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The mapping of the Coulomb problem into the oscillator[a)]

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The mapping of the Schrödinger equation for the "hydrogen atom" in one, two, and three dimensions into that for the oscillator is reviewed, with the purpose of relating two of the most basic, illustrative, and useful problems discussed in introductory quantum mechanics courses.

I. INTRODUCTION

The one- and two-dimensional "hydrogen atom" problems have been the subject of several papers in this Journal,[1–13] and have served to illustrate various aspects of quantum mechanics of considerable pedagogical value. The objective of this presentation is to highlight a different facet of these studies, namely, the mapping of the Schrödinger equation for the hydrogen atom problem into that for the oscillator in the one-, two-, and three-dimensional cases.

It is amusing to note that the inverse square law of force and the linear distance dependence have been linked from time to time in history. Robert Hooke,[14] in an attempt to understand the elliptic orbit of planets, drew attention to an analogy with a nonplanar pendulum which given an appropriate horizontal push in a suitable direction could be made to follow such a path. Later he realized that the inverse square force provided the basis for the correct motion, even though the full solution of the problem as we know it was due to Isaac Newton. Again the relationship between the Kepler problem and the harmonic oscillator was remarked upon by Bertrand[15] in 1875, who referring to a personal communication by "*notre illustre correspondant* M. Tchebychef," showed that a particle moving under the influence of a central force would move in closed and nonprecessing orbits only for force laws which are inverse square (Coulombic) or linear (harmonic oscillator). Erwin Schrödinger, during the early part of his stay in Dublin, published a paper in which the second-order operator of his equation was factored into the product of two mutually adjoint first-order differential operators. He then applied the method to the harmonic oscillator and Rydberg-atom problems in a subsequent publication,[16] and sent copies of these papers to Max Born who was then at Edinburgh. The publication pointed out the relationship of Schrödinger's approach to that of Heisenberg.

It was, however, Kustaanheimo and Stiefel[17] who brought out a more direct relationship between Kepler motion and the simple harmonic oscillator, transforming one problem into the other. The mapping they developed was also independently arrived at by Ravndahl and Toyoda,[18] and has subsequently been employed in several investigations.[19–22]

The objective of the present study is to illustrate these transformations through the "Coulomb" problem in one, two, and three dimensions, the purpose being to emphasize the pedagogical value residing in this interrelationship between two of the most practical applications of quantum mechanics which one considers in introductory courses.

II. THE ONE-DIMENSIONAL COULOMB PROBLEM

At the outset it should be mentioned that what is meant by the one- (or two-) dimensional Coulomb problem is (in the present context) the system in one (or two) dimensions consisting of a particle of mass m moving in the potential $-e/|x|$ (or $-e/\sqrt{x^2+y^2}$) and not the potential $(e/2)|x|$ [or $(e/2)\ln(x^2+y^2)$] corresponding to the solution of the Poisson equation in the relevant number of dimensions. Thus the Schrödinger equation of interest is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-\frac{e}{|x|}\right)\Psi(x)=-B\Psi(x), \tag{1}$$

where the binding energy B is the absolute value of the energy for the bound states. In view of the fact that the Hamiltonian is invariant under the parity operation $(x\to -x)$, it suffices to study the solutions for the semiline $(x\geqslant 0)$. The relevant mapping which shall lead us to the oscillator is

$$x=\rho^2, \tag{2}$$

which upon substitution throws Eq. (1) into this form:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{d\rho^2}\Psi-\frac{1}{\rho}\frac{d}{d\rho}\Psi\right)+4B\rho^2\Psi=4e^2\Psi. \tag{3}$$

Then upon putting $\Psi\equiv\rho^{1/2}\phi$ we have

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{d\rho^2}\phi-\frac{3/4}{\rho^2}\phi\right)+4B\rho^2\phi=4e^2\phi. \tag{4}$$

We may now compare this equation with the radial equation for an N-dimensional oscillator,[23,24] namely,

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}-\frac{(k-1)(k-3)}{8r^2}\right)u(r)+V(r)u(r)=Eu(r). \tag{5}$$

Here, $u(r)$ is related to the radial part of the wave function through $R(r) = r^{(N-1)/2}u(r)$, and $k=N+2l$ where l is the orbital angular momentum quantum number. The eigenvalue E is given by $E=\hbar\omega(k/4+n)$. Now comparison of Eqs. (4) and (5) tells us that $E=4e^2$, $\frac{1}{2}m\omega^2=4B$ and $k=4$ (refer to the Appendix). Therefore,

$$B=me^4/2\hbar^2n^2, \tag{6}$$

where n is an integer greater than or equal to unity. Here one is referred to the excellent discussion of this result based on the more conventional solution of the Schrödinger equation put against the background of the Bohr–Sommerfeld–Wilson quantization condition in an interesting paper by I. Richard Lapidus.[12] It is amusing to note how the Coulomb problem in the present approach was transmuted into the oscillator. The equation has been "turned inside-out," with what was the binding energy becoming the spring constant and the strength of the interaction e^2 now playing the role of the eigenvalue.

III. THE TWO-DIMENSIONAL COULOMB PROBLEM

The Schrödinger equation for the Coulomb problem in two dimensions,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi(x,y)-\frac{e^2}{r}\Psi(x,y)=E\Psi(x,y) \qquad (7)$$

(where E is the energy eigenvalue and $r^2 = x^2 + y^2$) is readily transmuted into an equation for an oscillator through the statagem of the conformal transformation below:

$$z\equiv x+iy=\xi^2 \Rightarrow x=\xi_1^2-\xi_2^2 \text{ and } y=2\xi_1\xi_2. \qquad (8)$$

Here, ξ_1 and ξ_2 are the real and imaginary parts of the complex variable ξ. The efficacy of this transformation in the context of classical mechanics in two dimensions was evidently remarked upon by Levi–Civita (see for example Ref. 17) who observed that a conic section with its center at the origin in the ξ plane by virtue of this substitution becomes a conic section in the z plane with one of its focii at the origin. In terms of these new variables the Schrödinger equation adopts the form

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial \xi_1^2}+\frac{\partial^2}{\partial \xi_2^2}\right)+\frac{m}{2}\left(\frac{8B}{m}\right)(\xi_1^2+\xi_2^2)\right]\Psi =4e^2\Psi, \qquad (9)$$

which describes a particle of mass m moving in two dimensions under the influence of an isotropic spring of Hooke's constant $8B$ (where B is the binding energy of the original problem) and the problem now becomes, as for the one-dimensional case, finding the values of B for which the eigenvalue is $4e^2$. This problem is most elegantly solved through the introduction of "annihilation" and "creation" operators:

$$a_1\equiv -i\left(\frac{\hbar}{2\sqrt{8mB}}\right)^{1/2}\left(\frac{\partial}{\partial\xi_1}+\frac{\sqrt{8mB}}{\hbar}\xi_1\right)$$

and

$$a_2\equiv -i\left(\frac{\hbar}{2\sqrt{8mB}}\right)^{1/2}\left(\frac{\partial}{\partial\xi_2}+\frac{\sqrt{8mB}}{\hbar}\xi_2\right), \qquad (10a)$$

$$a_1^\dagger= +i\left(\frac{\hbar}{2\sqrt{8mB}}\right)^{1/2}\left(-\frac{\partial}{\partial\xi_1}+\frac{\sqrt{8mB}}{\hbar}\xi_1\right)$$

and

$$a_2^\dagger= +i\left(\frac{\hbar}{2\sqrt{8mB}}\right)^{1/2}\left(-\frac{\partial}{\partial\xi_2}+\frac{\sqrt{8mB}}{\hbar}\xi_2\right), \qquad (10b)$$

which are easily seen to satisfy the commutation relations of

$$[a_i,a_j]=0=[a_i^\dagger,a_j^\dagger] \text{ and } [a_i,a_j^\dagger]=\delta_{ij} \text{ with } i,j=1,2. \qquad (11)$$

In terms of these operators the problem may be recast into the form

$$\hbar\sqrt{(8B/m)}(a_1^\dagger a_1+a_2^\dagger a_2+1)\Psi=4e^2\Psi, \qquad (12)$$

as is appropriate for an isotropic harmonic oscillator in two dimensions. The operators $\{a_i^\dagger a_i\}$ are number operators by virtue of the commutation relations [Eq. (11)], and have eigenvalues $\{n_i\}$ which are non-negative integers, and accordingly, the eigenvalues of the operator on the left-hand side of Eq. (12) are $\hbar\sqrt{(8B/m)}(n_1 + \frac{1}{2} + n_2 + \frac{1}{2})$. Now since $\xi = r^{1/2}e^{(1/2)i\theta}$, the single valuedness of the wave function (with $\theta\rightarrow\theta+2\pi$) requires that n_1 and n_2 are either both even or both odd, and thus the integer $n_1 + n_2$ is necessarily an even integer which we designate as $2n$. As a consequence Eq. (12) implies

$$B=(e^4m/2\hbar^2)[1/(n+\tfrac{1}{2})^2], \qquad (13)$$

which is the desired result. The occurrence of the factor $1/(n + \frac{1}{2})^2$ in place of the familiar $1/n^2$ (handed down to us by Bohr) is due to the topological peculiarity of two-dimensional space, wherein with a singular point (here being the origin which is also the potential source) there are two kinds of closed paths; namely, those that enclose the origin and those that do not. These two categories of paths cannot be deformed into each other.

IV. THE THREE-DIMENSIONAL COULOMB PROBLEM

The Schrödinger equation for the hydrogen atom,

$$\left(-\frac{\hbar^2}{2m}\nabla^2-\frac{e^2}{r}\right)\psi=E\psi,\ E<0, \qquad (14)$$

shall here be rewritten in terms of two complex coordinates ξ_1 and ξ_2, related to the old Cartesian coordinates through these equalities:

$$x+iy=2\xi_1\xi_2^*,\quad x-iy=2\xi_1^*\xi_2,\quad z=\xi_1^*\xi_1-\xi_2^*\xi_2. \qquad (15)$$

Equivalently, ξ_1 and ξ_2 relate to the spherical polar coordinates by

$$r=|\xi_1|^2+|\xi_2|^2,\quad \theta=\cos^{-1}\left(\frac{|\xi_1|^2-|\xi_2|^2}{|\xi_1|^2+|\xi_2|^2}\right),$$

$$\phi=\vartheta_1-\vartheta_2, \qquad (16)$$

where ϑ_1 and ϑ_2 are the phases of ξ_1 and ξ_2, respectively. It should be noted that the transformation being envisaged is one from three independent variables to four independent variables. Specifically, we go from the variables x, y, z (or r,θ,ϕ) to the two complex variables ξ_1 and ξ_2 (or $|\xi_1|$, $|\xi_2|$, ϑ_1, and ϑ_2). Thus an extra unphysical variable, $\sigma\equiv\vartheta_1 + \vartheta_2$, has made its appearance and we shall impose the requirement that physically interesting quantities such as the wave function cannot depend on this variable which is an artifact. Thus to summarize:

$$\xi_1=\sqrt{r}\cos(\theta/2)e^{(i/2)(\sigma+\phi)};$$

$$\xi_2=\sqrt{r}\sin(\theta/2)e^{(i/2)(\sigma-\phi)};\quad \sigma \text{ arbitrary}. \qquad (17)$$

Instituting this transformation and carrying out the necessary algebra one arrives at

$$r\nabla^2\psi=\left(\frac{\partial}{\partial\xi_1^*}\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2^*}\frac{\partial}{\partial\xi_2}\right)\psi. \qquad (18)$$

Here, it should be noted that the above identity involving the Laplacian operator is true provided ψ does not depend on the variable σ. Thus the Schrödinger equation translates into

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial\xi_1^*}\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2^*}\frac{\partial}{\partial\xi_2}\right)+B(\xi_1^*\xi_1+\xi_2^*\xi_2)\right]\psi=e^2\psi \tag{19a}$$

(where $B=-E$ is the binding energy) allowing the constraint:

$$\frac{\partial}{\partial\sigma}\psi=0\Rightarrow\left(\xi_1^*\frac{\partial}{\partial\xi_1^*}-\xi_1\frac{\partial}{\partial\xi_1}\right)\psi$$
$$=-\left(\xi_2^*\frac{\partial}{\partial\xi_2^*}-\xi_2\frac{\partial}{\partial\xi_2}\right)\psi. \tag{19b}$$

Thus the physical hydrogen atom problem in three dimensions has been mapped into the oscillator in the four variables contained through the real and imaginary parts of ξ_1 and ξ_2. Again we introduce creation and annihilation operators:

$$a_+=\frac{-i}{\sqrt{2}}\left(\frac{\hbar}{\sqrt{2Bm}}\right)^{1/2}\left(\frac{\partial}{\partial\xi_1}+\frac{\sqrt{2mB}}{\hbar}\xi_1^*\right),$$
$$a_-=\frac{-i}{\sqrt{2}}\left(\frac{\hbar}{\sqrt{2Bm}}\right)^{1/2}\left(\frac{\partial}{\partial\xi_1^*}+\frac{\sqrt{2mB}}{\hbar}\xi_1\right); \tag{20a}$$

$$a_+^\dagger=\frac{i}{\sqrt{2}}\left(\frac{\hbar}{\sqrt{2Bm}}\right)^{1/2}\left(-\frac{\partial}{\partial\xi_1^*}+\frac{\sqrt{2mB}}{\hbar}\xi_1\right),$$
$$a_-^\dagger=\frac{i}{\sqrt{2}}\left(\frac{\hbar}{\sqrt{2Bm}}\right)^{1/2}\left(-\frac{\partial}{\partial\xi_1}+\frac{\sqrt{2mB}}{\hbar}\xi_1^*\right); \tag{20b}$$

and similarly we introduce the creation and annihilation operators $b_\pm$ and their Hermitian adjoints which are arrived at by simply replacing ξ_1 by ξ_2 in the above equations. These operators satisfy the canonical commutation relations

$$[a_i,a_j^\dagger]=\delta_{ij},\quad [a_i,a_j]=0=[a_i^\dagger,a_j^\dagger],\ i,j=+,-; \tag{21}$$

and similarly for the b operators. Of course, the a operators commute with the b's. In terms of these operators the Schrödinger equation and the constraint equations (19) assume the form:

$$\hbar\sqrt{(B/2m)}(a_+^\dagger a_++a_-^\dagger a_-+b_+^\dagger b_++b_-^\dagger b_-+2)\psi=e^2\psi, \tag{22a}$$

$$(a_+^\dagger a_+-a_-^\dagger a_-)\psi=-(b_+^\dagger b_+-b_-^\dagger b_-)\psi. \tag{22b}$$

This system now is identical to the Schrödinger equation for dual two-dimensional oscillators, so constrained as to have equal and opposite angular momenta. Since the operators involved in the "Hamiltonian" occur in the quadratic Hermitian form, they have non-negative eigenvalues and we obtain

$$\hbar\sqrt{(B/2m)}(n_++n_-+m_++m_-+2)=e^2. \tag{23a}$$

We also have

$$n_+-n_-=m_--m_+, \tag{23b}$$

where the n's and m's are the eigenvalues of the number operators corresponding to the a and the b-type quanta; and the latter condition [Eq. (23b)] being a consequence of the constraint. We thus arrive at the Rydberg formula:

$$B=(me^4/2\hbar^2)[1/(n)^2] \tag{24}$$

where we have put $n_++m_++1=n_-+m_-+1=n$, n being the principal quantum number which is a positive integer.

V. CONCLUSION

Thus we have reviewed the intimate relationship between the Coulomb problem and the simple harmonic oscillator in one, two, and three dimensions which is revealed through the artifice of a suitable transformation which, so to say, turns the problem inside out, with the energy eigenvalue of the Coulomb problem becoming the spring constant for the oscillator which is now required to be such as to yield an eigenvalue equal to e^2, the strength of the Coulomb potential, thereby leading to the correct spectrum for the "hydrogen atom" in the corresponding dimensions.

APPENDIX

To discuss the Schrödinger equation in N dimensions we first introduce the polar coordinates which are defined as a simple generalization of the procedure in three dimensions, namely, with one "polar" type angle θ_1 and $N-2$ "azimuth" angles $\theta_2\cdots\theta_{N-1}$ so that just as we have

$$z=r\cos\theta,$$
$$x=r\sin\theta\cos\phi,$$
$$y=r\sin\theta\sin\phi,$$

in three dimensions, we shall now have

$$x_1=r\cos\theta_1,$$
$$x_2=r\sin\theta_1\cos\theta_2,$$
$$x_3=r\sin\theta_1\sin\theta_2,$$

and so on until:

$$x_{N-1}=r\sin\theta_1\sin\theta_2\cdots\cos\theta_{N-1},$$
$$x_N=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-1},$$

and the Schrödinger equation in N dimensions, namely

$$-(\hbar^2/2m)\nabla_N^2\Psi+V\Psi=E\Psi$$

is separable in polar co-ordinates, provided the potential is "central" viz., a function of r alone. Thus one arrives[23] at the radial equation:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}-\frac{l(l+N-2)}{r^2}\right)R(r)$$
$$=ER(r),$$

where $R(r)$ is the radial part of the wave function, E the energy eigenvalue, and l the "orbital angular momentum" quantum number. Carrying out the substitution

$$R(r)=r^{(1-N)/2}u(r)$$

we have the convenient form of the equation, namely,

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}-\frac{(k-1)(k-3)}{8r^2}\right)u(r)+\frac{1}{2}m\omega^2r^2u(r)$$
$$=Eu(r),$$

with $k\equiv N+2l$, where we have specialized to the case of the isotropic harmonic oscillator. To convert this equation to a standard form, it is convenient to make a change to dimensionless variable $\zeta \equiv (m\omega/\hbar) r^2$, whereupon we have

$$\zeta \frac{d^2u}{d\zeta^2} + \frac{1}{2}\frac{du}{d\zeta} - \frac{(k-1)(k-3)}{16\zeta} u - \frac{\zeta}{4} u + \frac{E}{2\omega\hbar} u = 0,$$

and setting $u = e^{-(1/2)\zeta}\chi$, in order to remove the term linear in ζ occurring in the coefficient of u, we arrive at the equation

$$\zeta \frac{d^2}{d\zeta^2}\chi - \left(\zeta - \frac{1}{2}\right)\frac{d}{d\zeta}\chi + \left(\frac{E}{2\omega\hbar} - \frac{1}{4} - \frac{(k-1)(k-3)}{16\zeta}\right)\chi = 0,$$

which is readily transformed into the standard form through the substitution $\chi = \zeta^{(k-1)/4}\phi$, whence

$$\zeta \frac{d^2}{d\zeta^2}\phi + \left(\frac{k}{2} - \zeta\right)\frac{d}{d\zeta}\phi + \left(\frac{E}{2\omega\hbar} - \frac{k}{4}\right)\phi = 0,$$

which is the confluent hypergeometric equation. The solution to this equation would vitiate the required boundary condition at large distances (namely, the wave function for a bound state must fall off) unless the series solution is made to truncate through the quantization condition

$$\frac{E}{2\omega\hbar} - \frac{k}{4} = n_r \Rightarrow E = \hbar\omega\left(2n_r + l + \frac{N}{2}\right) = \hbar\omega\left(n + \frac{N}{2}\right),$$

where n_r is the radial and n the principal quantum number. It may be noted that for the case being considered in the main text we have two possibilities, to wit; $k=0$ and $k=4$, the former case shall have to be ignored as then the solution would be singular at the origin.

[a]Dedicated to the memory of I. Richard Lapidus.

[b]On leave of absence from the Saha Institute of Nuclear Physics, Calcutta, India.

[1]R. Loudon, "One-Dimensional Hydrogen Atom," Am. J. Phys. **27**, 649 (1959).

[2]M. Andrews, "Ground State of the One-Dimensional Hydrogen Atom," Am. J. Phys. **34**, 1194–1195 (1966).

[3]T-I Shibuya and C. E. Wulfman, "The Kepler Problem in Two-Dimensional Momentum Space," Am. J. Phys. **33**, 570–557 (1965).

[4]B. Zaslow and M. E. Zandler, "Two-Dimensional Analog of the Hydrogen Atom," Am. J. Phys. **35**, 1118–1119 (1967).

[5]L. K. Haines and D. H. Roberts, "One-Dimensional Hydrogen Atom," Am. J. Phys. **37**, 1145–1154 (1969).

[6]M. Andrews, "Singular Potentials in One Dimension," Am. J. Phys. **44**, 1064–1066 (1976).

[7]J. W-K. Huang and A. Kozycki, "Hydrogen Atom in Two Dimensions," Am. J. Phys. **47**, 1005–1006 (1979).

[8]M. M. Nieto, "Hydrogen Atom and Relativistic Pi-Mesic Atom in *N*-space Dimensions," Am. J. Phys. **47**, 1067–1072 (1979).

[9]J. F. Gomes and A. H. Zimerman, "One-Dimensional Hydrogen Atom," Am. J. Phys. **48**, 579–580 (1980).

[10]G. Q. Hassoun, "One- and Two-Dimensional Hydrogen Atoms," Am. J. Phys. **49**, 143–146 (1981).

[11]M. Andrews, "Comments on One-Dimensional Hydrogen Atom," Am. J. Phys. **49**, 1074 (1981).

[12]I. R. Lapidus, "One Dimensional Bohr Atom," Am. J. Phys. **56**, 92–94 (1988).

[13]M. Andrews, "The One-Dimensional Hydrogen Atom," Am. J. Phys. **56**, 776 (1988).

[14]E. G. Valens, *The Attractive Universe* (World Publishing, Cleveland, OH, 1969), pp. 43.

[15]See, for example, L. S. Brown, "Forces Giving no Orbit Precession," Am. J. Phys. **46**, 930–931 (1978).

[16]E. Schrödinger, "A Method of Determining Quantum Mechanical Eigenvalues and Eigenfunctions," Proc. R. Irish. Acad. **46** A, 9–16 (1940); Proc. R. Irish Acad. **46** A, 183 (1941).

[17]P. Kustaanheimo and E. Stiefel, "Perturbation Theory of Kepler Motion based on Spinor Regularization," J. Reine Angew. Math. **218**, 204–219 (1965).

[18]F. Ravndahl and T. Toyoda, "A New Approach to the SU(2) ⊗ SU(2) symmetry of the Kepler Motion," Nucl. Phys. B **3**, 312–322 (1967).

[19]A. O. Barut and I. H. Duru, "Introduction of Internal Co-ordinates into the Infinite-Component Majorana Equation," Proc. R. Soc. London Ser. A **333**, 217–224 (1973).

[20]F. H. J. Cornish, "The Hydrogen Atom and the Four-Dimensional Harmonic Oscillator," J. Phys. A **17**, 323–327 (1984).

[21]R. Rockmore, "On the Isotropic Oscillator and the Hydrogen Atom in Classical and Quantum Mechanics," Am. J. Phys. **43**, 29–32 (1975).

[22]D. Bhaumik, B. Dutta-Roy, and G. Ghosh, "Classical Limit of the Hydrogen Atom," J. Phys. A **19**, 1355–1364 (1986).

[23]A. Chatterjee, "Large *N* Expansions in Quantum Mechanics," Phys. Rep. **186**, 249–370 (1990).

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A transformation of the type (1) may involve a reflection of the coordinate system in the three special dimensions and it may involve a time reflection, the direction du_0 in space-time changing from the future to the past. I do not believe there is any need for physical laws to be invariant under these reflections, although all the exact laws of nature so far found do have this invariance.

P. A. M. Dirac, "Forms of Relativistic Dynamics," Rev. Mod. Phys. **21**, 392–399 (1949).