Time dependence in quantum mechanics—Study of a simple decaying system

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itor screen. Analysis of the shape by Fourier decomposition then shows the presence of first, third, and, though close to the noise level, fifth harmonics.[9] The relative strength of the harmonics depends on the strength of the nonlinear force term and on any nonlinearities in the driver.

III. CONCLUSION

The simple nonlinear system described above can be used for a class demonstration of the jump phenomenon associated with the bent tuning curve. As an individual lab experiment, it provides the student with apparatus that may be adjusted to display either the behavior of a linear-damped harmonic oscillator or that of the forced Duffing oscillator through adjustment of the tension in the rubber string. The Duffing oscillator is readily introduced into a sophomore–junior level mechanics course and may be pursued at length in independent study at a higher level.

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Time dependence in quantum mechanics—Study of a simple decaying system

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A simple quantum-mechanical model consisting of a two-level system coupled to a continuum, where the interaction can be varied from perturbative to strong, is examined for what can be learned about quantum mechanical decay using elementary methods. The decay spectrum and time dependence of the amplitudes can be calculated in this model, without resorting to perturbation theory. The spectrum is not precisely Lorentzian nor is the decay precisely exponential, features that appear to be shared by all quantum-mechanical models of decaying systems. The conditions for exponential decay and some of the reasons for deviation from this at both short and long times are discussed.

I. INTRODUCTION

Whereas the behavior of quantum mechanical systems is normally studied or explained with the aid of simple solvable models, there is no such illustration for the system undergoing quantum mechanical decay. One of the reasons for this is that the distinctive feature of a decaying system, namely the exponential decrease of the probability of finding the system in its initial (undecayed) state, is not the characteristic time dependence of any simple model.

Why do we expect that excited mechanical systems, such as atoms, should decay exponentially? According to an elementary argument, when a system A decays to B with emission of c, then after a certain characteristic period of time τ, a fraction $1/e$ of an original sample of A's will survive. If at that time we were to separate from the remainder the undecayed A's, then these A's would suffer the same fate as the original sample, namely only a fraction e^{-1} would remain after a (further) time τ. Of the original sample a fraction e^{-2} is thus left after time 2τ, and we expect e^{-n} will be the residue after time $n\tau$: In other words, the number remaining reduces exponentially with time (t) as $e^{-t/\tau}$.

The problem with this argument is that it is essentially *classical*. It overlooks the peculiarity of quantum mechanics that one cannot allow, even for the sake of argument, the intrusion of an observer into the state of the system without sacrificing the orderly time development of the wave function. Thus at the point where we separately identify those atoms in state A and those in state B, their wave functions collapse from a coherent superposition of A and B to an incoherent mixture of so many A's and so many B's.

The validity of the exponential decay law has been discussed in the literature for a number of years. Khalfin[1,2] has shown in general that if the energy distribution is semifinite then there can be no pure exponential decay. In fact, the Fourier time transform $a(t)$ of any function $A(\omega)$ that vanishes for all ω less than some value ω_0, behaves for large t as a power of t.[3] Winter[4] shows in the special case of penetration through a delta-function barrier that the probability density does not decay exponentially except within a limited time interval. At large t again the decay rate is an inverse power of t. Other studies[5–8] show that the inverse power law is dependent on the structure of the initial state. Moreover, the breakdown of exponential decay at short times has also been noted.[9–12] It is a simple matter to show that the probability of decay is initially proportional to t^2. The probability of finding the system still in state A after a (short) time t is then $1 - at^2 + (\textit{higher powers})$. A system interrupted N times in an interval t would then be found to survive all N interrupts with probability $[1 - a(t/N)^2]^N$, a quantity which approaches unity as N approaches infinity. Thus over an arbitrary time interval t, the system would not decay at all if constantly interrupted. This has been described as the "watched pot" effect, or the quantum Zeno paradox.[11]

An experiment with the objective of displaying the quantum Zeno effect[9] for the rf transition between two $^9Be^+$ hyperfine levels, confirms that this is a real effect. As far as the long time deviations are concerned, from the studies cited above (Refs. 5–8) and others[13,14] there appears to be no formal obstacle to experimental observation of these large time deviations in nature, although the magnitude of the effect is likely to be very small. Yet experiments[15,16] including a recent study on the β decay of ^{56}Mn at large times have shown no sign of deviation from exponential decay.

Thus quantum mechanical decay is not exponential for very short times or for very long times, and indeed these seem to be its only *universally* proven features. Exponential decay, although apparently a rule of nature, is not a precise result according to quantum mechanics.

Textbooks contain many examples of how to calculate a decay rate w ($= 1/\tau$) using Fermi's Golden Rule,[17] which for a transition $j \to k$ we write:

$$w_{j\to k} = 2\pi |H^I_{jk}|^2 \rho(\omega_k), \tag{1}$$

where H^I is the interaction Hamiltonian and ρ is the density of states (notice that we have put $\hbar = c = 1$). The quantity w also appears as the *width* in the Lorentzian line shape, which is the characteristic energy distribution of the decay products c:

$$l(\omega_c) = w/\{2\pi[(\omega_c - \omega_{jk})^2 + w^2/4]\}. \tag{2}$$

But the arguments that result in Eqs. (1) and (2) must *assume* (sometimes only implicitly) that the decay rate is a constant (i.e., that the decay is exponential) and H^I can be treated as a *perturbation*.

In the following, we examine a model that does not rely on perturbation theory. It provides an alternative to the Lorentzian line shape with the desirable characteristic that it has a threshold [which Eq. (2) does not] and also enables us to examine the behavior in the limit of strong coupling. We are also able to shed some light on how the decay curve changes from its initial parabolic behavior, through an exponential phase, to a final inverse-power-law phase.

The system we originally had in mind was nucleon–antinucleon annihilation (for which the coupling is certainly strong) expressed as a particle falling into a hole (as in Dirac's hole picture of antiparticles) and accompanied by the emission of a scalar meson (no such particle exists, but a fictitious scalar meson, the σ, plays an important role in nuclear force calculations; see for example Ref. 18). But our purpose here is to examine this model for what it can reveal about decaying systems in quantum mechanics rather than to discuss any merits it may have as a model of particle–antiparticle annihilation; the original purpose will show only where we have chosen numerical values for the physical quantities involved.

II. MODEL OF A DECAYING SYSTEM

Consider a system that, in the absence of interactions, consists of a particle that can exist in a discrete excited particle-hole state $\psi_1\bar{\psi}_2$ with energy ω_0 ($= \omega_1 - \omega_2$), or in a continuum of free meson states with energy ω_k [$= \sqrt{(m^2 + k^2)}$]. An interaction couples these two with a scalar coupling constant g. We write in the interaction representation:

$$i\frac{\partial \Psi}{\partial t} = H_{\text{int}} \Psi, \tag{3}$$

where the wave function Ψ is a linear superposition of the particle-hole state $\psi_1\bar{\psi}_2$ and the meson plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$:

$$\Psi = a(t)\psi_1\bar{\psi}_2 + \int d^3k\, b(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

The interaction matrix element is

$$(H_{\text{int}})_{0k} = \int d^3r\, \psi_1\bar{\psi}_2^* g e^{i\mathbf{k}\cdot\mathbf{r}} e^{i(\omega_0 - \omega_k)t} \tag{5}$$

$$= V(\mathbf{k}) e^{i(\omega_0 - \omega_k)t}, \tag{6}$$

where Eq. (6) defines $V(\mathbf{k})$. At time $t = 0$ the system is launched in its undecayed state $(\psi_1\bar{\psi}_2)$ and no meson is present. The initial conditions are therefore

$$a(0) = 1, \quad b(\mathbf{k},0) = 0. \tag{7}$$

Equations for the coefficients $a(t)$ and $b(\mathbf{k},t)$ are extracted by substituting Eq. (4) into Eq. (3) and taking the inner products with $\psi_1\bar{\psi}_2$ and $e^{i\mathbf{k}\cdot\mathbf{r}}$:

$$i\frac{da}{dt} = \int d^3k\, V(\mathbf{k}) b(\mathbf{k},t) e^{i(\omega_0 - \omega_k)t}, \tag{8}$$

$$i\frac{\partial b}{\partial t} = V^*(\mathbf{k}) a(t) e^{i(\omega_k - \omega_0)t}. \tag{9}$$

Combining Eqs. (8) and (9):

$$\frac{da}{dt} = -\int d^3k\, |V(\mathbf{k})|^2 \int_0^t dt'\, a(t') e^{i(\omega_0 - \omega_k)(t - t')}. \tag{10}$$

Now after a long time ($t \to \infty$), $b(\mathbf{k},\infty)$ is the probability amplitude for emission of a meson of momentum $\mathbf{k}$, and $a(\infty)$ is the amplitude for the system to be found still in the state $\psi_1\bar{\psi}_2$ (and presumably zero, see Sec. IV however). It is convenient to define $a(t) = 0$ for $t < 0$, and to introduce the Fourier transform of $a(t)$

$$f(\omega) = \frac{1}{2\pi} \int_0^\infty dt\, a(t) e^{i\omega t} \tag{11}$$

so that

$$a(t) = \int_{-\infty}^{\infty} d\omega\, f(\omega) e^{-i\omega t}, \tag{12}$$

$$b(\mathbf{k},\infty) = -2\pi i V^*(\mathbf{k}) f(\omega_k - \omega_0). \tag{13}$$

To obtain an expression for $f(\omega)$ we integrate Eq. (10) as follows:

$$\int_0^\infty dt\, e^{(i\omega-\epsilon)t}\frac{da}{dt}$$
$$= -\int d^3k\,|V(\mathbf{k})|^2\int_0^\infty dt\, e^{[(i\omega-\epsilon)+i(\omega_0-\omega_k)]t}$$
$$\times\int_0^t dt'\, a(t')e^{-i(\omega_0-\omega_k)t'}, \tag{14}$$

where ϵ is infinitesimal. Applying integration-by-parts to both sides of Eq. (14) and rearranging, we get an explicit expression for $f(\omega)$:

$$f(\omega)=\left[2\pi i\left(-\omega+\int d^3k'\frac{|V(\mathbf{k}')|^2}{\omega+\omega_0-\omega_{k'}+i\epsilon}\right)\right]^{-1}. \tag{15}$$

Change the integral in Eq. (15) to one over energy by introducing the density of states $\rho(\omega_k)$ and defining:

$$Z(\omega_k)=\int d^3k'\frac{|V(\mathbf{k}')|^2}{\omega_k-\omega_{k'}+i\epsilon}$$
$$=\int_m^\infty d\omega_{k'}\frac{\rho(\omega_{k'})|V(\mathbf{k}')|^2}{(\omega_k+i\epsilon)-\omega_{k'}}. \tag{16}$$

If we regard Eq. (16) as an integral in the complex $\omega_{k'}$ plane then, for a physical energy, ω_k is real and greater than m; we can deform the path to that shown in Fig. 1 and separate it into a principal value integral and a contribution from the semicircular portion around the pole at ω_k:

$$\int_{\text{path 1}} d\omega_{k'}\frac{Q(\omega_{k'})}{\omega_k-\omega_{k'}}=P\int d\omega_{k'}\frac{Q(\omega_{k'})}{\omega_k-\omega_{k'}}-\pi i Q(\omega_k). \tag{17}$$

Since the integrand $Q=|V|^2\rho$ is a real function, then for real ω_k Eq. (17) divides the integral into real and imaginary parts; it is useful to identify them: Let

$$\Omega(\omega_k)=P\int_m^\infty d\omega_{k'}\frac{|V(\mathbf{k}')|^2\rho(\omega_{k'})}{\omega_k-\omega_{k'}}, \tag{18}$$

$$Q(\omega_k)=\rho(\omega_k)|V(\mathbf{k})|^2. \tag{19}$$

The spectrum of emitted mesons is then

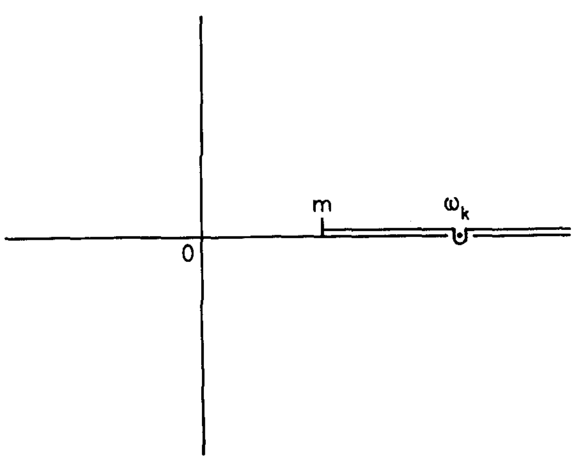

Fig. 1. Path used to evaluate integral in Eq. (16).

$$S(\omega_k)=|b(\mathbf{k},\infty)|^2\rho(\omega_k)$$
$$=\frac{Q(\omega_k)}{[\omega_0-\omega_k+\Omega(\omega_k)]^2+\pi^2Q(\omega_k)^2}, \tag{20}$$

which evidently reduces to the Lorentzian shape of Eq. (2) if we replace the function Q by its value at the approximate center of the distribution ω_0 and neglect Ω altogether. The corresponding time-dependent amplitude $a(t)$ is given by Eq. (12) where we now have an explicit expression for $f(\omega)$:

$$\text{for } t>0,\quad a(t)=\frac{1}{2\pi i}\int_{-\infty}^\infty d\omega\frac{e^{-i\omega t}}{Z(\omega)-\omega+\omega_0}. \tag{21}$$

In Eq. (21) we have moved the origin to ω_0 and discarded a factor $\exp(i\omega_0 t)$, which is irrelevant. Evaluation of the integral of Eq. (21) evidently calls for an explicit form for the function Z, which entails an explicit form for $|V|^2\rho$. We have selected a simple expression consistent with the requirements $|V|^2\rho=0$ at the threshold $\omega_k=m$, and $|V|^2\rho\approx o(\omega_k^{-2})$ as $\omega_k\to\infty$: Let

$$|V(k)|^2\rho(\omega_k)=g^2[(\omega_k^2-m^2)/(\omega_k^2-b^2)^2], \tag{22}$$

where b is real, $|b|<m$. Expression (22) has the added advantage that the integrals (16), (18) are easily carried out:

$$Z(\omega)=g^2\left(\frac{b^2+\omega m}{2b^2(b^2-\omega^2)}+\frac{(2b-\omega)m^2-b^2\omega}{4b^3(b-\omega)^2}\right.$$
$$\times\ln(m-b)+\frac{(2b+\omega)m^2+b^2\omega}{4b^3(b+\omega)^2}\ln(m+b)$$
$$\left.+\frac{\omega^2-m^2}{(b^2-\omega^2)^2}\ln(m-\omega)\right). \tag{23}$$

Here, ω ranges over the whole complex plane and the phase of the term $\ln(m-\omega)$ must be chosen to agree with $\Omega-i\pi Q$ on the positive real axis beyond $\omega=m$. It is fairly easy to see that this is

$$Z(\omega)=\Omega(\omega)+iQ(\omega)\arg(m-\omega),$$
$$-\frac{3\pi}{2}<\arg(m-\omega)<\frac{\pi}{2}, \tag{24}$$

where

$$\Omega(\omega)=g^2\left(\frac{b^2+\omega m}{2b^2(b^2-\omega^2)}+\frac{(2b-\omega)m^2-b^2\omega}{4b^3(b-\omega)^2}\right.$$
$$\times\ln(m-b)+\frac{(2b+\omega)m^2+b^2\omega}{4b^3(b+\omega)^2}\ln(m+b)$$
$$\left.+\frac{\omega^2-m^2}{(b^2-\omega^2)^2}\ln|m-\omega|\right), \tag{25}$$

$$Q(\omega)=g^2[(\omega^2-m^2)/(\omega^2-b^2)^2]. \tag{26}$$

Suppose we take the integral (21) from $-X$ to X (where ultimately we will let $X\to\infty$) and attempt to close the contour by a semicircular path of radius X. The exponential $e^{-i\omega t}$ (where $t>0$) will converge in the limit $X\to\infty$ only in the lower half-plane. Now the function $Z(\omega)$ has a branch point at the threshold value m, and so the semicircle will not close; it is necessary to circumvent the branch point as shown in Fig. 2. Take the cut along the line $\omega=m-iy$, $\infty>y>0$, so that the contour consists of two quarter-circles. In the limit $X\to\infty$ the contribution from the arcs vanishes leaving only that along the real axis, and the integral up and down the cut (or, equivalently, the inte-

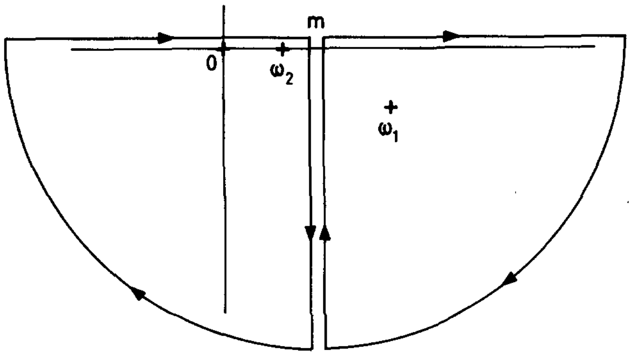

Fig. 2. Contour used to evaluate integral in Eq. (21).

gral of the discontinuity across the cut). The latter, owing to the placement of the cut, converges like a real exponential e^{-yt}. Thus the integral (21) is of the form:

$$a(t) = -\sum_{\text{poles}\, j} R_j e^{-i\omega_j t} + \frac{1}{2\pi i}\int_m^{m-i\infty} d\omega \left(\frac{1}{Z(\omega)-\omega+\omega_0}\right)_{\omega-\epsilon}^{\omega+\epsilon} e^{-i\omega t}, \tag{27}$$

where j labels all poles of the function $[Z(\omega)-\omega+\omega_0]^{-1}$ on or below the real axis and R_j is the corresponding residue. From Eq. (24) we can write the contribution from the cut more explicitly as

$$-ie^{-imt}\int_0^\infty dy\,\frac{Q(m-iy)e^{-yt}}{[\Omega(m-iy)-\frac{1}{2}i\pi Q(m-iy)-m+iy+\omega_0]^2+[\pi Q(m-iy)]^2}. \tag{28}$$

This integral cannot be expressed in closed form (as far as we know) but it is a slowly varying function of time and easily tabulated for any set of parameters.

III. NUMERICAL EXAMPLES

For the constants in the model we take the following set of values;

$$\omega_0 = 1000\text{ MeV},\quad m = 300\text{ MeV},$$
$$a(=\sqrt{m^2-b^2}) = 50\text{ MeV}, \tag{29}$$
$$g = 750\text{–}4000\text{ MeV}^{3/2},$$

notice that we give a range of values for the coupling constant g in order to explore the transition from a perturbative type of situation, where the golden rule should be a good approximation, to the case of strong coupling.

To calculate the decay curve it is necessary to know the analytic structure of $a(t)$ from Eq. (27). For the smaller values of g, one pole dominates the decay amplitude for many half-lives. The decay curve, $|a(t)|^2$, is consequently very near to being exponential, which would appear as a straight line on a semilogarithmic plot (see Fig. 3). Nevertheless, a departure in the form of a slight oscillation is apparent after five or six lifetimes. This is the result of interference between the pole term and the small contributions of the cut. At large values of t, the cut term should dominate because its asymptotic form is proportional to t^{-2} as can be seen by expressing the variable in Eq. (28) as $\omega = m - iy$, expanding the rational part of the integrand as a power series in y, and integrating. This gives us

$$\int_0^\infty dy\, e^{-yt} \times\left(-\frac{g^2 m}{(m^2-b^2)^2[\Omega(m)-m+\omega_0]^2}y + O(y^2)\right) \sim -\frac{mg^2}{(m^2-b^2)^2[\Omega(m)-m+\omega_0]^2}t^{-2}. \tag{30}$$

Thus in the limit of large t, $|a(t)|^2 \sim t^{-4}$. An estimate for weak/moderate coupling of the time at which exponential decay breaks down is given by

$$t^2\exp[\text{Im}(\omega_1)t] = \frac{mg^2}{(m^2-b^2)^2[\Omega(m)-m+\omega_0]^2}, \tag{31}$$

where ω_1 is the position of the pole. This expression is obtained by equating the approximate amplitude of the pole term to the asymptotic expression for the cut term.

To consider the short time behavior, note that both terms in Eq. (27) contribute, and both are dominated by oscillatory (imaginary exponential) factors: $\exp(-imt)$ for the cut term [see Eq. (28)], and $\exp[-i\,\text{Re}(\omega_1 t)]$ for the single pole term. The probability, $|a(t)|^2$, thus has a component which behaves like $\cos[\text{Re}(\omega_1)-m]t$, which is a rapid oscillation unless the energy of the transition is very low, and which approaches unity in the prescribed parabolic fashion as $t\to 0$ as shown in Fig. 4. But the para-

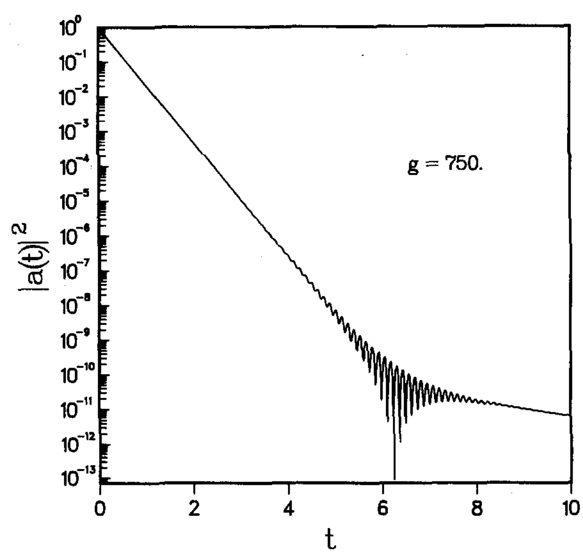

Fig. 3. Decay curve for weak coupling ($g = 750$) shows exponential decay for five to six lifetimes but eventually switches over to a power-law (t^{-4}) dependence. Time (t) in Figs. (3), (4), and (5) is in units $\hbar$/MeV.

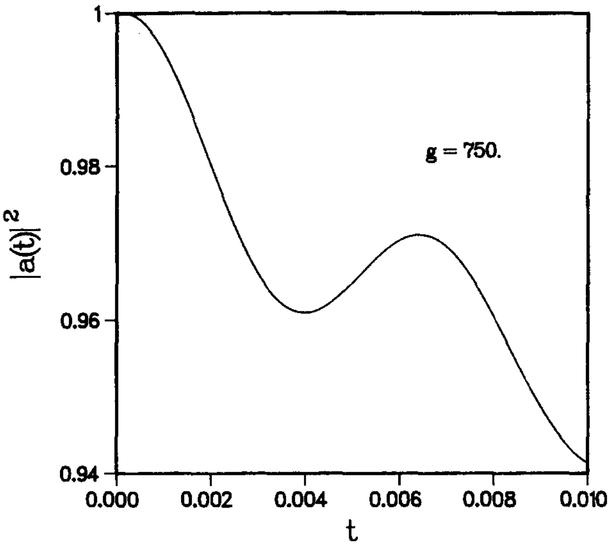

Fig. 4. Short-time behavior with $g = 750$, showing initial oscillations and parabolic behavior as $t \to 0$.

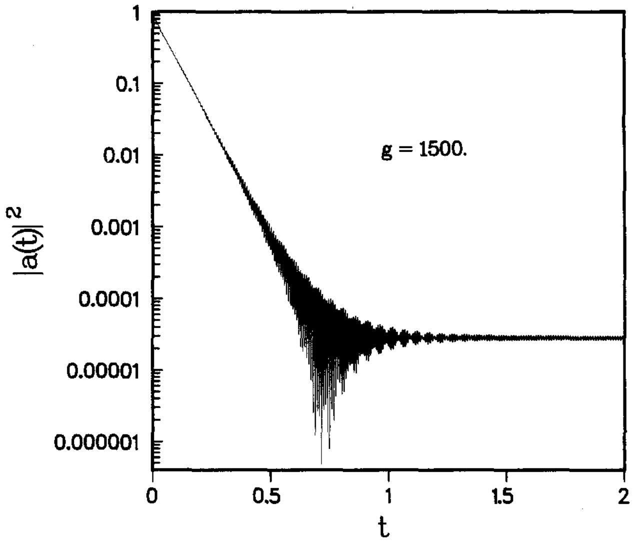

Fig. 6. Decay curve for intermediate coupling ($g = 1500$) shows exponential decay initially but eventually approaches a nonzero value.

bolic phase lasts only a very short time, approximately rendered in terms of the input parameters as $(\omega_0 - m)^{-1}$. The basic behavior is oscillatory with a decreasing mean value which does, indeed, fall exponentially with time; moreover the amplitude of the oscillations decreases as shown in Fig. 5.

For large g (> 1337) a second pole is evident on the real axis (see ω_2 in Fig. 2) and the decay now approaches a nonzero value as in Fig. 6. For very strong coupling ($g = 4000$) all traces of exponential decay disappear and the decay curve proceeds in an oscillatory fashion to its new value (Fig. 7). In both of these cases, the state approached as $t \to \infty$ is dictated by the second pole, which represents a stable composite state, part particle-hole and part meson cloud, and which is now the lowest energy state of the interacting system. Thus the end product of the decay process has a small particle-hole component and results in the nonzero limit for $|a(t)|^2$. The wave function of this state and its relationship to the pole on the real axis, although not essential to the present argument, are nevertheless interesting and given in the Appendix.

The corresponding line shapes [Eq. (20)] for the energy of the emitted meson are shown in Fig. 8; they do not show any eccentricities corresponding to the marked changes in the behavior of the decay curve. In fact aside from a growing width (Q_0) and displacement of the center (Ω_0) as g grows from 750 to 4000, a neat bell-shaped curve is retained. The bell is slightly asymmetrical but this is hardly

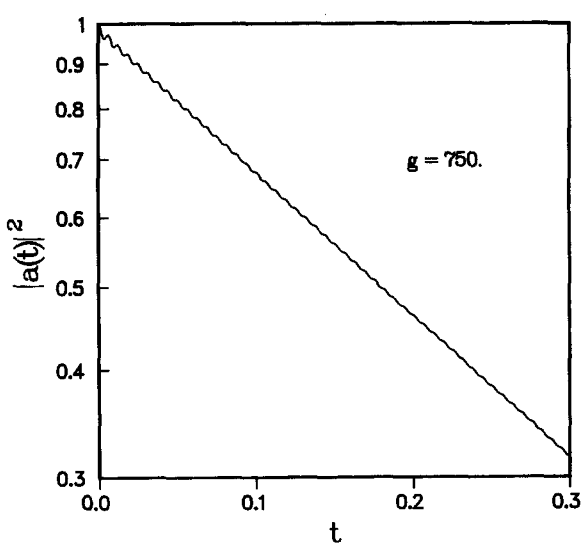

Fig. 5. Behavior of oscillations with $g = 750$, over a period comparable with the lifetime ($\tau = 0.26$ on this scale).

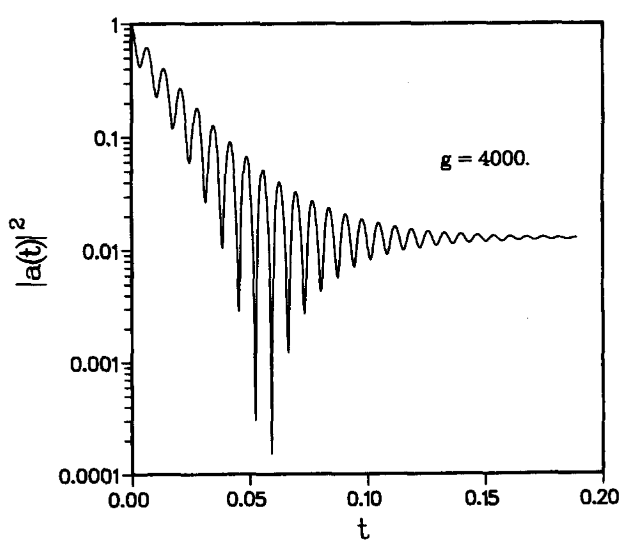

Fig. 7. Decay curve for strong coupling ($g = 4000$) shows entirely nonexponential time behavior.

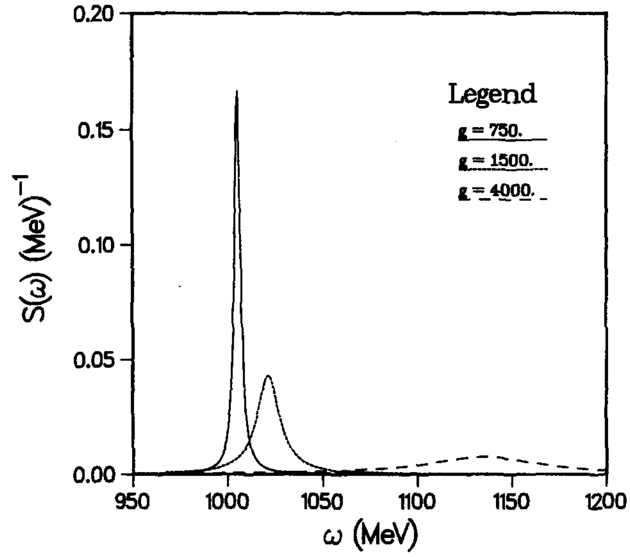

Fig. 8. Line shapes for $\omega_0 = 1000$ MeV show bell-shaped curves shifted and slightly asymmetric as coupling increases.

noticeable. Departure from the Lorentzian shape is more evident if we move the center of the line ($\omega_0 + \Omega_0$) nearer to the threshold (m). In Fig. 9 the value of ω_0 has been reduced to 400 MeV which is only 100 MeV above the value of m. Now the line shapes are seen to exhibit a characteristic cusp at threshold, so strong in the case of weak coupling ($g = 750$) that a slight second maximum is created.

Although our example is not intended as a universal model of decaying systems, we thought it interesting to fit the parameters to reproduce some real physical systems: To do so we need the threshold (m), the lifetime (τ) or width (w), and a dimension (a^{-1}) typical of the size of the system. As an example we chose the short-lived component of the kaon K_S because its uncanny ability to survive past

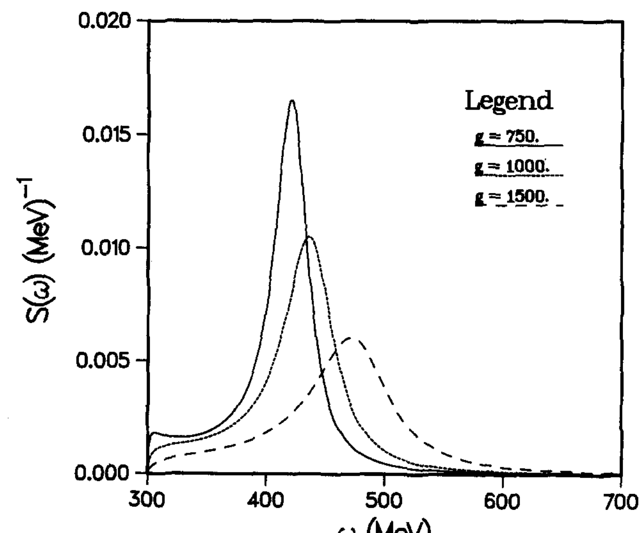

Fig. 9. Line shapes for $\omega_0 = 400$ MeV (close to threshold at 300 MeV) show cusp-shaped modification at threshold.

its exponential lifetime is the origin of the discovery of CP violation (might this be in part attributable to the natural breakdown of exponential decay)? To represent strong coupling we chose the Δ—the most venerable of nucleon-pion resonances which decays so rapidly it barely exists as a particle at all. We found the values $\omega_0 = 500$, $m = 300$, $a = 200$, $g = 5.4 \times 10^{-4}$, for the K_S system; these put it comfortably in the region where perturbation theory applies and the onset of nonexponential decay in our model does not occur until 90 lifetimes have passed. To reproduce the Δ resonance we find we need $\omega_0 = 1200$, $m = 1070$, $a = 200$, $g = 2650$; we note this implies a coupling strong enough that the decay curve would seriously deviate from exponential. Whereas no one has seriously looked for exponential decay of the Δ, our warning is not without content; it would also be questionable to represent the Δ by a single pole, for example.

IV. REMARKS AND DISCUSSION

The nonexponential nature of models of decaying systems has been remarked upon earlier; we could compare, for example, with recent notes on the solution of the barrier penetration problem,[19,20] which is a popular model for alpha decay and fission. All show these essential features in common: The long-term decay curve follows a power-law type behavior and not exponential, the earlier part of the curve has an oscillatory component superimposed on the exponential decay.

Possibly some of these features arise from unrealistic initial conditions. It is difficult to describe in a universal way the conditions under which a sample of unstable material is separated and identified, and translate these into the initial conditions on the wave function or density matrix. But one of the reasons for choosing the present model is its simplicity; it should then be possible to visualize what is going on (with the caveat that such a vision is likely to be classical in nature, and somewhat suspect for that reason).

The system we have described starts as a pure particle-hole. As it begins to decay, the particle leaks into the hole and a cloud of mesons starts to form around it. In weak coupling the cloud dissipates and has little effect, but with strong coupling a significant meson wavepacket forms whose presence near the source retards the decay of the system by giving it the opportunity of reabsorbing the meson and returning to the initial state. In either case there is an oscillatory exchange between the two parts of the system which, in the case of weak coupling, is effectively damped. The initial behavior is indeed parabolic, as Ref. 10 would predict, but only in the sense that it is the opening of an oscillation (comparable with a pendulum being released from its extreme position). When the meson system is an adequate match for the particle-hole system, the two can reach some form of equilibrium, albeit oscillatory, and the decay proceeds adiabatically, so that one would expect the population to fall only as fast as the magnitude of the wavepacket in the neighborhood of the source. Wavepacket spreading is not exponential but typically follows a power law, hence we may understand the power-law type dependence of the tail of the decay curve. For sufficiently strong coupling, part of the meson wavepacket fails to escape altogether and is retained to form a new composite ground state. The stronger the coupling, the sooner this happens, along with the accompanying leveling of the decay curve.

The second effect noted in the time dependence is oscillation about the exponential line. These would wash out in an unphased assembly of systems with starting times spread over a period large compared with that of the oscillations. Such an assembly might very well represent many physical samples we would be able to prepare, and hence give some credence to the position that under realistic conditions exponential decay still holds. Under conditions where the initial oscillations have a long period (in our model this means very low energy transitions), and where it is possible to trap and study individual systems, these oscillations should be observable. There also remains the problem of observing the nonexponential (power-law) decay in the extreme long-time limit.

V. SUMMARY AND CONCLUSIONS

We have a straightforward quantum mechanical model for a system that consists of a discrete level coupled to a continuum; it is therefore a model of quantum mechanical decay. The time-dependent Schrödinger equation can be solved without approximation. For weak coupling we can see how the Lorentzian line shape is closely reproduced and how the time dependence, dominated by a single "pole" term, is very near to the classical decaying exponential, over many lifetimes. We can also see the breakdown at large times and trace this to a second term, a "cut" contribution, which interferes with the pole. The interference also produces rapid oscillatory contributions. For strong coupling a second pole appears, which we can identify with a new ground state of the system, and which deforms the time dependence still further so that the decay amplitude no longer approaches zero (there being an overlap between the original state and the new ground state). The oscillatory contribution persists and the exponential behavior is increasingly swamped as the coupling increases. The line-shape is no longer a simple Lorentzian but is shifted and is slightly lopsided. Noticeable also is the behavior at the threshold, where the distribution rises vertically in a typical cusp-like fashion.

Far from being peculiar to this model, these features seem to be typical of such models of decaying systems in quantum mechanics (a particle escaping through a barrier is another example[4,19,20]). Whereas there are many respects in which these results seem to be at odds with nature (where decay appears always to proceed in an unhampered exponential fashion), one cannot yet fault the predictions by experimental counterexamples. The most one can say is that the predicted behavior seems to be counterintuitive.

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APPENDIX: GROUND STATE OF THE INTERACTING SYSTEM

We are looking for an eigenstate of the interacting system, which we identify with the label q:

$$(H_0 + H_I)\psi_q = \omega_q \psi_q. \tag{A1}$$

Express ψ_q in terms of the unperturbed eigenstates [similar to Eq. (4)]:

$$\psi_q = a_q \psi_1 \bar{\psi}_2 + \int d^3k\, b_q(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{A2}$$

Now take the inner product of Eq. (A1) first with the state $\psi_1\bar{\psi}_2$, we get

$$\omega_0 a_q + \int H^I_{0k} b_q(k) d^3k = \omega_q a_q \tag{A3}$$

then with the meson state $e^{i\mathbf{k}\cdot\mathbf{r}}$

$$\omega_k b_q(\mathbf{k}) + H^I_{k0} a_q = \omega_q b_q(\mathbf{k}) \tag{A4}$$

combining Eqs. (A3) and (A4) we get

$$\omega_q - \omega_0 = \int \frac{|H^I_{0k}|^2}{\omega_q - \omega_k} d^3k = Z(\omega_q), \tag{A5}$$

where the function $Z(\omega)$ is defined by Eq. (16). Now with reference to Eq. (21) we see that the expression for $a(t)$ has a pole wherever $Z(\omega) = \omega + \omega_0$, and this, according to Eq. (A5), will be true at $\omega = \omega_q - \omega_0$, on the real ω axis. This is the pole associated with the bound state. To get the wave function of the state combine Eqs. (A3) and (A2):

$$\psi_q = a_q \left(\psi_1 \bar{\psi}_2 + \int \frac{H^I_{k0}}{\omega_q - \omega_k} e^{i\mathbf{k}\cdot\mathbf{r}} d^3k \right), \tag{A6}$$

where the coefficient a_q may be determined by normalization. Since it is a bound state we require

$$(\omega_q, \omega_q) = 1,$$

which results in

$$|a_q|^2 \left(1 + \int \frac{|H^I_{k0}|^2}{(\omega_q - \omega_k)^2} d^3k \right) = 1. \tag{A7}$$

This can be written most compactly

$$|a_q|^2 = 1/[1 - Z'(\omega_q)], \tag{A8}$$

where the prime indicates the derivative of $Z(\omega)$. In this form it is easily related to the residue [say R_q, which will be one of those appearing in Eq. (27)] of the integrand $[Z(\omega_0 + \omega) - \omega]^{-1}$ at the bound-state pole $\omega = \omega_q - \omega_0$:

$$R_q = 1/Z'(\omega_q) \tag{A9}$$

and hence

$$|a_q|^2 = R_q/(R_q - 1). \tag{A10}$$

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Fourier transform construction by vector graphics

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The learning process is invariably improved by alternative, complementary views of a concept. This paper's pictorial representation of the discrete Fourier transform (DFT) is helpful in that students can understand internal steps of development, as well as the final result, in terms of vectors. It builds upon the widely known fact that the exponential terms in the transform pair are roots of unity in the complex plane; or in alternative physics terminology, they are unit vectors. After weighting these vectors by sampled values of the function to be transformed, using a simple recipe, the DFT is obtained through vector addition.

I. INTRODUCTION

The Fourier transform has received greater attention in experimental physics as instruments based on it have become popular.[1-3] Meyer-Arendt[4] displays boldness in his statement: "Fourier transform spectroscopy is the superior method. Even more important, Fourier spectroscopy is not simply the application of another little invention; rather, it marks a turning point in philosophy, away from high-precision delicate optics, toward a simple, rugged sensor coupled with sophisticated electronic data processing." Many disciplines other than optics have also been assisted by this powerful mathematical tool, since it is now possible to perform rapid conversion of time traces to the frequency domain, using the fast Fourier transform (FFT).[5] For example, power spectra have become central to the study of systems displaying deterministic chaos.[6,7] In the realm of image analysis, some engineers have focused much of their career on computer techniques based on the two-dimensional discrete Fourier transform (DFT).[8] In all of these examples, which represent a very small fraction of the whole world of Fourier processing, the advent of inexpensive digital computers was prerequisite to making the necessary computations practical. Numerous algorithms based on the FFT are now available to take an analog to digital converted voltage versus time record and produce a spectrum from it. Typically, these records are at least 1024 samples, if the resolution is to be reasonable.

The present paper describes an unconventional way of viewing the DFT. It facilitates understanding for those who are best served by visual aids. It has been successfully used by the author for the past 4 years in teaching an undergraduate optics course, as well as an experimental laboratory in computational physics. As opposed to the "abstraction" of the complex exponential representation, it is based on vectors. Most professionals with whom the matter has been discussed have recognized that the resultant produced by any DFT algorithm can be thought of as a set of vectors. The author is not aware, however, of anyone else having used vectors in this way for its development.

II. THEORY

For the present paper, the Fourier transform pair in x and k is defined as

$$G(k) = \int_{-\infty}^{\infty} g(x) e^{-ikx}\,dx, \tag{1a}$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k) e^{ikx}\,dk. \tag{1b}$$

As noted in Guenther,[9] there are alternative forms in which (1) the pair of equations is symmetric, by associating $(2\pi)^{-1/2}$ with each one; and/or (2) the positive and negative exponentials are interchanged. The six different forms (all acceptable) have been the source of confusion for many students through the years.

The variable k, which is conjugate to the position variable x, involves the spatial frequency f_x as follows:

$$k = 2\pi f_x = 2\pi/X, \tag{2}$$

