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Reply to "Comment on 'In what frame is a current-carrying conductor neutral?'"

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Paradoxes in special relativity are often formulated in such a way that an observer at rest in one (S) frame finds that "A will happen" and an observer at rest in another frame (S') finds that "A will not happen," where A is some effect. The resolution of such paradoxes can take the form of showing that both observers are correct, consistency being achieved by some nonintuitive aspect of the transformation between frames. In general it does not help in understanding the problem to argue that because A happens in the S frame, A must also happen in the S' frame, particularly if that occurrence is not reasonable from the point of view of the S' observer.

In this example the S frame is one in which the lattice of a current-carrying conductor is at rest. The effect is the deflection of the conducting electrons in the magnetic field created by the flowing charges, resulting in a transverse electric field and a negative volume charge density in the end. Of course we understand this charge separation occurs at the same time as the steady current is being set up in the conductor. Therefore, although the process is pictured as a consisting of two steps, actually there is no time at which the current is steady and the electrons are undeflected or partially deflected.

In the S' frame, in which the conducting electrons are at rest and the lattice is moving, there is no magnetic force on the electrons (because the velocity is zero), and therefore no transverse electric field and no volume charge density in the steady state. Because the bulk of the conductor was neutral before the current started and is neutral in the steady state, it would seem reasonable that the S' observer would expect neutrality in between.

In the scenario described in the preceding comment, the S' observer would find a transverse electric field in the conductor as the steady current is set up, the field subsequently being canceled by movement of electrons in this field. No explanation is given as to why the S' observer would expect such a field to appear (only to be canceled out later), other than that is what is implied by the transformation of the two-step process from the S frame to the S' frame. However, because the two steps occur together and are not separated in time, a more reasonable view of what happens in the S' frame is that no transverse electric field appears in the conductor as the steady current is being established and no field implies no deflection.

One-dimensional Bohr atom

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The one-dimensional hydrogenic atom with the potential energy

$$V(x) = -Ze^2/|x| \qquad (1)$$

has been the subject of a number of papers in this Journal.[1-4] Solutions of the Schrödinger equation have been found with energy levels which, for $n > 0$, are numerically equal to those for the three-dimensional potential energy

$$V(r) = -Ze^2/r. \qquad (2)$$

However, there has been a controversy regarding the validity of some of the solutions that have been obtained. In particular, it has been asserted that in this one-dimensional problem the odd and even solutions are degenerate.

None of the authors who have investigated the one-dimensional hydrogenic atom have noted that the energy levels obtained by using the Bohr model for this system do not agree with those obtained for the three-dimensional atom. This may be seen as follows.

The energy of an atom with the potential energy given by Eq. (1) is

$$E = p^2/2m - Ze^2/|x|. \qquad (3)$$

The Bohr–Sommerfeld–Wilson quantization condition is

$$\begin{aligned} nh &= \oint p\,dx = 4\int_0^{x_m} p\,dx \\ &= 4\int_0^{x_m}\left(2mE + \frac{2mZe^2}{x}\right)^{1/2} dx \\ &= 2\pi(2mZe^2x_m)^{1/2} = 2\pi(-2mZ^2e^4/E)^{1/2}. \end{aligned} \qquad (4)$$

Thus

$$E = -4Z^2E_1/n^2, \qquad (5)$$

where $E_1 = me^4/2\hbar^2$.

The energy levels given in Eq. (5) are four times those of the corresponding model in three dimensions. This may be understood as follows. In one dimension the classical motion is oscillatory and the total energy equals the maximum potential energy, which is a negative quantity. When the potential energy is a maximum, the kinetic energy, which is a positive quantity, vanishes. Thus one obtains an additional factor of 2 in the energy compared with the three-dimensional model. The amplitude of the oscillation is $x_m = n^2 a_0/2Z$, where $a_0 = \hbar^2/me^2$. This distance is half the radius of the three-dimensional model, and provides another factor of 2 in the expression for the energy.

Also, one may note that the period of oscillation of the electron in the one-dimensional Bohr model of the hydrogenic atom is

$$T = 2\pi n^3(\hbar^3/me^4)4Z^2, \tag{6}$$

which is one fourth of the period of the three-dimensional atom. This is consistent with Eq. (5).

For a relativistic atom the linear momentum is obtained from the relation

$$p^2c^2 = (E + Ze^2/|x|)^2 - m^2c^4. \tag{7}$$

In this case the quantization condition is

$$\begin{aligned} nh &= \oint p\,dx = 4\int_0^{x_m} p\,dx \\ &= \frac{4}{c}\int_0^{x_m}\left[\left(E + \frac{Ze^2}{x}\right)^2 - m^2c^4\right]^{1/2} dx \\ &= (4Ze^2/c)I(\epsilon), \end{aligned} \tag{8}$$

where $\epsilon = Ze^2/mc^2x_m = (1 - E/mc^2)$ and

$$I(\epsilon) = \epsilon^{-1/2}\int_0^1 [\epsilon + 2(1-\epsilon)u - (2-\epsilon)u^2]^{1/2}\frac{du}{u}, \tag{9}$$

with $u = x/x_m$.

The integral $I(\epsilon)$ is logarithmically divergent at $u = 0$ and the areas enclosed by the phase space trajectories for the relativistic problem are infinite. In order to calculate the lowest-order relativistic correction to Eq. (5) one may cut off the contribution of the very high momentum by replacing the lower limit of integration in Eq. (9) by the small number δ. This is equivalent to replacing Eq. (1) by the truncated potential energy

$$V(x) = -Ze^2/(|x| + \delta x_m). \tag{10}$$

Then one obtains the implicit expression

$$\begin{aligned} \frac{n\pi}{2Z\alpha} &= \left(\frac{1-\epsilon}{(2\epsilon - \epsilon^2)^{1/2}}\right)\left(\frac{\pi}{2} + \sin^{-1}(1-\epsilon)\right) \\ &\quad + \ln(\epsilon) + \eta, \end{aligned} \tag{11}$$

where $\alpha = e^2/\hbar c$ and $\eta = \ln(2\delta e)$ is an infinite constant which is independent of the energy.

To lowest order in ϵ one again obtains Eq. (5). If one chooses $\delta \sim \alpha^2$, the logarithmic terms may be neglected for the purpose of computing the radiation spectra. Then the energy including the first relativistic correction is

$$E = mc^2[1 - (2Z^2\alpha^2/n^2)(1 - 4Z\alpha/n\pi)]. \tag{12}$$

Nieto[5] and Spector and Lee[6] have obtained solutions of the relativistic Schrödinger (Klein–Gordon) equation for the one-dimensional hydrogenic atom with energy levels given by

$$E = mc^2[1 + Z^2\alpha^2/(n+s)^2]^{1/2}, \tag{13}$$

where n is a non-negative integer and $s = [1 \pm (1 - 4Z^2\alpha^2)^{1/2}]/2$. Expanding Eq. (13) to include the first relativistic correction one obtains[6]

$$E = mc^2(Z\alpha)(1 - Z\alpha/2), \qquad (n = 0)$$

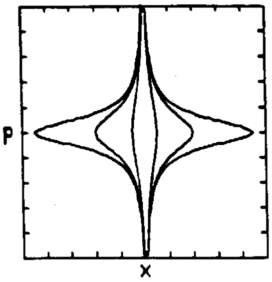

Fig. 1. Plots of phase space trajectories for the one-dimensional hydrogenic atom for $n = 1, 2, 3$. The phase space areas are proportional to n. The units are $P = pa_0/\hbar$ and $X = x/a_0$.

$$E = mc^2\left\{1 - \left(\frac{Z^2\alpha^2}{2n^2}\right)\left[1 - Z^2\alpha^2\left(\frac{2}{n^2} - \frac{3}{4n^3}\right)\right]\right\}, \quad (n > 0). \tag{14}$$

It is of interest to examine the phase space trajectories that are unusual in appearance compared to the more familiar one-dimensional problems. These are plotted in Fig. 1 for the first three energy levels. As n increases, the amplitude of the oscillations increases proportional to n^2 and the phase space area enclosed by the trajectories is proportional to n. The logarithmically infinite contribution to the phase space areas for the relativistic trajectories comes from the infinite momenta at $x = 0$. For finite momenta the trajectories are essentially identical for the relativistic and nonrelativistic cases. The figure was computed and plotted using a simple BASIC program on a Digital Equipment Corporation Professional 350 microcomputer.

The status of the solutions of the Schrödinger equation with even parity and odd quantum number has been a matter of controversy for more than 25 years. If one allows only even values of the quantum number, the Bohr model predictions of the energy levels of the one-dimensional nonrelativistic hydrogen atom are in agreement with those of the Schrödinger equation. Alternatively, if the singularity at the origin divides the real axis into two independent regions, then motion in the Bohr model is restricted to the regions $x < 0$ or $x > 0$. For these conditions the phase space is one half that considered above and the energy is reduced by a factor of 4 factor in agreement with the Schrödinger equation result.

The agreement between the energy levels obtained from the solutions of the nonrelativistic three-dimensional Schrödinger equation and the predictions of the Bohr model is somewhat fortuitous. In the Bohr model the principal quantum number is replaced by the angular momentum quantum number. But the solutions of the Schrödinger equation are degenerate with respect to the angular momentum quantum number. Furthermore, the solutions of the Schrödinger equation for the potential energy in Eq. (2) are not the same in two[7] and three dimensions, while for the classical orbits in the Bohr model there is no difference. (The difference between the two- and three-dimensional case vanishes in the limit of large quantum numbers.)

The solution of the problem presented here is straightforward. Because of the unusual phase space it may be of interest to present this material in an introductory quantum mechanics course as a supplement to the usual examples.

[1]R. Loudon, Am. J. Phys. 27, 649 (1959).
[2]M. Andrews, Am. J. Phys. 34, 1194 (1966); 44, 1064 (1976); 49, 1074 (1981).
[3]L. K. Haines and D. H. Roberts, Am. J. Phys. 37, 1145 (1969).
[4]J. F. Gomes and A. H. Zimerman, Am. J. Phys. 48, 579 (1980); 49, 1074 (1981).
[5]M. M. Nieto, Am. J. Phys. 47, 1067 (1979).
[6]H. N. Spector and J. Lee, Am. J. Phys. 53, 248 (1985). [Note that their definition of s is missing the square root.)
[7]B. Zaslow and M. E. Zandler, Am. J. Phys. 35, 1118 (1967), W. J.-K. Huang and A. Kazycki, Am. J. Phys. 47, 1005 (1979).

Erratum: "Problem: The effective weight of a sliding block on a recoiling inclined plane (wedge)" [Am. J. Phys. 55, 777 (1987)]

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The following correction should be made in the solution to the problem cited above: On p. 847, the graphs are labeled in reverse order; the one closest to the vertical axis should be $M = 0.01m$ and the one farthest away should be $M = 5m$.

Erratum: "Order-of-magnitude 'theory' of stellar structure" [Am. J. Phys. 55, 804 (1987)]

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In a personal communication, Ralph Baierlein has kindly pointed out an error in the derivation of the nuclear reaction rate [Eq. (19)]. This derivation improperly combined $\mathscr{P}_{\text{tunnel}}$, the quantum-mechanical tunneling probability *at* a given energy E, with $\mathscr{P}(>E)$, the statistical probability of a particle possessing any energy *greater than* E.

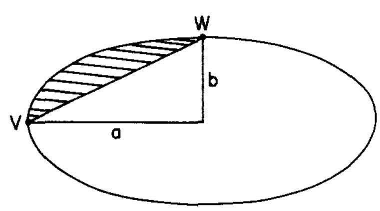
The correct method is to multiply $\mathscr{P}_{\text{tunnel}}$ by $\mathscr{P}$ [Eq. (11)], the statistical probability of a particle possessing energy E. One then determines the height and width of the resulting Gamow peak in the usual manner, and finds the nuclear reaction rate to be a factor $(E_{\text{peak}}/kT)^{1/2}$ greater than that given in Eq. (19). This alters the final results as follows: The stellar radius [Eq. (22)] and mass [Eq. (24)] are multiplied by $(E_{\text{peak}}/kT)^{1/8}$, while the luminosity [Eq. (23)] is multiplied by $(E_{\text{peak}}/kT)^{3/8}$. Because E_{peak}/kT is not large (it is of order 8 in the Sun), the correction is not numerically significant.

SOLUTION TO THE PROBLEM ON PAGE 39

Let τ denote the time to be calculated. Let R, T_0 denote the radius and period of the original orbit. ($T_0 = 1$ yr.) Take the Earth's initial velocity as having an infinitesimal component perpendicular to the initial position vector. The Earth will thus go into an elliptical orbit with the Sun at one focus. We will eventually take the limit as this perpendicular component goes to zero and the ellipse becomes a straight-line segment, a degenerate ellipse with eccentricity, $e = 1$.

In the elastic collision with the mirror, the Earth's total energy does not change. Hence the semimajor axis and orbital period do not change: $a = R$ and $T = T_0$. In order to exploit Kepler's second law, the constancy of areal velocity, we show in Fig. 1 a nondegenerate ellipse. In the limit as $b \to 0$, the Sun moves to the left-hand vertex (V) and the point W moves to the center of the ellipse. Since the distance of the Earth from the Sun when the collision occurs is $R = a$, we see that W is appropriately taken as the position of the Earth immediately after the collision. To reach the Sun, the Earth must traverse $\frac{1}{4}$ of the perimeter of the ellipse, but this is the fast part of the orbit and much less than $\frac{1}{4}$ of a year is required. In the time of interest, τ, the Earth sweeps out the shaded area shown, whose value is $(\pi ab/4 - ab/2)$. The time required to sweep out the total area of the ellipse, πab, is just T_0. Fortunately, the ratio of these two areas is independent of the eccentricity of the ellipse. By Kepler's second law,

$$\tau = (1/4 - 1/2\pi) T_0 = 0.091 \text{ yr}.$$



We note that this time is satisfyingly less than the time required to reach the Sun if the Earth is simply released from rest at a distance R, this latter time being $\tau' = 2^{-5/2} T_0 = 0.177$ yr. [See S. Van Wyk, Am. J. Phys. 54, 913 (1986).]

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