2[x-x_0(t)]^2\exp(\lambda t)\}H_n(\beta[x-x_0(t)]\exp(\tfrac{1}{2}\lambda t)),$$

where p_0 is $m\dot{x}_0\exp(\lambda t)$, the momentum conjugate to x_0. The probability density again moves as a whole in the classical rhythm, simultaneously deforming by a scale change $\exp(\frac{1}{2}\lambda t)$ in the x-scale. Asymptotically for $t \gg \lambda^{-1}$, $|\psi_n|^2$ becomes a delta function centered at $x_0(t)$: for sufficiently long times the classical and quantal motions coincide. This does not, formally, contravene the uncertainty principle, for, as a calculation shows, the momentum becomes wholly indeterminate in this limit. On the other hand we recognize the gradual transition into the classical regime by the disappearance of $\hbar$ from the commutation rule

$$x(m\dot{x})-(m\dot{x})x = i\hbar\exp(-\lambda t)$$

connecting position and *mechanical* momentum $m\dot{x}$.

Suppose the damped oscillator is hit by any transitory disturbing force F, $F(t) \simeq 0$ $(t > T)$. The oscillator motion subsequent to time T decays classically according to

$$x_{0T} = x_0(T)\exp(-\tfrac{1}{2}\lambda t)\cos\omega t.$$

If the oscillator is charged with charge e, we may interpret $-\gamma\dot{x}$ as the approximate force of radiation damping, and may examine the Maxwell fields emanating from the quantal charge–current distribution (Schrödinger 1926). The charge density $e|\psi_n|^2$ has a dipole moment ex_{0T}; an observer in the radiation zone will see a dipole line centered at ω with a width λ; that is, he will see the classical result of a shifted and broadened line. Quite generally, in fact, the dipole field will be the same as that calculated classically. In the case that the oscillator is damped but not forced the quasi-stationary states ψ_n give a nearly static charge distribution having no dipole or other odd moments. What radiation there is is confined mainly to the induction field; so the oscillator as it damps out creates variable fields predominantly in its immediate neighborhood, i.e. is surrounded by a "virtual" field of varying energy content, one that can affect closely approaching particles but not distant ones.

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