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A three-particle problem in one dimension

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The problem of three particles of identical mass interacting with each other via a delta-function potential is studied within a simple framework inspired by the 2-particle system. We discuss the S matrix and the possibility of bound states arising for both bosons and fermions. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The availability of simple models is a great aid in the teaching and learning of quantum theory. The delta-interaction between equally massive particles has appeared in a number of introductory texts[1,2] and advanced monographs[3] and still continues to be relevant in current research work.[4] In this article, we consider the case of three particles each of mass m, in one dimension and interacting with each other via a delta function potential. Although the two-body problem is generally solvable, the n-body problem has resisted efforts directed at it and the present case should provide a beginner with a preview of some of the difficulties inherent in the n-body case.

We solve completely in Sec. II the two-body problem in a unified way by making use of Feynman's transition kernel.[5] This provides a unified approach to both the scattering and bound states of the system which does not seem to have been done previously from an elementary standpoint. Most calculations rely on the T matrix instead. In Sec. III, we study the three-particle system by following our intuition for the time development of the system and applying the calculational rules of the two-body problem. Our treatment bears a faint resemblance to the perturbative method which led to the Feynman diagrams. As offshoots of our work, we look into bound states, rearrangement scattering, and the effect of statistics in the succeeding two sections (IV and V). We feel that the simplicity of the discussion should make it a useful topic for an introductory course. The same problem was originally treated by McGuire[6] who used a geometric construction to solve it. An exactly soluble three-body atomic model has appeared in this journal before.[7] Recently, interest in the stability of a system of three arbitrary charges has arisen in the context of few-body mechanisms.[8] The n-particle case for the delta potential has been completely solved by many authors.[6,9,10] We refer the interested reader to the literature for this very interesting problem.

II. THE TWO-BODY PROBLEM

We start by giving in this section a complete discussion of the two-body problem. Consider two particles, each of mass m, moving in one dimension and interacting via the delta potential $g\delta(x_1-x_2)$, $g\neq 0$, where x_1 and x_2 are the positions of the particles. In terms of the center-of-mass coordinate X and relative separation x, defined by

$$X=\frac{1}{2}(x_1+x_2), \quad x=x_1-x_2 \tag{1}$$

the Schrödinger equation of the system is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dX^2}\Psi-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\Psi+g\delta(x)\Psi=E\Psi, \tag{2}$$

where $M=2m$, $\mu=(1/2)m$. For our wave function we write

$$\Psi(X,x)=Ae^{iKX}e^{\pm ikx}, \tag{3}$$

where $K=k_1+k_2$, $k=(1/2)(k_1-k_2)$ and $\hbar k_1$, $\hbar k_2$ are the particle momenta. The first factor in Eq. (3) describes the free motion of the center of mass (CM), while the second together with the constant A describes the wave function corresponding to the relative motion of the particles. Denoting this latter factor by $\phi_{\pm k}(x)$, the boundary conditions may be expressed as

$$\begin{aligned}&\phi(+0)=\phi(-0),\\&\phi'(+0)-\phi'(-0)=\frac{mg}{\hbar^2}\phi(0),\end{aligned} \tag{4}$$

which specify that the wave function is continuous when the particles touch and that there is a jump in the derivative of ϕ due to the delta interaction. These conditions yield

$$\Psi(X,x)=e^{iKX}\begin{cases}e^{ikx}+Re^{-ikx} & x<0\\ Te^{ikx} & x>0\end{cases} \tag{5}$$

with

$$T=\frac{s}{1+s}, \quad R=\frac{-1}{1+s}, \quad s\equiv\frac{2\hbar^2k}{igm} \tag{6}$$

for the transmission and reflection coefficients. It is straight forward to verify that the following orthonormality relation holds

$$(\phi_k,\phi_{k'})\equiv\int_{-\infty}^{\infty}\phi_k\phi_{k'}^*\,dx=\delta(k-k'). \tag{7}$$

A unified treatment of the system can be given by appealing to Feynman's formula for the amplitude of the system to evolve in time $T(>0)$ from (X,x) to (X',x'):

$$\begin{aligned}K(X',x';X,x;T)&=\sum_{K,k}\Psi_{K,k}^*(X,x)\Psi_{K,k}(X',x')e^{-i(E_K+E_k)T/\hbar},\\&=\sum_K\Psi_K^*(X)\Psi_K(X')e^{-iE_KT/\hbar}\sum_k\phi_k^*(x)\phi_k(x')e^{-iE_kT/\hbar},\end{aligned} \tag{8}$$

where $E_K=[\hbar^2/2(2m)]K^2$, $E_k=[\hbar^2/2(1/2m)]k^2$ are the CM and relative motion energies.[5] The first sum on the sec-

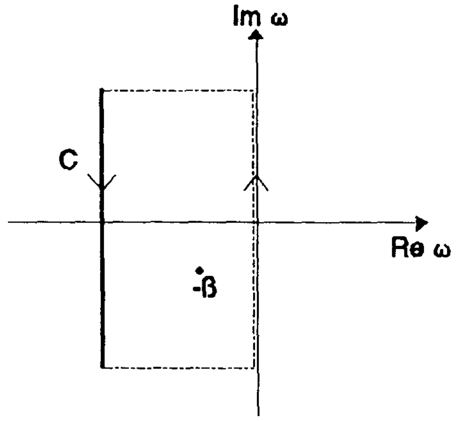

Fig. 1. Integration contour in the ω plane.

ond line of Eq. (8) is just the amplitude for a free particle of mass $2m$ to go from X to X' in time T. Using the prescription $\Sigma_K \rightarrow \int (dK/2\pi)$ we find[5]

$$\sum_k \Psi_K^*(X)\Psi_K(X')e^{-iE_K T/\hbar} = \left(\frac{m}{\pi i \hbar T}\right)^{1/2} \exp \frac{im(X'-X)^2}{\hbar T}. \tag{9}$$

To evaluate the second sum, let us suppose for a moment that $x<0$ and $x'>0$. Then $\phi_k(x)$ describes an incident plane wave from the left of the origin, while $\phi_k(x')$ is the transmitted plane wave to the right of the origin. From Eq. (5)

$$\phi_k^*(x)\phi_k(x') = e^{ik(x'-x)}\left(1-\frac{1}{1+s}\right), \tag{10}$$

and introducing the quantities

$$\alpha \equiv \frac{m}{ih}, \quad a \equiv \alpha^{1/2}(x'-x),$$
$$\beta \equiv i\,\frac{g\alpha^{1/2}}{2\hbar}, \quad \hbar = i\alpha^{1/2}\omega, \tag{11}$$

we may write the second sum in Eq. (8) as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\hbar\, e^{-i(\hbar k^2/m)T} e^{ik(x'-x)} + i\,\frac{\alpha^{1/2}\beta}{2\pi}\int_C \frac{d\omega}{\omega+\beta}\, e^{-a\omega+\omega^2 T}, \tag{12}$$

where C is the solid line parallel to the imaginary axis in Fig. 1. By the residue theorem

$$\int_C \frac{d\omega}{\omega+\beta}\, e^{-a\omega+\omega^2 T} = 2\pi i \times \text{Residue} - \int_{-i\infty}^{i\infty} \frac{d\omega}{\omega+\beta}\, e^{-a\omega+\omega^2 T}. \tag{13}$$

The residue is the contribution coming from the simple pole at $\omega=-\beta$ which occurs provided $g<0$, i.e., the delta potential is attractive (since C is closed by the imaginary axis and the lines at $\omega=\pm i\infty$). We find

$$\text{Residue} = e^{-a\omega+\omega^2 T}\big|_{\omega=-\beta} = \exp\left\{-\frac{mg}{2\hbar^2}(x'-x)\right\}\exp\left\{i\,\frac{mg^2}{4\hbar^3}\,T\right\}\theta(-g) \tag{14}$$

which corresponds to a bound state of energy $-[(mg)/(4\hbar^3)]$ (the coefficient of $-iT/\hbar$ in the exponent) and a normalized wave function of

$$\left(\frac{m|g|}{2\hbar^2}\right)^{1/2} \exp\left\{-\frac{m|g|}{2\hbar^2}\,|x'-x|\right\}$$

(the time independent factor).

The remaining integral may be expressed in terms of the error function by converting the denominator into an exponential

$$\int_{-i\infty}^{i\infty} \frac{d\omega}{\omega+\beta}\, e^{-a\omega+\omega^2 T} = \int_0^{\infty} d\mu\, e^{-\mu(\omega+\beta)} \int_{-i\infty}^{i\infty} d\omega\, e^{-a\omega+\omega^2 T} = -i\pi e^{\beta^2 T + a\beta}\,\mathrm{erfc}(\beta T^{1/2} + \tfrac{1}{2} a T^{-1/2}), \tag{15}$$

where

$$\mathrm{erfc}(y) \equiv 2\pi^{-1/2}\int_y^{\infty} e^{-x^2}\, dx. \tag{16}$$

Thus our final result is

$$\sum_k \phi_k(x')\phi_k^*(x)e^{-iE_k T/h} = \left(\frac{\alpha}{4\pi T}\right)^{1/2} \exp\left(-\frac{1}{4T}\,\alpha^{1/2} a\right) - \tfrac{1}{2}\,\alpha^{1/2}\beta\, \exp(\beta^2 T + a\beta)\,\mathrm{erfc}(\beta T^{1/2} + \tfrac{1}{2} a T^{-1/2}) + \alpha^{1/2}\beta\, \exp(-\beta a + \beta^2 T)\,\theta(-g). \tag{17}$$

The general case for any x,x' is done similarly. Our result will be exactly the same as Eq. (17) except that our definition of a in Eq. (11) is now replaced by

$$a \equiv \alpha^{1/2}(|x'| + |x|). \tag{18}$$

Equation (17) has been obtained before by a number of authors who used the path integral approach.[11–13] The extension of Eq. (17) to any number of delta potentials and one particle is straightforward. This has been done by Bauch[11] and Crandall[14]

III. THE THREE-PARTICLE PROBLEM

Section II provided us with the basic elements for studying the three-particle case. To deal with this problem we stipulate first that each particle should collide with the other two. Since the particles have identical masses and interaction strengths, momentum conservation assures us that any two-particle collision has only two possible outcomes for the momenta: Either the particles retain their respective momenta or they exchange momenta. Thus no new momentum is ever generated in a collision although a switch might occur. Hence, we will require that no pair has zero relative momentum at any time. Clearly the collision cannot be inelastic.

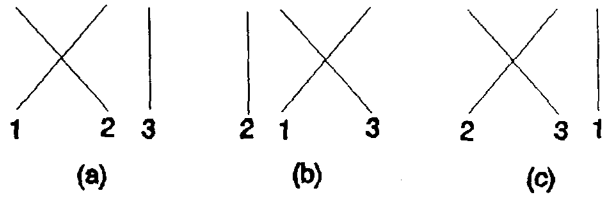

Fig. 2. Events in the collision of particles 1, 2, 3.

For a given two-particle collision event the incident wave function is given by

$$\Psi_{\text{inc}}^{(2)}=e^{iKX}e^{ikx}=e^{ik_1x_1}e^{ik_2x_2}. \tag{19}$$

Particles are identified by their momenta. From Eq. (5) the scattered wave function is

$$\begin{aligned}\Psi_{\text{sc}}^{(2)}&=e^{iKX}(T_{12}e^{ikx}+R_{12}e^{-ikx})\\&=T_{12}e^{ik_1x_1}e^{ik_2x_2}+R_{12}e^{ik_1x_2}e^{ik_2x_1},\end{aligned} \tag{20}$$

$$T_{12}=\frac{s_{12}}{1+s_{12}},\quad R_{12}=\frac{-1}{1+s_{12}},\quad s_{12}=\frac{\hbar^2(k_1-k_2)}{igm}. \tag{21}$$

The coefficient T_{12} gives the amplitude for the exchange of positions of particles 1 and 2, while R_{12} given the amplitude that the original order of particles is retained after the collision.

The entire three-body scattering process may be described as follows. Label the incident particles 1, 2, 3 from left to right. A first collision takes place between particles 1 and 2 with particle 3 as spectator. Then follows a collision between particles 1 and 3 with particle 2 undisturbed. Finally particles 2 and 3 collide in the presence of particle 1. This description is depicted schematically in Fig. 2. Starting with the incident wave function

$$\Psi_{\text{inc}}^{(3)}=e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3} \tag{22}$$

the outcome of the first collision is

$$\Psi_{\text{sc}}^{(3)}[\text{Fig. 2(a)}]=e^{ik_3x_3}\{R_{12}e^{ik_2x_1}e^{ik_1x_2}+T_{12}e^{ik_1x_1}e^{ik_2x_2}\} \tag{23}$$

and after the second collision we find

$$\begin{aligned}\Psi_{\text{sc}}^{(3)}&[\text{Fig. 2(b)}]\\&=e^{ik_2x_1}R_{12}\{T_{13}e^{ik_1x_2}e^{ik_3x_3}+R_{13}e^{ik_1x_3}e^{ik_2x_3}\}\\&\quad+e^{ik_2x_2}T_{12}\{T_{13}e^{ik_1x_1}e^{ik_3x_3}+R_{13}e^{ik_1x_3}e^{ik_3x_1}\}.\end{aligned} \tag{24}$$

The final result for these collisions is

$$\begin{aligned}\Psi_{\text{sc}}^{(3)}=&\,e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}T_{12}T_{13}T_{23}\\&+e^{ik_1x_1}e^{ik_2x_3}e^{ik_3x_2}T_{12}T_{13}R_{23}\\&+e^{ik_1x_3}e^{ik_2x_2}e^{ik_3x_1}(T_{12}R_{13}T_{23}+R_{12}R_{13}R_{23})\\&+e^{ik_1x_3}e^{ik_2x_1}e^{ik_3x_3}(T_{12}R_{13}R_{23}+R_{12}R_{13}T_{23})\\&+e^{ik_1x_2}e^{ik_2x_1}e^{ik_3x_3}R_{12}T_{13}T_{23}\\&+e^{ik_1x_2}e^{ik_2x_3}e^{ik_3x_1}R_{12}T_{13}R_{23}.\end{aligned} \tag{25}$$

Table I. Summary of Eq. (25).

Wave function	Amplitude	Particle order
$e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}$	$T_{12}T_{13}T_{23}=s_1s_2s_3\Delta^{-1}$	321
$e^{ik_1x_1}e^{ik_2x_3}e^{ik_3x_2}$	$T_{12}T_{13}R_{23}=-s_1s_2\Delta^{-1}$	231
$e^{ik_1x_3}e^{ik_2x_2}e^{ik_3x_1}$	$T_{12}R_{13}T_{23}+R_{12}R_{13}R_{23}=-(1+s_1s_3)\Delta^{-1}$	123
$e^{ik_1x_3}e^{ik_2x_1}e^{ik_3x_2}$	$T_{12}R_{13}R_{23}+R_{12}R_{13}T_{23}=-(s_1+s_3)\Delta^{-1}$	213
$e^{ik_1x_2}e^{ik_2x_1}e^{ik_3x_3}$	$R_{12}T_{13}T_{23}=-s_3s_2\Delta^{-1}$	312
$e^{ik_1x_2}e^{ik_2x_3}e^{ik_3x_1}$	$R_{12}T_{13}R_{23}=s_2\Delta^{-1}$	132

One can see the intermediate processes involved in arriving at the outcome. For instance, the first term of Eq. (25) describes the transmission of 1 through 2, then 1 through 3 and finally the transmission of 2 through 3. In this process the original order 1 2 3 is altered to 3 2 1. All this information is summarized in Table 1, where we have introduced the quantities

$$\begin{aligned}&s_1=\frac{\hbar^2(k_1-k_2)}{igm},\quad s_2=\frac{\hbar^2(k_1-k_3)}{igm},\\&s_3=\frac{\hbar^2(k_2-k_3)}{igm}=s_2-s_1,\\&\Delta=(1+s_1)(1+s_2)(1+s_3).\end{aligned} \tag{26}$$

We can gather from Table I that the probability for particles 1 and 2 to exchange is the same as that for particles 2 and 3 exchanging. The results summarized in Table I are usually called the S-matrix elements of the scattering problem.

IV. BOUND STATES

The discussion of Sec. III centered on pure scattering. We study here the possible appearance of bound states. Based on Eqs. (13) and (14), bound states appear at the poles of the transition amplitude. Examination of Eq. (25) or Table I shows that this is so when any of the s is equal to -1. But it is not possible for all three to be simultaneously equal to -1 because of the constraint $s_1+s_3=s_2$. Moreover, we cannot have $s_1=s_2=-1$ since that would imply $s_3=0$ and hence particles 2 and 3 would have zero relative momentum between them, contrary to the stipulation given at beginning of Sec. III. Thus the case of two s's both having value -1 is specified by $s_1=s_3=-1$. In the CM frame where $k_1+k_2+k_3=0$ we find

$$k_1=k_3=-ig,\quad k_2=0 \tag{27}$$

and the bound state wave function can be written as

$$\exp-\frac{g}{2}\{|x_1-x_3|+|x_1-x_2|+|x_2-x_3|\} \tag{28}$$

corresponding to a three-particle bound state of energy $-g^2$.

If only one s is equal to -1 (for definiteness let this be s_1) then we have the scattering of a free particle (3) against a two-particle bound system. The outcome of this scattering can be a simple elastic one, or a rearrangement in which particle 3 replaces either particle 1 or 2 in the bound system, or a crack-up in which there is no bound state in the final system. This last possibility is not allowed in our case because of momentum conservation. Thus there are only the first two possibilities and these are indicated in Fig. 3. In drawing these diagrams it is necessary to include only dis-

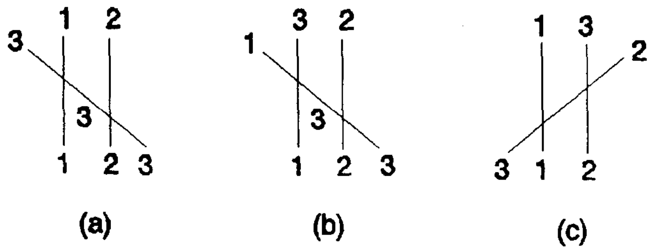

Fig. 3. Events in the collision of a 2-particle bound state with a free particle.

tinct processes. For instance, in Fig. 3(a) the scattering of 3 off 1 first and then off 2 next, is not a process distinct from the one depicted. With $s_1=-1$ the corresponding amplitude are

$$3(a):\ T_{13}T_{23}=\frac{s_3}{s_3+1}\frac{s_2}{s_2+1}=\frac{s_2-1}{s_2+1},$$

$$3(b):\ R_{13}T_{23}=\frac{-1}{s_2+1}\frac{s_2}{s_3+1}=-\frac{1}{s_3+1}, \tag{29}$$

$$3(c):\ R_{32}T_{23}=\frac{-1}{s_3+1}\frac{s_2}{s_2+1}=\frac{-1}{s_3+1},$$

where in the last result we used the fact that $|s_2/(s_2+1)|=1$ [see (30) below]. In the system in which $k_1+k_2=0$ we have

$$s_3=-i\frac{k_3}{g}+\frac{1}{2},\quad s_2=-i\frac{k_3}{g}-\frac{1}{2}. \tag{30}$$

V. FERMIONS

For spin 1/2 fermions much of the above analysis still applies but we must take statistics into account. First of all, if the particles are all spin-up (or down) then the delta interaction has no effect on them because the exclusion principle prevents any two from being together. The case of a pair of spin-up particles and one spin-down is more interesting. Analogous to Eq. (22), the wave function in this case is

$$\Psi=\frac{1}{\sqrt{2}}e^{ik_1x_1}\begin{vmatrix} e^{ik_2x_2} & e^{ik_2x_3}\\ e^{ik_3x_2} & e^{ik_3x_3}\end{vmatrix}$$

$$=\frac{1}{\sqrt{2}}(e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}-e^{ik_1x_1}e^{ik_2x_3}e^{ik_3x_2}), \tag{31}$$

where we assume that the particles 2 and 3 are both spin-up while particle 1 is spin down. If we have initially a bound state it can only be between 1 and 2 or 1 and 3. A rearrangement scattering such as that Figs. 3(b) or 3(c) is not permitted by the exclusion principle. Hence there can only be an elastic collision for the case of a bound pair and a free particle; ionization is not possible.

The first term of Eq. (31) gave us the entries of Table I. A similar calculation for the second term leads to Table II. In this second table the initial order is 1 3 2 and indices refer to the momenta (not the particles). The outgoing wave function is just the sum of the entries of the two tables. Hence,

Table II. S-matrix elements of the second term of Eq. (31).

Wave function	Amplitude	Particle order
$e^{ik_1x_1}e^{ik_2x_3}e^{ik_3x_3}$	$T_{12}T_{13}R_{23}=-s_1s_2\Delta^{-1}$	321
$e^{ik_1x_1}e^{ik_2x_3}e^{ik_3x_2}$	$T_{12}T_{13}T_{23}=s_1s_2s_3\Delta^{-1}$	231
$e^{ik_1x_3}e^{ik_2x_2}e^{ik_3x_1}$	$R_{12}T_{13}R_{23}=s_2\Delta^{-1}$	123
$e^{ik_1x_3}e^{ik_2x_1}e^{ik_3x_2}$	$R_{12}T_{13}T_{23}=s_2s_3\Delta^{-1}$	213
$e^{ik_1x_2}e^{ik_2x_1}e^{ik_3x_3}$	$R_{12}R_{13}T_{23}+T_{12}R_{13}R_{23}=s_2\Delta^{-1}$	312
$e^{ik_1x_2}e^{ik_2x_3}e^{ik_3x_1}$	$R_{12}R_{13}R_{23}+T_{12}R_{13}T_{23}=-(1+s_1s_3)\Delta^{-1}$	132

$$\Psi_{sc}^{(3)}=\frac{1}{\sqrt{2}}\frac{s_1s_2}{(1+s_1)(1+s_2)}e^{ik_1x_1}\begin{vmatrix} e^{ik_2x_2} & e^{ik_2x_3}\\ e^{ik_3x_2} & e^{ik_3x_3}\end{vmatrix}$$
$$-\frac{1}{\sqrt{2}}\frac{s_2}{(1+s_1)(1+s_3)}e^{ik_2x_1}\begin{vmatrix} e^{ik_1x_2} & e^{ik_1x_3}\\ e^{ik_3x_2} & e^{ik_3x_3}\end{vmatrix}$$
$$+\frac{1}{\sqrt{2}}\frac{1}{(1+s_2)}e^{ik_3x_1}\begin{vmatrix} e^{ik_1x_2} & e^{ik_1x_3}\\ e^{ik_2x_2} & e^{ik_2x_3}\end{vmatrix}. \tag{32}$$

VI. CONCLUSIONS

We have studied a system of three equally massive particles interacting via a delta potential of identical strength for each pair. For bosons, it was found that no new velocities are generated as a result of scattering and that the model does not allow inelastic scattering between one particle and a bound two-particle pair, although a rearrangement can take place. There is one bound state for three particles in the case of an attractive potential. The scenario is more complicated for fermions. For instance, if all are spin-up then the delta potential is ineffective; if a fermion scatters off a bound pair, no rearrangement is possible.

We did not succeed in finding a closed form of the three-body kernel analogous to Eq. (17). It might be possible to obtain one, perhaps along the lines of Sec. II and with the aid of the discontinuity calculus elaborated on by Crandall.[14]

ACKNOWLEDGMENT

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The Ricci tensor of a diagonal metric

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Formulas in suffix notation are obtained for the general diagonal component and the general off-diagonal component of the Ricci tensor of a pseudo-Riemannian manifold of arbitrary dimension with diagonal metric. Two applications to general relativity are briefly considered. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

To obtain the standard solutions of the Einstein field equations, such as the Schwarzschild exterior solution or the Robertson–Walker cosmological model, it is necessary at some stage to compute the components $R_{\beta\delta}$ of the Ricci tensor corresponding to a space–time metric $(g_{\alpha\beta})$ with some degree of symmetry. This is usually done by first working out the Christoffel symbols

$$\Gamma^{\gamma}_{\alpha\beta}=\tfrac{1}{2}\,g^{\gamma\delta}(-g_{\alpha\beta,\delta}+g_{\beta\delta,\alpha}+g_{\delta\alpha,\beta}) \tag{1}$$

and then using the formula

$$R_{\beta\delta}=\Gamma^{\alpha}_{\beta\delta,\alpha}-\Gamma^{\alpha}_{\beta\alpha,\delta}+\Gamma^{\epsilon}_{\beta\delta}\Gamma^{\alpha}_{\alpha\epsilon}-\Gamma^{\epsilon}_{\beta\alpha}\Gamma^{\alpha}_{\delta\epsilon}, \tag{2}$$

where the commas denote partial differentiation with respect to the local coordinates (x^{α}). This is a very laborious process, only slightly alleviated by rewriting Eq. (2) in the form

$$R_{\beta\delta}=|g|^{-1/2}(|g|^{1/2}\Gamma^{\alpha}_{\beta\delta})_{,\alpha}-(\ln|g|)_{,\beta\delta}-\Gamma^{\epsilon}_{\beta\alpha}\Gamma^{\alpha}_{\delta\epsilon}, \tag{3}$$

where $g=\det(g_{\alpha\beta})$. However, in the two classic cases mentioned above, the conventional local coordinate systems are such that the metric is *diagonal*, i.e., $g_{\alpha\beta}=0$ for $\alpha\neq\beta$, so some simplification should be possible.

In fact, as long ago as 1933, Dingle[1] gave explicit formulae for the components of the closely related *Einstein tensor*

$$G_{\beta\delta}=R_{\beta\delta}-\tfrac{1}{2}\,(g^{\alpha\gamma}R_{\alpha\gamma})g_{\beta\delta}$$

valid for any space–time metric. Unfortunately, he omitted to use suffix notation for labeling the (diagonal) components of the metric, and he therefore needed ten separate formulas for the ten independent components, which, as Synge[2] has remarked, "are naturally rather formidable." It has been pointed out by Rindler[3] that one really only needs two formulas, one for the diagonal components and one for the off-diagonal ones. He therefore derives explicit formulas for R_{11} and R_{12}, and indicates that the remaining formulas may be obtained "by making the obvious permutations on these two." He also sets the formulas out in such a way that one can easily read off corresponding formulas for manifolds of lower dimension.

We offer here two formulas in suffix notation for the off-diagonal and diagonal components respectively of the Ricci tensor in a pseudo-Riemannian manifold of *arbitrary* dimension with diagonal metric. These enable one to find all the components easily by *direct* substitution. In particular, the relatively simple formula (8) for the off-diagonal elements makes short work of the otherwise tiresome chore of checking that $R_{\beta\delta}=0$ whenever $\beta\neq\delta$ in the two classic cases mentioned previously.

It could be argued that this is a pointless exercise, since the "old-fashioned" tensor approach to general relativity has now been superseded by Cartan's approach in terms of exterior differential forms.[4,5] This not only gives deeper insights into the differential geometry of general relativity but also provides a superior calculational technique[6] for problems such as those considered here. The snag is that most students find this approach difficult to grasp in the initial stages, so that constraints of time often necessitate the more traditional approach. In such cases the formulas derived here should help to shorten the calculations.

II. CHRISTOFFEL SYMBOLS

Let us *abandon the summation convention*, and write

$$\lambda_{\alpha}=\tfrac{1}{2}\ln(|g_{\alpha\alpha}|)\quad\text{for }\alpha=1,2,\ldots,n, \tag{4}$$

