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An aging harmonic oscillator

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The adiabatic theorem has been investigated by studying an analytic example of an aging harmonic oscillator. Closed-form solutions to the corresponding time-dependent Schrödinger equation are available and the validity of the theorem can be explicitly tested. The results are consistent with the conventional proof by means of the series expansion method in the perturbation theory.

I. INTRODUCTION

The adiabatic theorem[1] is one of the backbones of the time-dependent perturbation theory in quantum mechanics. The essence of the theorem is as follows: Consider a quantum system whose Hamiltonian contains a parameter f, which varies slowly with time. The requirement of "slow" variation means that the time variation of $f(t)$ should not cause a substantial variation of the Hamiltonian in a time of the order of the natural periods of the system with constant f. Consequently, at any instant of time, the Hamiltonian may be treated as constant and an approximate solution can be obtained by regarding the Schrödinger equation as time independent,

$$H(f)u_n(f,x) = E_n(f)u_n(f,x), \quad (1)$$

with $u_n(f,x)$ and $E_n(f)$ being the eigenfunctions and eigenvalues of the Hamiltonian $H(f)$. Here, x stands for all the independent variables of the problem. In other words, if Eq. (1) can be solved at each instant of time, we expect that a system that is in a nondegenerate state $u_n(f(0),x)$ with energy $E_n(f(0))$ at $t=0$ is likely to be in the state $u_n(f(t),x)$ with energy $E_n(f(t))$ at time t, provided that $H(f(t))$ changes very slowly with time. Conventionally, the adiabatic theorem is proved by means of the series expansion method in perturbation theory. It is therefore of interest to give examples in which closed-form solutions are available and the validity of the theorem can be explicitly tested.

The very simplest example that comes naturally to mind is a harmonic oscillator with a slowly aging spring constant. In this case, we may model the Hamiltonian by

$$H(t) = \begin{cases} (p^2+x^2)/2, & t<0, \\ [p^2+x^2\exp(-\epsilon t)]/2, & t>0, \end{cases} \quad (2)$$

where $f(t)=\exp(-\epsilon t)$ with $\epsilon \ll 1$. For convenience, we have set the units such that $\hbar=1$, $m=1$, and $\omega=1$. In this article, analytic solutions to the time-dependent Schrödinger equations are shown. Implications of the results are discussed as well.

II. ANALYSIS

The model Hamiltonian of an aging harmonic oscillator is

$$H(t) = \begin{cases} (p^2+x^2)/2, & t<0, \\ [p^2+x^2\exp(-\epsilon t)]/2, & t>0\,. \end{cases}$$

For $t<0$, we suppose that the system is in one of the stationary states of a harmonic oscillator. Our task is then to find the solutions to the time-dependent Schrödinger equation for $t>0$, which satisfy the continuity boundary condition of the wavefunction at $t=0$, i.e.,

$$\psi(x,t=0^-) = \psi(x,t=0^+). \quad (3)$$

For simplicity, we consider only two cases; namely,

$$\text{(i)} \quad \psi(x,t<0) = \varphi_0(x)\exp(-iE_0t),$$

and

$$\text{(ii)} \quad \psi(x,t<0) = \varphi_1(x)\exp(-iE_1t), \quad (4)$$

where $\varphi_0(x) = (1/\pi)^{1/4}\exp(-x^2/2)$ is the harmonic oscillator ground state wavefunction with energy $E_0=\frac{1}{2}$, and $\varphi_1(x) = (4/\pi)^{1/4}x\exp(-x^2/2)$ is the first excited state wavefunction with energy $E_1=\frac{3}{2}$.

The time-dependent Schrödinger equation for $t>0$ is given by

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\exp(-\epsilon t)\right)\psi(x,t) = i\frac{\partial}{\partial t}\psi(x,t). \quad (5)$$

Let us consider the first case now. With the ansatz

$$\psi_0(x,t) = \exp[a(t)x^2 + c(t)], \quad (6)$$

where $a(t)$ and $c(t)$ are some arbitrary functions of t, we obtain, by equating the coefficients, the following equations:

$$2\dot a(t) = i4a^2 - i\exp(-\epsilon t), \quad (7a)$$

$$\dot c(t) = ia. \quad (7b)$$

By Ricatti transformation,

$$2a(t) = i\dot F(t)/F(t), \quad (8a)$$

Eq. (7a) yields

$$\ddot F(t) + \exp(-\epsilon t)F(t) = 0. \quad (8b)$$

Introducing the variable $s(t)$,

$$s(t) = (2/\epsilon)\exp(-\epsilon t/2), \quad (9a)$$

we can reduce Eq. (8b) to a zeroth-order Bessel's differential equation

$$\frac{d^2F}{ds^2} + \frac{1}{s}\frac{dF}{ds} + F = 0, \quad (9b)$$

whose solution is, of course, given by

$$F(s) = C_1J_0(s) + C_2Y_0(s), \quad (9c)$$

for some arbitrary constants C_1 and C_2. Here, $J_0(s)$ and $Y_0(s)$ are the zeroth-order Bessel functions of the first and second kind, respectively. Thus $a(t)$ can be expressed as

$$a(t) = (i\epsilon s/4)\{[C_1J_1(s) + C_2Y_1(s)]/[C_1J_0(s) + C_2Y_0(s)]\}. \quad (10)$$

With Eqs. (7b), (8a), and (9c), we can obtain

$$c(t) = \tfrac{1}{2}\log[C_3/F(s)], \tag{11}$$

where C_3 is an arbitrary constant. In order to satisfy the boundary condition, we must require

$$\text{(i)}\ A \equiv C_1/C_2 = -[Y_0(s(0)) + iY_1(s(0))]/[J_0(s(0)) + iJ_1(s(0))]$$

and (12)

$$\text{(ii)}\quad C_3 = (1/\pi)^{1/2}F(s(0)).$$

Hence, with all these results, we are able to determine the time-dependent wavefunction for $t>0$,

$$\psi_0(x,t>0) = (1/\pi)^{1/4}\{[AJ_0((0)) + Y_0(s(0))]/[AJ_0(s) + Y_0(s)]\}^{1/2} \times \exp((i\epsilon s/4)\{[AJ_1(s) + Y_1(s)]/[AJ_0(s) + Y_0(s)]\}x^2). \tag{13}$$

Now, we turn to the second case. Here, we will try the ansatz

$$\psi_1(x,t) = x\exp[b(t)x^2 + d(t)], \tag{14}$$

with $b(t)$ and $d(t)$ being some arbitrary functions of time. Following the same procedure as in the first case, we will then obtain the time-dependent wavefunction for $t>0$,

$$\psi_1(x,t>0) = (4/\pi)^{1/4}\{[AJ_0(s(0)) + Y_0(s(0))]/[AJ_0(s) + Y_0(s)]\}^{3/2} \times \exp[(i\epsilon s/4)\{[AJ_1(s) + Y_1(s)]/(AJ_0(s) + Y_0(s)]\}x^2), \tag{15}$$

where A is given by Eq. (12).

Up to now all the calculations shown above are exact. In Sec. III we will discuss the time evolution of the wavefunctions under the adiabatic approximation; i.e., $\epsilon \ll 1$.

III. DISCUSSION

In this section we examine the time evolution of the wavefunctions under the assumption that the process is adiabatic, i.e., $\epsilon \ll 1$. With the large argument asymptotic expansion of the Bessel functions, it can be easily shown that

$$A \simeq i. \tag{16}$$

In the small t limit, namely, $s(t) \gg 1$, the wavefunctions are given by

$$\psi_0(x,t>0) \sim (1/\pi)^{1/4}\exp(-\epsilon t/8) \times \exp[-(x^2/2)\exp(-\epsilon t/2)] \times \exp\{-i[1-\exp(-\epsilon t/2)]/\epsilon\}$$

and

$$\psi_1(x,t>0) \sim (4/\pi)^{1/4}\exp(-3\epsilon t/8)x \times \exp[-(x^2/2)\exp(-\epsilon t/2)] \times \exp\{-i3[1-\exp(-\epsilon t/2)]/\epsilon\}. \tag{17}$$

It is apparent that these wavefunctions are exactly the results of applying the adiabatic theorem; that is, solving Eq. (1). Beyond the small t limit, the spatial dependence of the wavefunctions is basically the same, but extra phase factors appear. This can be easily recognized if we rewrite the wavefunctions as follows:

$$\psi_0(x,t>0) = (\epsilon/\pi^2)^{1/4}[J_0^2(s) + Y_0^2(s)]^{1/4} \times \exp\{-i[(1/\epsilon) - (\pi/8) - (\theta/2)] \times \exp(-(\epsilon s/4)\{[J_1^2(s) + Y_1^2(s)]/[J_0^2(s) + Y_0^2(s)]\}^{1/2} \times \exp[i(2\phi - \pi)/2]x^2)$$

and

$$\psi_1(x,t>0) = (2^{1/2}/\pi)\epsilon^{3/4}[J_0^2(s) + Y_0^2(s)]^{3/4} \times \exp\{-i[(3/\epsilon) - (3\pi/8) - (3\theta/2)]x \times \exp(-(\epsilon s/4)\{[J_1^2(s) + Y_1^2(s)]/[J_0^2(s) + Y_0^2(s)]\}^{1/2} \times \exp[i(2\phi - \pi)/2]x^2),$$

where

$$\tan\theta = Y_0(s)/J_0(s)$$

and

$$\tan\phi = [J_1(s)Y_0(s) - J_0(s)Y_1(s)]/[Y_0(s)Y_1(s) + J_0(s)J_1(s)]. \tag{18}$$

Of course, its time dependence is very complicated.

All these analyses explicitly show that the adiabatic theorem is valid only in the limit $s(t) \gg 1$, i.e., the small t limit, in this example. This criterion can be reexpressed in terms of the Hamiltonian $H(t)$ of the system as follows:

$$\left|\frac{\partial H(t)}{\partial t}\right| \ll |\omega(t)H(t)|, \tag{19}$$

where $\omega(t) = \exp(-\epsilon t/2)$. This means that the Hamiltonian undergoes a small fractional change in a typical period of the system. Of course, this is exactly the basic criterion of the validity of the adiabatic theorem stated in Sec. I. The physical rationale behind the criterion is that beyond the limit $s(t) \gg 1$, the energy levels become closer and closer together so that the system can undergo a lot of transitions between energy levels very easily; this implies that the aging process of the harmonic oscillator is, in fact, no longer adiabatic.

All in all, we have shown in the above an investigation of the adiabatic theorem by studying an analytic example of an aging harmonic oscillator. Closed-form solutions to the corresponding time-dependent Schrödinger equation were obtained and the validity of the theorem was explicitly tested. The results are consistent with the conventional proof by means of the series expansion method in the perturbation theory.

[1]M. Born and V. Fock, Z. Phys. **51**, 165 (1928). Discussions of the adiabatic theorem can be found in most standard textbooks on quantum mechanics, e.g., R. L. Liboff, *Introductory Quantum Mechanics* (Holden-Day, San Francisco, 1980), L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), and D. Park, *Introduction to the Quantum Theory* (McGraw-Hill, New York, 1974).