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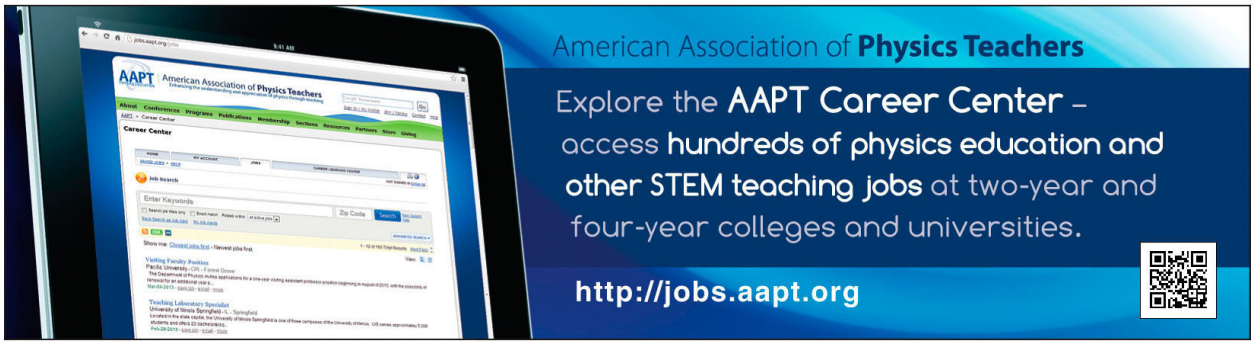

These values of the parameters correspond closely with the measured values for the reflector that produced the caustic of Fig. 2. In addition, these values satisfy Eqs. (28)–(29) and, although the curved inner region focuses at about $z_w = 2.5$, its small angular extent ensures that by $z_w = 4.625$, the inner region rays will not have substantially diverged. Again, for each of the 6400 incident rays, the interception point (x_w, y_w) of the reflected ray on the viewing screen was numerically calculated and plotted in Fig. 6. This spot diagram pictorially indicates the clustering of the lights rays on the viewing screen produced by the reflection from the metallic surface. Again, it is qualitatively similar to the caustic of Fig. 2.

The point of view that we have taken is that the machinery of geometrical optics provides a useful method for understanding the physical mechanisms that produce nearfield caustics. We find that caustics that look qualitatively similar but are produced by reflectors of very different geometries owe their existence to very different properties of the reflector surfaces. It is also pleasing to see that these physical mechanisms, astigmatic focusing from a circular surface and the focusing from the intersecting troughlike structures of a square surface, can be numerically tested in a simple way by the calculation of spot diagrams.

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The quantum mechanical few-body problem

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A solution to quantum mechanical problems of two, three, and four bodies is discussed from the point of view of two-body harmonic oscillator forces. An exact solution to the three- and four-body problems is found for harmonic oscillator forces, assuming equal masses. The aim of the discussion is to make students aware of the real difficulties of the few-body problem.

I have often explained the classical and quantum solutions to the two-body problem to undergraduate students. However, I find that they often do not appreciate the assumptions one needs to make in order to obtain a solution, nor do they have much inkling of how to go beyond two bodies, so I decided in my last quantum mechanics course to introduce students to the three-body problem. To set the stage, I set an exercise asking the students to solve the three-body problem in analogy to the two-body solution. I later told the students that a general solution cannot be found, but that I can solve it by making some assumptions. We then went through the derivations presented below, and it became very clear to the students what one must assume to obtain a solution. The net result of this exercise was that the students now *really* understood the *two*-body problem and also had some experience with the more general few-body problem. I think, therefore, that it is considerably worthwhile to introduce the material, to be discussed below, in introductory quantum mechanics courses. It is also very easily adapted to the few-body Lagrangian problem in classical mechanics.

For orientation and completeness I shall also present an outline of the standard[1,2] two-body solution.

I. TWO-BODY PROBLEM

The time-independent two-body Schrödinger equation is

$$[-(\hbar^2/2m_1)\nabla_1^2 - (\hbar^2/2m_2)\nabla_2^2 + V(\mathbf{r}_1,\mathbf{r}_2)]\Psi(\mathbf{r}_1,\mathbf{r}_2) = E_{12}\Psi(\mathbf{r}_1,\mathbf{r}_2), \quad (1)$$

where E_{12} is the total energy of the two bodies. Introducing the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the center of mass (c.m.) coordinate $\mathbf{R} = (m_1/M)\mathbf{r}_1 + (m_2/M)\mathbf{r}_2$ (with $M = m_1 + m_2$), Eq. (1) reduces to

$$[-(\hbar^2/2M)\nabla_R^2 - (\hbar^2/2\mu)\nabla_r^2 + V(r)]\Psi(\mathbf{r},\mathbf{R}) = E_{12}\Psi(\mathbf{r},\mathbf{R}), \quad (2)$$

assuming that the potential $V(\mathbf{r}_1,\mathbf{r}_2)$ depends only on the distance between the two particles $V(|\mathbf{r}_1-\mathbf{r}_2|)=V(r);\mu$. Equation (2) is solved by separating the variables in Ψ to $w(\mathbf{r})W(\mathbf{R})$ to obtain

$$-(\hbar^2/2M)\nabla_R^2 W(\mathbf{R})=E_R W(\mathbf{R}) \tag{3}$$

and

$$[-(\hbar^2/2\mu)\nabla_r^2+V(r)]w(\mathbf{r})=E_r w(\mathbf{r}). \tag{4}$$

Thus $W(\mathbf{R})$ is just the solution for a free particle and $w(\mathbf{r})$ is the standard harmonic oscillator solution for an harmonic potential $V(r)$.[1-2]

The reasons that we are able to solve the two-body problem are that (1) the momentum operator separates into ∇_r^2 and ∇_R^2 (with no cross term) and (2) $V(r)$ is assumed to depend on r only.

II. THREE-BODY PROBLEM

The approach presented below is widely used for the three-quark baryon problem.[3] An alternate approach is discussed by Vladimiroff.[4]

The time-dependent three-body Schrödinger equation is

$$\left(\frac{-\hbar^2}{2m_1}\nabla_1^2-\frac{\hbar^2}{2m_2}\nabla_2^2-\frac{\hbar^2}{2m_3}\nabla_3^2+V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\right)\times\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=E_{123}\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3). \tag{5}$$

We shall hereby assume that the masses of all the particles are equal, such as the nucleons in a nucleus or the quarks in a nonflavored hadron. We define the overall c.m. coordinate as

$$R=(1/\sqrt{3})(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3) \tag{6a}$$

and the relative coordinate between particles 1 and 2 as

$$\boldsymbol{\rho}=(1/\sqrt{2})(\mathbf{r}_1-\mathbf{r}_2) \tag{6b}$$

and the relative coordinate between particle 3 and the c.m. of particles 1 and 2 as

$$\boldsymbol{\lambda}=(1/\sqrt{6})(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3) \tag{6c}$$

so that

$$\nabla_1^2+\nabla_2^2+\nabla_3^2=\nabla_R^2+\nabla_\lambda^2+\nabla_\rho^2, \tag{7}$$

which is a necessary prerequisite for obtaining a solution to (5). Note that when the particle masses are not assumed equal, one obtains cross terms (such as $\nabla_R\cdot\nabla_\lambda$) in (7) thus preventing a solution via separation of variables. If we now assume a harmonic oscillator potential acting between each pair of particles

$$V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\tfrac{1}{2}k(r_{12}^2+r_{13}^2+r_{23}^2), \tag{8}$$

with

$$r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|, \tag{9}$$

then

$$V=\tfrac{3}{2}k(\rho^2+\lambda^2), \tag{10}$$

which, when combined with (7), allows for an exact solution in each coordinate. That is, upon separating variables

$$\Psi(\mathbf{R},\boldsymbol{\rho},\boldsymbol{\lambda})=W(\mathbf{R})w(\boldsymbol{\rho})u(\boldsymbol{\lambda}) \tag{11}$$

gives the uncoupled equations

$$-(\hbar^2/2m)\nabla_R^2 W(\mathbf{R})=E_R W(\mathbf{R}), \tag{12a}$$

$$[-(\hbar^2/2m)\nabla_\rho^2+\tfrac{3}{2}k\rho^2]w(\boldsymbol{\rho})=E_\rho w(\boldsymbol{\rho}), \tag{12b}$$

$$[-(\hbar^2/2m)\nabla_\lambda^2+\tfrac{3}{2}k\lambda^2]u(\boldsymbol{\lambda})=E_\lambda u(\boldsymbol{\lambda}). \tag{12c}$$

The harmonic oscillator potential is the only potential where this remarkable separation occurs. Our assumptions have been (1) equal mass particles; (2) two-body potentials; and (3) harmonic oscillator forces. An interparticle harmonic oscillator force is used in the confining of quarks in baryons[3] and can also be used for a qualitative understanding of nuclei.[5]

When teaching this material to students, I asked them to show that the momentum operator does not separate into a solvable form when the particles are of unequal mass. I also asked them to consider a Coulomb and cubic potential and to show where the difficulty in solution lies. These exercises greatly enhance the students' appreciation of the solution presented above.

III. FOUR-BODY PROBLEM

Remarkably, we can extend our analysis even further. Define the c.m. vector

$$\mathbf{R}=(1/\sqrt{4})(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4) \tag{13a}$$

and the relative vector between the c.m. of particles 1 and 2 and of particles 3 and 4 as

$$\boldsymbol{\lambda}=(1/\sqrt{4})(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4) \tag{13b}$$

and the relative vectors

$$\boldsymbol{\rho}=(1/\sqrt{2})(\mathbf{r}_1-\mathbf{r}_2) \tag{13c}$$

and

$$\boldsymbol{\eta}=(1/\sqrt{2})(\mathbf{r}_3-\mathbf{r}_4). \tag{13d}$$

Equations (13) imply that

$$\nabla_1^2+\nabla_2^2+\nabla_3^2+\nabla_4^2=\nabla_R^2+\nabla_\lambda^2+\nabla_\rho^2+\nabla_\eta^2 \tag{14}$$

and

$$V=\frac{1}{2}k\sum_{\substack{i,j=1\\ i\neq j}}^{4}r_{ij}^2=\frac{4}{2}k(\lambda^2+\rho^2+\eta^2), \tag{15}$$

which again we can solve! We have used the same assumptions as in the three-body problem.

IV. COMPLETE SOLUTION INCLUDING EXTERNAL FIELD

The present article is intended to be primarily pedagogical rather than a complete discussion of the few-body problem. However, the reader interested in persuing this topic further is encouraged to look at the articles by Bergmann and Holz.[6,7] In the present work, I have assumed that the masses of the particles were equal in order to avoid cross terms in the derivative operators. The absence of these cross terms ensures a solution via separation of variables. As pointed out by a referee, the assumption of equal masses is not really necessary. One can still obtain an exact solution even in a uniform magnetic field. This is fully discussed by Bergmann and Holz.[6,7]

V. DISCUSSION

I have not extended the above approach to more than four bodies; however, keen students should be encouraged to try. The trick is always to pick appropriate coordinates. However, the assumption of two-body forces starts break-

ing down when we consider the more general many-body problem.

Furthermore, one should realize that, in the real world, nonseparability is the rule. Usually the forces are not of the harmonic oscillator types, nonseparability follows, and one must seek approximate solutions to the few-body problem.

In conclusion, the student now has a very clear understanding of the *two*-body problem (obtained via contact with the few-body solution), and also has some appreciation of the more general problem of a few bodies. I recommend physics instructors to include the above material in their quantum mechanics courses, and also to present similar techniques in classical mechanics courses.

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An extension of the Eisberg–Resnick treatment for electron energies in many-electron atoms

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Eisberg and Resnick present a simple argument for the energy of an electron in a multielectron atom using the concept of shielding from electrons in inner shells. The results of such a treatment are unfortunately confined so as to be out of range of experimental values. Here, the effect of electrons in outer shells is included, and, in the nonrelativistic region, energies are obtained for electrons in the first and second shells in reasonable agreement with experiment.

I. THE EISBERG–RESNICK MODEL

This article refers to the treatment of the Hartree theory for multielectron atoms given by Eisberg and Resnick[1] in their popular textbook on quantum physics. Following an account of the method itself, the authors present a highly elegant discussion of how the general features of the results obtained numerically may be reproduced using an extremely simple model. Essentially, the Bohr idea of orbits (or orbitals) is used. The charge corresponding to all the electrons in the nth shell is assumed to be spread evenly over the surface of a spherical shell. The electrons are taken to be moving in a Coulomb potential corresponding to an effective atomic number Z_n for that particular shell. For shells other than the outermost one, the value of Z_n is obtained by subtracting from the actual atomic number Z the number of electrons in inner shells. For argon, with Z equal to 18, Z_1 is given as 16, and Z_2 is given as 8. As a special rule, it is suggested that, for the outermost shell, a suitable value of Z_n is just n. The value of the energy of an electron in the nth shell is then given by $-(E_H)\ (Z_n/n)^2$ where E_H is the ground-state energy of the hydrogen atom, 13.6 eV.

Such a procedure sums up a great deal of information on atomic physics. Clearly, it would be inappropriate to "improve" this model, since its essential feature is its simplicity. It may be suggested, though, that the simplest of models may, and should, be analyzed fully. From this point of view there do seem to be two ways in which the discussion of Ref. 1 may be strengthened.

A. Neglect of outer shielding

In the calculation of Ref. 1 for the electrostatic energy of an electron in the nth shell, the electrons in shells outside the nth are ignored. This does not appear to be correct. Inside a spherically symmetric shell of charge $-N_m|e|$ and radius r_m, the electrostatic field is zero, but the electrostatic potential is given by $-N_m|e|/(4\pi\epsilon_0 r_m)$. From the point of view of an electron in the nth shell ($n<m$), this is a constant addition to potential energy, the wavefunction of the electron will not be affected, but the energy will be increased. (Slater[2] is a good general reference for the area, and a discussion of this point is given on pp. 227–229. The terms inner and outer shielding are used for the effect of electrons in shells inside and outside the one under consideration.)

B. Electrons in the same shell

It does not seem possible to obtain the values of Z_n for argon given in Ref. 1 from the model of these authors. Let us consider, for example, an electron in the second shell. It is not clear in what way to consider the other electrons in this shell; this will be a source of uncertainty throughout this article. One might imagine any number between 0 and the full 7 could be regarded as shielding the electron under consideration (but not 8; an electron cannot shield itself). This would lead to a value of Z_2 between $Z-2$ and $Z-9$, or 16 and 9, while Ref. 1 gives the value 8. Similarly, Z_1 would be between Z and $Z-1$, 18 and 17, while Ref. 1 gives 16.