Schrödinger equation in two dimensions for a zero-range potential and a uniform magnetic field: An exactly solvable model

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which for analytic potentials may be calculated exactly or suitably approximated.

[7] For the special case $\alpha_j = 0$, the solution (4) is replaced by $\psi_j(x) = A_j x + B_j$ leading to the matrix

$$M[x,\alpha=0] \equiv \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

This is *not* the $\alpha = 0$ limit of the matrix (6) which itself becomes singular in this limit. However, the combination

$$K[\alpha=0,w] \equiv M[x,\alpha=0]M^{-1}[x+w,\alpha=0]$$

$$= \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}$$

is the *same* as the $\alpha = 0$ limit of the $K[\alpha,w]$ matrix defined in (10) and so justifies setting $\alpha = 0$ in that definition whenever necessary.

[8] Some samples from the Cold Fusion literature are the paper by J. S. Cohen and J. D. Davis, "The cold fusion family," Nature **338**, 705–707 (1989) and the three related articles of K. Ross and S. Bennington, T. Greenland, and D. Morrison in "Solid state fusion (?)" Phys. World **2**, 15–18 (1989).

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Schrödinger equation in two dimensions for a zero-range potential and a uniform magnetic field: An exactly solvable model

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The spectrum and eigenfunctions of a particle moving in two dimensions under the influence of an external uniform magnetic field and in the presence of a "point interaction" is determined. This is done after an elementary discussion of how to construct a "point interaction" in two dimensions circumventing the well-known difficulties with the Dirac $\boldsymbol{\delta}^2(\mathbf{r})$ interaction.

The delta-function potential is used in the one-dimensional Schrödinger equation to illustrate a number of interesting features. In two and three dimensions, however, the δ-function potential is problematic: The usual limiting procedure for its construction does not work (see discussion below). Recently, however, the theory of point interactions (also called zero-range or contact interactions in the literature) and its application to solid-state physics have been subject of extensive studies.[1-3]

The purpose of this paper is to solve the problem of a charged particle moving in a plane subjected to a magnetic field perpendicular to it and acted upon by an "impurity" represented by a two-dimensional contact interaction. A system like this has been used by Prange[4] in connection to the quantized Hall effect. Prange,[4] however, uses for this contact interaction a delta function that, as we will see, remains "too strong," even in the presence of a magnetic field.

We first review the problem of defining a point interaction in two dimensions without magnetic field. Our treatment is very pedestrian and therefore should serve as an introduction to more powerful methods presented in Ref. 2.

Consider a particle moving in a plane. Let's introduce polar coordinates (ρ,θ) and suppose that the interaction is a square well of depth V_0 and radius δ. Later, we shall let V_0 go to infinity and δ to zero in such a way that the specified energy of the unique bound state remains constant.

We shall consider only S waves, as in the limit $\delta \to 0$ the states with $l \neq 0$ are unaffected by the potential.

The Schrödinger equation for the S waves reads

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\rho}\right) = E\psi, \quad \text{for} \quad \rho > \delta, \tag{1}$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\rho}\right) - |V_0|\psi = E\psi, \quad \text{for} \quad \rho < \delta. \tag{2}$$

Suppose there is a bound state of energy $|E_b|$, and as usual call $k = \sqrt{2m|E_b|/\hbar^2}$ and $k_0 = \sqrt{2m/\hbar^2(|V_0| - |E_b|)}$.

The solutions of (1) and (2) are Bessel functions:

$$\psi = K_0(k\rho), \text{ for } \rho > \delta, \tag{3}$$

$$\psi = J_0(k_0\rho), \text{ for } \rho < \delta. \tag{4}$$

Now we have to match the functions and derivatives at $\rho = \delta$. Using[5] the relations

$$J_0'(z) = -J_1(z), \tag{5}$$

$$K_0'(z) = -K_1(z), \tag{6}$$

we get

$$-k_0[J_1(k_0\delta)/J_0(k_0\delta)] = -k[K_1(k\delta)/K_0(k\delta)]. \tag{7}$$

If we now try to let $\delta \to 0$, assuming E_b to remain finite and in such way that $|V_0|\delta^2 \to \Delta$ [so that our potential would approach $-\Delta\,\boldsymbol{\delta}^2(\mathbf{r})$] we find that the left-hand side of Eq. (7) behaves as[5]

$$-k_0\frac{J_1(k_0\delta)}{J_0(k_0\delta)}\to-\frac{\sqrt{(2m/\hbar^2)\Delta}}{\delta}\frac{J_1(\sqrt{(2m/\hbar^2)\Delta})}{J_0(\sqrt{(2m/\hbar^2)\Delta})},\tag{8}$$

whereas the right-hand side behaves as[5]

$$-k[K_1(k\delta)/K_0(k\delta)]\to\{\delta[\ln(k\delta/2)+\gamma]\}^{-1},\tag{9}$$

where $\gamma=0.5772$ is the Euler constant. This shows that the hypothesis that E_b remains finite cannot be maintained and that as mentioned before the delta function in two dimensions is "too strong." So if we insist on having a bound state with finite energy the only remedy is to make the left-hand side of Eq. (7) to diverge at the same rate. For that we let $k_0=\sqrt{(2m/\hbar^2)(|V_0|-|E_b|)}$ diverge slightly slower than $1/\delta$; that is, we write

$$k_0(\delta)=(1/\delta)f(k\delta),\tag{10}$$

where we want $f(k\delta)\to 0$ as $\delta\to 0$.

Replacing (10) in Eq. (7) and taking into account Eq. (9) we have

$$-(1/\delta)f(k\delta)[J_1(k_0\delta)/J_0(k_0\delta)]$$
$$=\{\delta[\ln(k\delta/2)+\gamma]\}^{-1}+\cdots,$$

from which we get

$$f^2(k\delta)=-2/[\ln(k\delta/2)+\gamma],\tag{11}$$

where k is determined by the chosen value of the bound-state energy through $k=\sqrt{(2m/\hbar^2)|E_b|}$. This choice of $f(k\delta)$ amounts to taking

$$\frac{2m}{\hbar^2}(|V_0|-|E_b|)=\frac{1}{\delta^2}\frac{-2}{\ln(k\delta/2)+\gamma},\tag{12}$$

which means, for $\delta\to 0$ that $V_0(\delta)$ is

$$\frac{2m}{\hbar^2}V_0(\delta)\simeq\frac{2}{\delta^2\ln(k\delta/2)}\left(1-\frac{\gamma}{\ln(k\delta/2)}+\cdots\right).\tag{13}$$

So we see that we have a logarithmically weakening version of the usual $\boldsymbol{\delta}^2(\mathbf{r})$ function. Scattering can now be calculated. For $E>0$, Eqs. (1) and (2) became

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}+k'^2\psi=0,\quad\rho>\delta,\tag{14}$$

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}+k_0'^2\psi=0,\quad\rho<\delta,\tag{15}$$

where

$$k'=\sqrt{(2m/\hbar^2)E},\quad k_0'=\sqrt{(2m/\hbar^2)(|V_0|+E)}.$$

Solutions of (14) and (15) are

$$\psi=AJ_0(k'\rho)+BN_0(k'\rho),\quad\rho>\delta,\tag{16}$$

and

$$\psi=J_0(k_0'\rho),\quad\rho<\delta.\tag{17}$$

Matching (16) and (17) and their derivatives at $\rho=\delta$, and taking into account that when $\delta=0$ Eq. (12) holds, we get

$$B/A=\pi/(2\ln\sqrt{E_b/E}).\tag{18}$$

Considering the behavior of $J_0(k'\rho)$ and $N_0(k'\rho)$ for large values[5] of ρ we see that Eq. (16) behaves as $A(2/\pi k'\rho)^{1/2}\cos[k'\rho-\pi/4+\delta_0]$ where δ_0 is the S-wave phase shift.[6,7] So we have that $\tan\delta_0=-B/A$.

Consider now a particle moving in a plane and subjected to a magnetic field $\mathbf{B}$ perpendicular to the plane.

Choosing the rotationally invariant gauge $\mathbf{A}=-\mathbf{r}\times\mathbf{B}/2$ and polar coordinates the Schrödinger equation reduces[8] (for S waves) to

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}+\lambda\psi-\alpha^2\rho^2\psi=0,\tag{19}$$

where

$\lambda=(2m/\hbar^2)E$ and $\alpha=eB/2c\hbar$.

Let's now introduce a square well of depth V_0 and radius δ and since the radius goes to zero we may neglect the term $\alpha^2\rho^2$ inside the well. The depth $V_0(\delta)$ will be chosen as given by Eq. (13).

The solution for $\rho>\delta$ that goes to zero as $\rho\to\infty$ is

$$\psi=e^{-1/2\alpha\rho^2}U\left[\tfrac{1}{2}(1-\lambda/2\alpha),1,\alpha\rho^2\right],\tag{20}$$

where $U(a,b,x)$ is the Kummer's function[9] which is the regular solution at infinity. For $\rho<\delta$ we have

$$\psi=J_0(k_0\rho),\tag{21}$$

where

$$k_0=\sqrt{(2m/\hbar^2)(|V_0|+E)}.$$

Using[10] $(d/dz)U(a,c,z)=-aU(a+1,c+1,z)$ we can calculate

$$\lim_{\delta\to 0}\frac{d\Psi/d\rho}{\Psi}\to\frac{2}{\rho}\frac{1}{[\log\alpha\rho^2+\psi(\frac{1}{2}-\lambda/4\alpha)-2\gamma]},\tag{22}$$

where $\psi[z]$ is the digamma function, that is, the logarithmic derivative of the $\Gamma[z]$ function.[11] So matching the logarithmic derivatives,

$$\log\delta+\frac{1}{2}\log\alpha+\frac{1}{2}\psi\left(\frac{1}{2}-\frac{\lambda}{4\alpha}\right)-\gamma$$
$$=\lim_{\delta\to 0}(-)\frac{1}{k_0\delta}\frac{J_0(k_0\delta)}{J_1(k_0\delta)}.\tag{23}$$

With our choice of $V_0(\delta)$, we get from Eq. (13) as $\delta\to 0$

$$\psi\left(\frac{1}{2}-\frac{\lambda}{4\alpha}\right)=\ln\frac{k^2}{4\alpha}+4\gamma,\tag{24}$$

which determines the value of λ, thus solving the problem.

It is easy to see that when $k\to 0$ we get the energies of the S states of an harmonic oscillator (the S states of the Landau levels). On the other hand, when $\alpha\to 0$ we get, of course, a bound state at the energy E_b.

The energy spectrum predicted by Eq. (24) can be deduced from a graph of the bigamma function $\psi[z]$.[12] From this graph it can be seen that each λ solving Eq. (24) corresponds to an S state that is displaced from its Landau level and lies between two Landau levels except for one level that lies below all Landau levels. The other states in the Landau levels are of course the unaffected $l\neq 0$ states. This is exactly similar to the spectrum found by Prange.[4]

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A student experiment on optical bistability

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A hybrid system that consists of an oscilloscope and a photodiode is used to demonstrate optical bistability. The details of the experimental setup and its graphical solution will be presented.

I. INTRODUCTION

Optical bistability is a rapidly expanding field of current research because of its potential application to all-optical logic and because of the interesting phenomena it encompasses.[1] An optical system that possesses two different steady-state transmission states for the same input intensity is said to be optically bistable.

Thus a system having the transmission curve of Fig. 1 is said to be bistable between $I_\downarrow$ and $I_\uparrow$. Such a system is clearly nonlinear, i.e., I_T is not just a multiplicative constant times I_I. In fact, if I_I is between $I_\downarrow$ and $I_\uparrow$, knowing I_I does not reveal I_T. Nonlinearity alone is not sufficient to assure bistability. It is feedback that permits the nonlinear transmission to be multivalued, i.e., bistable.

An undergraduate experiment on optical bistability has been reported using a PLZT modulator as the nonlinear switch.[2] Here, we would like to present an interesting experiment in which an oscilloscope and a photodiode are used to demonstrate optical bistability.

II. EXPERIMENTAL SETUP

The experimental setup is shown in Fig. 2. The hybrid optically bistable system consists of a student oscilloscope (SCOPE-1) and a photodiode (DET) facing the screen of the oscilloscope. The signal detected by the photodiode is amplified by a preamplifier (AMP) whose circuit is shown in Fig. 3. The amplified signal is then feedback to the Y input of the SCOPE-1.

The optical bistability can be observed by varying the intensity or the horizontal position of the light spot on the screen. In order to monitor the optical hysteresis curve of Fig. 1, a signal generator (OSC) with sine output is connected to the X or Z input of SCOPE-1 to vary the horizontal position or the intensity of the light spot on the screen. For convenience, another oscilloscope (SCOPE-2) is used to observe the optical switching. Under proper conditions we can observe a hysteresis curve similar to Fig. 1; here, the horizontal coordinate could be the horizontal position or intensity of the spot, and the vertical coordinate is the signal detected by the photodiode. In order to have a clean

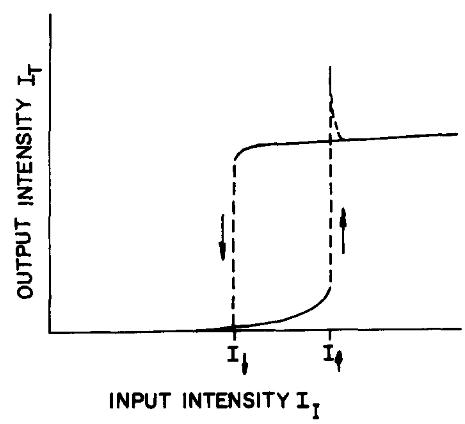

Fig. 1. Characteristic curve for an optical bistable system.

