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A hydrogenic atom in d-dimensions

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The laws of physics in d spatial dimensions are interesting and often lead to insights concerning the laws of physics in three spatial dimensions. A hydrogenic bound system in d-dimensions is investigated where d is any real positive number, not necessarily an integer, and where a d-dimensional radial Schrödinger equation is defined by the analytic continuation of the Schrödinger equation used in integer dimensions, with a d-dimensional potential defined by Gauss' law. The fundamental constants and physical relationships in d dimensions are arrived at by assuming the electric flux obeys a Gausslike law in d dimensions. This fixes the potential and makes it possible to write down the d-dimensional Schrödinger equation. This equation is then solved numerically for the ground-state energy eigenvalues, for several different dimensions, using a fourth-order Runge–Kutta predictor algorithm. The solutions to this equation are compared to the analogous Bohr model energies in d dimensions and to the results obtained by others at well-known integer dimensions. There is good qualitative agreement between the two theories in all dimensions. In addition, the wave functions for the Schrödinger equation are numerically determined and normalized in order to locate the most probable orbital radius. These values are compared to the d-dimensional Bohr radii.

I. INTRODUCTION

There has been considerable recent interest in physics in dimensions other than the usual four space-time dimensions.[1-7] In field theory and superstring theory, space-times of higher dimension are frequently used where the extra spatial dimensions are taken to be the same as the three we currently experience directly. In thin films one uses two dimensions, and in field theory the 2-D Schwinger model is often used to teach QED. Many students have seen or read *Flatland*[8] or some similar book describing life in a universe with other than three spatial dimensions. The case of two spatial dimensions has received considerable attention[8-10] and an entire society of workers have been busy examining its features, including a full periodic chart.[11] With the recent explosion of interest in chaos and fractals,[12,13] people have also developed an interest in noninteger dimensions. These fractional dimension sets have found a variety of applications and several popular accounts of their uses are now available.[14] A significant body of literature exists on these subjects and many students are very interested in these results and are capable of grasping the main ideas readily. This recent work, coupled with the curiosity of our students, led us to calculate and interpret the bound-state energies and radii for hydrogenic systems in d spatial dimensions where d is any real number (d can be complex but the results aren't as interesting).

The hydrogen atom in other than three spatial dimensions is a well-studied problem. Many authors treat this problem by using the standard Coulomb potential,[1-3,7,15-18] $V(r) = -ke^2/r$, for the electron–proton interaction in all dimensions. This is both valuable and appropriate for understanding many physical phenomena, especially surface interactions,[3] and several exact solutions were given by Alliluev[19] and extended by Herrick.[20] Other workers, however, have pointed out that Gauss' law gives a

different potential, one that describes the flux due to a point source, which depends upon the dimension. The potential can be extended in a covariant way to other dimensions as treated by Landau and Lifshitz[21] and Reiser[22] in two dimensions. It is this work that we are building upon; especially the results of Ehrenfest[23] and Sommerfeld,[24] and the two-dimensional results of Asturias and Aragon[11] and Reiser;[22] the well-known three-dimensional results as found in any standard text such as Merzbacher;[25] and the extensively studied one-dimensional results, particularly Langhoff,[26] Flamm and Schoberl,[27] and Nieto and Simmons.[27]

As a precursor to any analysis in arbitrary dimensions, and to make perfectly clear what our approach relies upon, a method of choosing what laws of physics hold is important. The key issue for this paper is the form of the electrostatic force law in d dimensions (d dimensions will always mean d spatial dimensions). We are not selecting a Coulomblike potential for all dimensions. Instead, we begin by considering the question, how do things spread out?[28] Given a point source, how will the gravitational force change with distance? How will the illumination from a point light source change with increasing distance? In this paper we keep this consideration primary by using Gauss' law to describe the way that fields behave as a function of distance in d dimensions. By keeping the unit of charge equal to the Coulomb, and the unit of distance equal to the meter, Gauss' law is then restrictive enough to fix the functional form of the d-dimensional potential and the units of the permittivity of free space. Furthermore, we take the speed of light as still fundamental in all dimensions, fixing the permeability of free space. With the classical equations of motion established, the quantum dynamics equations are arrived at by replacement of the Poisson brackets with the corresponding commutators and by introducing a factor of $i\hbar$. This is sufficient to establish the units of Planck's con-

stant in d dimensions. The details of this unit selection and further discussions are given in Sec. II.

Once units have been established we make use of the Bohr theory of the atom (Sec. III). Every student is familiar with the Bohr model and the resulting picture of atoms as tiny "planetary" systems. The derivation of radii and energies of this system in three dimensions only involves some algebra and can be done by any student. With an understanding of the electric potential in d dimensions, students can generalize the Bohr model to d dimensions, and can derive the radii and orbital energies of a d-dimensional Bohr atom.

The original three-dimensional Bohr model led to definite predictions concerning the spectrum of hydrogen. When Bohr first proposed this model in three dimensions, the agreement with experiment was quite good and gave physicists a hope of understanding the atom.[29] Later, quantum physics provided the foundation for understanding the successes of the Bohr model. The Schrödinger equation, and later the Dirac equation, helped establish a rigorous mathematical foundation for understanding the nature of quantum phenomena. In this article, we will follow this historic order: We first (Sec. III) discuss the Bohr model; we then (Sec. IV) use the classical d-dimensional electric potential that follows from Gauss' law in the d-dimensional Schrödinger equation; we then finally compare the Bohr and Schrödinger results for both atomic radius and ground-state energy.

II. UNITS AND EQUATIONS IN d DIMENSIONS

A. Meaning of d dimensions

It is customary to think of the universe as being composed of a space-time with an integer number of dimensions. Casual observation indicates that there are at least three large spatial dimensions and one temporal dimension. However, our current laws of physics may not be the laws of physics on a simple 4-D space-time, they may be the laws of physics on a complex 10-D space-time manifold with six dimensions conspiring to give us the dynamics of interacting particles.[30] It is difficult and sometimes controversial to formulate a consistent physics for other dimensions, especially noninteger dimensions. In order to analyze physical laws in a space with a different number of dimensions, we have concentrated on Gauss' law as a means of establishing useful laws that reduce to the standard laws in the limit as the number of spatial dimensions approaches three. Many questions immediately come to mind concerning the formulation of laws of physics in other dimensions; what would a 2-D universe be like in terms of the interactions between particles; how would the laws of physics behave in 2.6 dimensions; and what does it mean to have a noninteger number of dimensions?

We use Gauss' law to fix the way fields may spread out from a point source in integer dimensions. To extend this law to noninteger dimensions we simply analytically continue the arguments that depend upon dimension to those real values between the fixed integers. This extension to noninteger dimensions is quite different from having an underlying fractal[31] space that is self-similar. A fractal space can be everywhere continuous but nowhere differentiable, making it difficult to perform the normal operations used in calculus. An instructive way to understand the existence of a continuous fractional dimension space can be found by considering a simple example. It is common to

continue integration analytically to d-dimensional spaces by replacing the volume element with a d-dimensional volume element and the corresponding functional variables by d such variables,

$$\int f(x_i) d^3x \rightarrow \int f'(x_i) d^dx, \tag{1}$$

where care must be taken to be sure that $f'(x)$ is the appropriate form of $f(x)$ in d dimensions, and i runs over all dimensions. If $f(x)$ is dimensional, then dimensional analysis can be used to help fix the form of $f'(x)$. The integral is well defined for all values of d and, although it may be difficult to imagine, d is *continuous*. This technique is commonly used for defining generating functions of orthogonal polynomials and for convergence analysis.[32,33]

We also must be clear about how we intend to treat our time variable. Time is considered to be a continuous one-dimensional variable regardless of the dimension of the adjoining spatial manifold. This provides us with a clear and unambiguous method for establishing time-independent eigenvalue equations arrived at by separation of variables. As a result, the Schrödinger equation takes on its well-known eigenvalue form.

B. Gauss' law in d dimensions

It is now possible to discuss the laws of physics in a space of arbitrary dimensions. To do this we follow Ehrenfest[23] and defer to Gauss' law, which describes how fields "spread out" in space. We demand that the flux of the electric field from a charge distribution be equal to the net charge enclosed

$$\epsilon_{0d} \int \mathbf{E} \cdot d\mathbf{A} = \int_v \rho \, dV,$$

$$\epsilon_{0d} \nabla \cdot \mathbf{E} = \rho. \tag{2}$$

Substituting in the potential gives Poisson's equation

$$\nabla^2 \phi = \rho/\epsilon_{0d}, \tag{3}$$

which gives solutions

$$-e\phi = V(r) = -e^2 r^{2-d}/\epsilon_{0d} A_d (d-2), \quad d \neq 2,$$

$$V(r) = (e^2/\epsilon_{0d} A_d) \ln(r/r_0), \quad d = 2, \tag{4}$$

where r_0 is a scale constant. Equations (4) correspond to a Coulomb force law of the form

$$F = q_1 q_2/\epsilon_{0d} A_d r^{d-1}. \tag{5}$$

Here, A_d is a dimensionless constant that is equal to the angular measure subtended by any closed $d-1$-dimensional surface in d space. In three spatial dimensions this is 4π steradians, in two dimensions this is 2π radians, and in d dimensions we have

$$A_d = 2\pi^{d/2}/\Gamma(d/2), \tag{6}$$

where $\Gamma(d/2)$ is the Gamma function[34] of $d/2$.

Equation (5) describes the way the electric flux from a point source "spreads out" in a d-dimensional space. The field falls off in accordance with the way the area of a d-dimensional spherical shell changes with radius. In this paper d-dimensional physics always refers to the potentials and force law given by Eqs. (4) and (5) and is of the type used by some, but not all, other authors working on hydrogenic atoms in d dimensions.[11,22-27] This is to be contrasted with other authors who are interested in the Coulomb problem in d dimensions. They take the potential to be of

an inverse distance type, even though the number of spatial dimensions is different from three.[1-3,7,15-18]

The Coulomb force is still the negative gradient of the potential for all dimensions; however, we must be careful of the fundamental constants that appear in the law. The units on the permittivity of free space are dependent upon the dimension

$$[\epsilon_{0d}] = C^2 \, s^2/kg \, m^d, \tag{7}$$

where we have selected the Coulomb, kilogram, and meter to be our fundamental charge, mass, and length in all dimensions. The charge density also depends on the dimension since the volume element does, hence

$$[\rho] = C/m^d. \tag{8}$$

Having fixed our defining equations, and demanding that the speed of light be the same in all spatial dimensions, we can fix the permeability of free space;

$$c = (\mu_{0d}\epsilon_{0d})^{-1/2},$$
$$[\mu_{0d}] = kg \, m^{d-2}/C^2. \tag{9}$$

The prescription for quantizing the classical equations of motion calls for changing the classical Poisson brackets into commutators and introducing a factor of $i\hbar$. In doing this in d dimensions we introduce Planck's constant and fix its units. The classical equations of motion are given by the Poisson brackets,

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\} \tag{10}$$

which, after quantization, become

$$i\hbar\dot{q}_i = [q_i, H], \quad i\hbar\dot{p}_i = [p_i, H] \tag{11}$$

for canonical coordinates q_i, momenta p_i, and Hamiltonian H. This forces h to be treated in the same way as c, and its units are unchanged in going from one dimension to another; h retains the units of Joule second.

We now have a complete set of consistent units and equations of motion for our d-dimensional universe. Before turning to explicit calculation with the Bohr model, it is worth mentioning this interesting implication of the dimensions (examined by Lapidus[35] for integer dimensions): Taking mass, length, and time as fundamental, e^2, $\hbar$, and c have the units

$$\begin{pmatrix} \cdots & M & L & T \\ e^2 & 1 & d & (-2) \\ \hbar & 1 & 2 & (-1) \\ c & 0 & 1 & (-1) \end{pmatrix}. \tag{12}$$

This matrix has determinant $d-3$. This means that if $d=3$, the determinant is zero, hence the rows are not linearly independent and it is possible to take a combination of rows in which M, L, and T all cancel. Specifically, if $d=3$, then the elementary row operation $R_1 - R_2 - R_3 = 0$, where R_i is the ith row, hence, e^2/hc is dimensionless. In contrast, if $d \neq 3$, the matrix has rank 3, so the rows are linearly independent, hence no fine structure constant can be made from e^2, h, and c. In other than three dimensions, the mass of the electron is needed to form a dimensionless combination. One is reminded that $d=3$ is special when one sees $m^{(d-3)}$ in Eq. (20) for the electron velocity. Similarly, one is reminded that $d=2$ and $d=4$ are special when one sees various quantities raised to powers involving $(d-2)$ and $(d-4)$ in Sec. III.

Of course, only so much can be gleaned from dimension-al analysis. Already it is clear that some surprises are on the way. In Sec. III the Bohr model will be solved in d dimensions and those results later compared to the information revealed via solving Schrödinger's equation.

III. THE BOHR MODEL

Following the early work of Ehrenfest,[23] we calculate the Bohr energies and radii, using methods similar to those of an introductory physics class.

A. Case 1: $d \neq 2$; $d \neq 4$

The energy of the atom is

$$E = T + V = \tfrac{1}{2} mv^2 - k_d e^2 r^{2-d}/(d-2), \tag{13}$$

where the first of Eqs. (4) has been used for the potential and

$$k_d = 1/\epsilon_{0d} A_d. \tag{14}$$

Using the uncertainty principle,

$$v = \Delta p/m = \hbar/mr \tag{15}$$

in Eq. (13) and minimizing by setting dE/dr equal to zero, we find

$$r_1 = (\hbar^2/mk_d e^2)^{1/(4-d)}. \tag{16}$$

Alternatively, one may start with

$$F = ma = m(v^2/r) \tag{17}$$

and assume the angular momentum is quantized,

$$mvr = n\hbar, \quad n = 1,2,3,\ldots, \tag{18}$$

where n is the principal quantum number. Using Eq. (5) for the force and eliminating v between Eqs. (17) and (18) yields

$$r_n = (n^2\hbar^2/mk_d e^2)^{1/(4-d)} \tag{19}$$

in agreement with Eq. (16). This method has the advantage of including the dependence on the principal quantum number; but the disadvantage of relying on $a = v^2/r$ with no obvious visual image for that when $d < 2$.

The "velocity" of the electron in the ground state of the (admittedly naive) Bohr model is given by using Eq. (19) in Eq. (18),

$$v_n = (k_d e^2 m^{d-3}/\hbar^{d-2})^{1/(4-d)} n^{(2-d)/(4-d)}. \tag{20}$$

Finally, the energies can be obtained by substituting Eqs. (19) and (20) into (13), yielding

$$E_n = [(d-4)k_d e^2/2(d-2)]$$
$$\times (n^2\hbar^2/mk_d e^2)^{(2-d)/(4-d)}. \tag{21}$$

Equation (21) is plotted for the ground state in Fig. 1 and several numerical values are listed in Table I. Note, however, that Fig. 1 must be interpreted with care: Eq. (4) shows that $V(r) = 0$ at $r = 0$ for $d < 2$, but $V(r) = 0$ at $r = \infty$ for $d > 2$. Hence the energy is being compared to different points for the two branches in Fig. 1.

With the calculated velocity, it is a straightforward matter to calculate the kinetic energy T. This results in a Bohr version of the virial theorem,

$$V = -[2/(d-2)]T. \tag{22}$$

This result is valid for all dimensions except for $d = 2$ where the logarithmic potential is needed. We will refer to this result as the Bohr version of the virial theorem since it does *not* agree with the general result from basic quantum
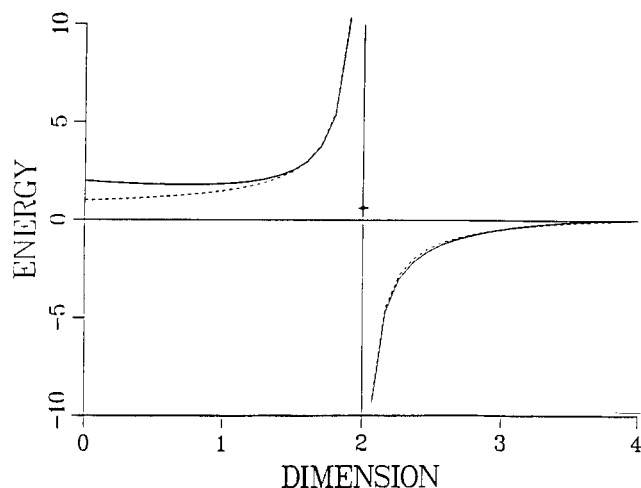
GROUND STATE ENERGY

Fig. 1. Ground-state energy predicted by the Bohr (dashed line) and Schrödinger (solid line) theories, in units with $m_c = \hbar = k_d e^2 = 1$. The $d = 2$ line separates disjoint energy regimes. For $d > 2$, $V(\infty) = 0$, for $d < 2$, $V(0) = 0$, for $d = 2$, $V(r_0) = 0$. Comparisons in different regions must be made with this in mind.

mechanics. This is because the expectation value of the kinetic energy and potential energy become infinite and the assumptions needed to derive the virial theorem are invalidated. Liebman and Yorke have examined this question in some detail.[36]

B. Case 2: $d = 2$

The two-dimensional Bohr atom must be considered separately; the calculation of the Bohr radii is identical, but the energy levels turn out to be very different. First, consider the radii. The equations used to calculate the radii still hold, so the radii are given by Eq. (19), which simplifies to

$$r_n = n\hbar/e\sqrt{mk_2}. \tag{23}$$

Similarly, Eq. (20) also holds and simplifies to

$$v = e\sqrt{k_2/m}. \tag{24}$$

Table I. Several ground-state energies, in units with $m_c = \hbar = k_d e^2 = 1$.

Dimension	Bohr, Eq. (21)	Schrödinger, Eq. (37) $\pm 2\times 10^{-4}$	Other authors
0.4	1.1250	1.8526	
0.8	1.3333	1.8095	
1.0	1.5000	1.8557	1.5775[27]
1.4	2.1667	2.2839	
1.8	5.5000	5.3839	
2.0	0.5000	0.5101	0.526 64[11]
	$\pm 2\times 10^{-3}$	$\pm 2\times 10^{-3}$	$\pm 3\times 10^{-5}$
			0.726 99[22]
			$\pm 3\times 10^{-3}$
2.4	-2.0000	-2.1678	
2.8	-0.7500	-0.8110	
3.0	-0.5000	-0.5000	-0.5000[25]
3.4	-0.2143	-0.1501	
3.8	-0.0554	-0.0087	

The energies are different because in two dimensions the second (rather than the first) of Eqs. (4) must be used. The r_0 in Eq. (4) is simply an integration constant that can be chosen for convenience as the first Bohr orbit,

$$r_0 = \hbar/e\sqrt{mk_2}. \tag{25}$$

(Comparison of $d = 2$ with the familiar three-dimensional results therefore requires keeping in mind that it was *not* possible to use the usual convention of $V = 0$ at infinity for $d = 2$.) With this choice of r_0, the two-dimensional Bohr energies are

$$E_n = \tfrac{1}{2} mv^2 + V = k_2 e^2 \left[\tfrac{1}{2} + \ln(n)\right]. \tag{26}$$

C. Case 3: $d = 4$

The four-dimensional Bohr model is totally pathological. Attempting to begin in the usual way yields

$$F = k_4 e^2/r^3 = mv^2/r, \tag{27}$$

$$k_4 e^2 = mr^2 v^2 = n^2 \hbar^2/m, \tag{28}$$

hence

$$k_4 e^2 = n^2 \hbar^2/m. \tag{29}$$

Equation (29) is overdetermined. The model used here suggests that a four-dimensional Bohr atom cannot exist unless the constants of nature have the relation expressed in Eq. (29); even then, it could not change states.

D. Discussion

The Bohr atom behaves differently, even qualitatively, depending on the dimension. Various cases are described briefly below. In each case, the key idea will involve checking how the energy depends on n.

1. $d < 2$ and $d > 4$

For these cases, Eq. (21) shows that the energy expression has a positive coefficient, multiplied by n raised to a positive power; hence the energy increases with n. The $n = 1$ state is the ground (lowest energy) state. An isolated atom cannot be ionized because there are an infinite number of bound states whose energies increase without limit. The impossibility of ionizing can be verified from another point of view: For $d < 2$ the atom cannot be ionized because increasing n increases r without limit [Eq. (19)], which in turn increases V without limit [Eq. (4)]. In a related way, for $d > 4$, the atom cannot be ionized because increasing n *decreases* r without limit, which in turn increases the kinetic energy without limit.

2. $d = 2$

These results are qualitatively similar to the preceding paragraph. Equation (26) shows that the energy increases with n. The $n = 1$ state is the ground state. An isolated atom cannot be ionized because there are an infinite number of bound states whose energies increase without limit. Again, Eq. (4) shows that the potential is infinite at infinity.

3. $2 < d < 4$

For this case, Eq. (21) shows that the energy expression has a negative coefficient, multiplied by n raised to a negative power. Hence the atom is qualitatively like the familiar

three-dimensional Bohr atom: The $n = 1$ state is the ground state. There are an infinite number of bound states, all with energy less than zero; the atom can be ionized.

4. $d = 4$

The result that the equations are overdetermined here is unusual and in marked contradiction to the $d = 4$ Schrödinger result, where an inverse-square potential can be simply treated like a dipole potential.[37] These Bohr results do signal a warning to the cautious student: Bohr theory is greatly oversimplified and the results, although often intuitive, are not to be taken too seriously.

IV. THE SCHRÖDINGER EQUATION

Following Louck[38] and Nieto[18] we express the Schrödinger equation in d dimensions, where d is confined to integer values, as

$$-\frac{\hbar^2}{2m}\nabla_d^2\psi + V_d\psi = i\hbar\frac{\partial\psi}{\partial t}, \qquad (30)$$

where we are in the center-of-mass frame, r is the relative coordinate between the proton and electron, m is the reduced mass of the system, and $\hbar$ is Planck's constant divided by 2π. We are calculating energy eigenstates and since our time coordinate is a full 1-D continuous dimension, we can separate variables to arrive at the time-independent eigenvalue equation for any integer dimension,

$$(-\hbar^2/2m)\nabla_d^2\psi + V_d\psi = E\psi. \qquad (31)$$

The potential is a pure radial function that has the form given by Eq. (4). Our goal is to separate Eq. (31) in spherical coordinates, for integer dimensions, to arrive at a single 1-D radial equation. We then analytically continue the solutions of this radial equation to noninteger values of the dimension.

Going to d-dimensional spherical coordinates, with $d - 1$ angular coordinates and 1 radial coordinate, the variables can be separated[18,24,38] for any integer value of d, and the wave function can be expressed as the product,

$$\psi = R_{nl}(r)Y_l^m(\theta_i) \qquad (32)$$

where the symbol Y_l^m represents the contributions from the hyperspherical harmonics that arise in higher integer dimensions (or spherical harmonics in three spatial dimensions). This leaves us with a single 1-D radial equation

$$\left[\frac{-\hbar^2}{2m}\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\right)\right.$$
$$\left. + \frac{L^2}{2mr^2} + V(r)\right]R_{nl} = ER_{nl}, \qquad (33)$$

where L is the angular momentum operator for the $d - 1$ angular coordinates. It has eigenvalues of $l(l + d - 2)$ for any given d.[24,38] Introducing the dimensionless variable

$$z = \beta^{1/(4-d)}r = (2e^2m/\hbar^2\epsilon_{0d}A_d)^{1/(4-d)}r, \qquad (34)$$

we can express Eq. (33) as

$$\left[\frac{1}{z^{d-1}}\frac{d}{dz}\left(z^{d-1}\frac{d}{dz}\right) - \frac{\gamma}{z^2}\right.$$
$$\left. + \frac{z^{2-d}}{d-2} + \lambda\right]R_{nl}(z) = 0, \qquad (35)$$

where $\gamma = l(l + d - 2)$ is the dimensionless d-dimensional eigenvalue[24,38] of L^2, and $\lambda = 2mE\beta^{2/(d-4)}/\hbar^2$. The ap-

parent difficulty in using Eq. (34) when $d = 4$ is again a result of the fundamental constants forming an overdetermined set of equations. In four dimensions β is dimensionless and the equation has an irregular singularity at the origin. When $d = 2$ the potential term takes on the form $V(z) = \ln(z)$ after the transformation, this then corresponds to the equation used by Asturias and Aragon[11] and Reiser.[22] Equation (35) is self-adjoint for all d; thus we can eliminate the first-order derivative term by the transformation

$$R_{nl}(z) = P(z)z^{(1-d)/2}, \qquad (36)$$

where the n,l labels have been absorbed in our definition of $P(z)$. Substituting Eqs. (36) and (4) into (35) gives a second-order differential equation for $P(z)$,

$$\frac{d^2P(z)}{dz^2} - \left[\frac{l(l+d-2)-(1/4)(1-d)(d-3)}{z^2}\right.$$
$$\left. - \left(\frac{z^{2-d}}{d-2} + \lambda\right)\right]P(z) = 0, \qquad (37)$$

where l is the orbital angular momentum. It is to be understood that for $d = 2$ we are using the logarithmic potential in Eq. (37). As presented, Eq. (37) is only valid for integer values of the dimension. We will now relax this condition by analytically continuing the solutions of Eq. (37) to noninteger values. As a result, our definition of quantum mechanics in noninteger dimensions amounts to finding solutions to Eq. (37) for noninteger values of d. Unfortunately, Eq. (37) is not easily solved for arbitrary d; however, the form of the equation is enough to tell us about specific values of d that are particularly important.

This can be seen by examining the singularity structure of the equation as discussed by Sommerfeld.[24] The bracketed expression in Eq. (37) has two singular terms at the origin for $d > 2$. Even for the ground state ($l = 0$) the effective potential retains a dimensionally dependent singular term that only vanishes for $d = 1$ and $d = 3$. The vanishing of this term helps in finding exact closed-form solutions in $d = 1$ and $d = 3$. These correspond to Airy's differential equation, which is investigated in detail by Langhoff[26] and the well-known associated Laguerre functions.[25] It is important to notice that the $d = 0$ case is not the simple harmonic oscillator since the additional term in the effective potential does not vanish. This singularity structure has critical values where the singularities become essential (or irregular) and effect the method of solution. For $d > 4$ the essential singularity at the origin assures us that a unique solution expanded about the origin doesn't exist. The singularity is such that there do not exist any stable orbits for $d > 4$; all orbits eventually cross the origin. Barrow[39] has pointed out that, in general, the existence of stable periodic orbits requires $r^3F(r) \to 0$ as $r \to 0$ and $r^3F(r) \to \infty$ as $r \to \infty$. This difficulty is mimicked in the Bohr model where the atom cannot be ionized for $d > 4$. The only other difficult case is $d = 2$. Here, the singularity is changed by switching to the logarithmic potential, as demanded by Gauss' law, and a stable solution can be found. The search for ground-state solutions is therefore limited to $d < 4$.

Fortunately we don't need to solve Eq. (37) analytically for $P(z)$ in order to compare ground-state energy eigenvalues to those of the Bohr model. We have incorporated a numerical fourth-order Runge–Kutta predictor corrector algorithm[40] to generate numerical solutions for the eigenvalues. Here, a trial eigenvalue is selected and the wave
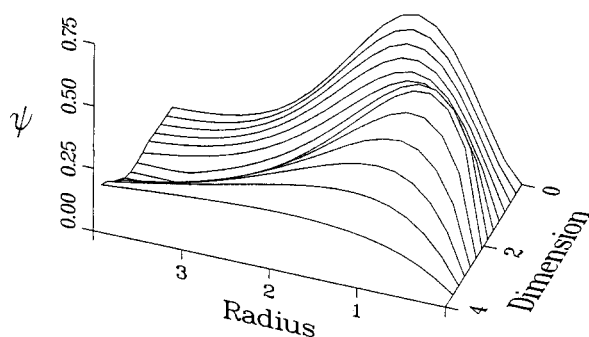
Wave Function

Fig. 2. The ground-state wave function $\psi(r)$, where $r = 1$ corresponds to the Bohr radius in all dimensions.

function and its second derivative are calculated from Eq. (37). Corrections to the trial eigenvalue are made depending upon the behavior of the wave function. Iterations are continued until a tolerance of $1:10^{-4}$ is met. Although this provides a convergent method for determining the eigenvalues, the rate of convergence is very low in the neighborhood of the critical dimensions of 2 and 4 and the tolerance is weaker, $1:10^{-3}$. A graphical comparison between the Schrödinger and Bohr results for the ground-state energies is displayed in Fig. 1, where we have calculated the energy eigenvalues for each 0.1 increment in dimension. Table I gives a list of numerical values obtained for the ground-state eigenvalues at several dimensions along with a comparison to previous work. The difference between Asturias and Aragon[11] and Reiser[22] appears rather large but Asturias and Aragon have done a later and more accurate treatment. Both our Schrödinger and Bohr results agree quite well with the results of Asturias and Aragon. As expected, the $d = 3$ results agree with the well-known solution to the 3-D hydrogen atom. There is a difference in energies compared to the work of Nieto and Simmons[27] due in part to the approximation they used in their expansion of the energy and in using this expansion for small values of n. In addition to the ground-state energy eigenvalues we have numerically determined the normalized wave functions resulting from the fourth-order Runge–Kutta solutions to Eq. (37). These wave functions are plotted in Fig. 2 where the value of 1.0 corresponds to the Bohr radius in every dimension.

V. CONCLUSION

The solutions to the Bohr model and the Schrödinger equation for ground-state energies reveal a fair match for all dimensions (Table I and Fig. 1). This is not too surprising since the singularity structures of Eqs. (21) and (37) are identical. This immediately reduces the stable bound states to dimensions between 0.0 and 4.0. Coupled with the same behavior at $d = 2$, and the same asymptotic behavior of the potential, a stronger parallelism is produced. The ground-state energies, however, only agree exactly at $d = 1.61$, 3.0, and 4.0. For experiments done at today's precision, the Bohr model would easily be discarded in all other dimensions. Moreover, comparisons across the $d = 2$ line cannot be taken too seriously. The $d = 2$ line separates the solutions into three distinct and disjoint classes; those

with $d < 2$, those with $d = 2$, and those with $d > 2$. The reason these solutions are distinct is due to the zero of the potential. For $d > 2$ it is possible to take the zero at infinity. This is not possible for the other dimensions. This means that the regions for $d < 2$ and $d = 2$ can be scaled vertically without changing the results. Within each region, however, where the same zero is taken, the energy curves are similar. Thus the even space dimensions of $d = 2$ and $d = 4$ represent critical values that separate the solutions into disjoint regions for both the Bohr and Schrödinger results. Furthermore, the Bohr and Schrödinger radii also display a rough qualitative agreement with the wave-function maximum being near the Bohr radius of $r = 1$ (Fig. 2). It would be interesting to do a more detailed study of these wave functions in all dimensions. Further work on states other than the ground state would also be of interest.

ACKNOWLEDGMENTS

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The two-dimensional hydrogen atom with a logarithmic potential energy function

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Recently, a "shooting" method has been used to obtain exact expressions for eigenvalues and eigensolutions of the two-dimensional hydrogen atom. This paper shows that the shooting method is easy for undergraduate students to understand and implement numerically. The highly accurate approximations for both eigenvalues and eigensolutions are then used to contrast the two-dimensional and three-dimensional hydrogen atoms. Finally, previous methods for solving the two-dimensional hydrogen atom are compared with the shooting method.

I. INTRODUCTION

Understanding the physical world is inextricably linked with a clear understanding of the quantum mechanical model that has proved so successful in describing atomic processes. Recently, applications of this model to physical systems in other spatial dimensions[1] have been made in the hope of both illuminating the special characteristics of the three-dimensional world and obtaining rare "exact" solutions to the quantum theory. One such attempt has been the study of the hydrogen atom in two-dimensional space. This is the atom formed by the attraction of two electrically charged particles, an electron of two-dimensional charge $-q$ and a proton of two-dimensional charge q. It is common knowledge[2-4] that the correct mutual electrostatic potential energy function, the one satisfying Gauss' theorem, for such a system in two-dimensional space is a logarithmic

function of the distance r separating the two particles. Neglecting relativistic and magnetic effects, this means that the time-independent Schrödinger equation in polar coordinates for the relative motion of this system is

$$[-\hbar^2 \nabla^2/(2\mu) + q^2 \ln(r/r_0)]\psi(r,\phi) = E\psi(r,\phi), \quad (1)$$

where $\mu = m_p m_e/(m_p + m_e)$ is the reduced two-dimensional mass of the proton–electron system, $\hbar$ is the two-dimensional Planck constant divided by 2π, and r_0 is a scale constant. The goal is to find the energy eigenvalues E and the corresponding (nontrivial) eigensolutions ψ of Eq. (1) which are bounded and square integrable in two-dimensional space:

$$\int_0^{2\pi} \int_0^\infty |\psi(r,\phi)|^2 r \, dr \, d\phi < \infty. \quad (2)$$