Scattering from an attractive delta-function potential

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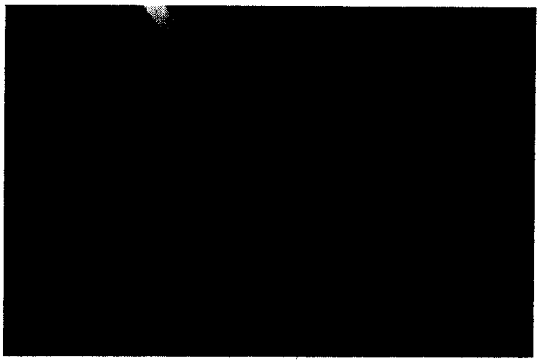

Fig. 3. Reflections of light inside the glass cross section.

the width of the refraction is narrower, resulting in a better measurement of α. However, the refraction intensity is lower, making the α measurement more difficult. Therefore, the measurement of α requires an alignment of the incident beam such that the refraction from D provides the optimal condition for determining α . The pattern of refracted light depends strongly on the illuminated surfaces of the bottle. These surfaces should be clean, free of scattering centers, and should have uniform curvature. The refracted light approaches the undisturbed incident beam as the thickness of the bottle decreases. These two light components cannot be resolved when the thickness reaches a critical value. Thus there is a low limit where the thickness of a bottle can be measured with this method. This limit is about 0.25 cm.

Another way to determine the thickness of the bottle is to measure the separation between the points where reflections occur inside the glass. These internal reflections are illustrated by the dotted line of Fig. 1. The length of the arc between any two adjacent points on the outer surface can be used to determine α. For example, the length of the arc between A and D divided by the outer radius is equal to α. After α is evaluated, Eq. (1) gives the thickness T. Measurements were made on the same glass bottle. The reflections of the laser beam are shown in Fig. 3. The arc length between two adjacent reflections at the outer surface was determined by wrapping a piece of graph paper around the bottle. This arc length was found to be 0.73 cm and α to be 14.9°. According to Eq. (1), T was calculated to be 0.34 cm, which is in good agreement with the measurement made by a caliper.

The optical demonstration may be more impressive when it is conducted with a glass bottle having a narrow neck. With a small neck bottle it appears to be impossible to insert any measuring device into the bottle to make the necessary measurement. However, the optical method provides the solution with ease.

The thickness measurement requires knowledge of the index of refraction of the glass. Most ordinary glass has an index of refraction of about 1.5. With the index of refraction taken as 1.5, the thickness measurement has been demonstrated to have sufficient accuracy to be convincing. However, there is a way to eliminate the requirement of knowing the index of refraction of glass beforehand. For this purpose, the incident laser beam can be rotated counterclockwise around point A (Fig. 1) until light going from A to B has an incident angle that equals the critical angle (θ_c). Let the angle of rotation be ϕ. Snell's law states that $\sin \phi = n \sin(\theta_c - \alpha/2)$. Then Eq. (1) becomes $T = R[1 - 1/\sin \phi]$. As a result, the thickness measurement no longer requires prior knowledge of the index of refraction of the glass. However, the demonstration becomes more complex.

ACKNOWLEDGMENTS

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[1]See, for example, E. Hecht and A. Zajac, *Optics* (Addison–Wesley, Reading, MA, 1974), p. 81.

[2]A. F. Leung, Phys. Teach. **22**, 94 (1984).

Scattering from an attractive delta-function potential

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A recent article[1] in this Journal derives the quantum-mechanical solution to the problem of a particle incident on a one-dimensional well formed by a repulsive delta-function barrier in front of an impenetrable wall. The stationary states obtained are used to determine the relation between scattering resonances and the time evolution (exponential decay) of states initially located entirely within the well. Massmann notes that his results are modified only slightly when the repulsive δ barrier is replaced by an attractive δ well, due to the fact that a delta-function potential of either sign acts as a barrier.

It is interesting to consider further the case of a quantum-mechanical particle in the latter potential that is depicted in Fig. 1. The time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + W_0\delta(x - x_0) - E \right)\psi(x) = 0 \qquad (1)$$

must be solved with the boundary condition $\psi(0) = 0$. Use of the continuity equation[2]

$$\psi_{\rm I}'(x_0) - \psi_{\rm II}'(x_0) = -(2\mu/\hbar^2)W_0\psi(x_0) \qquad (2)$$

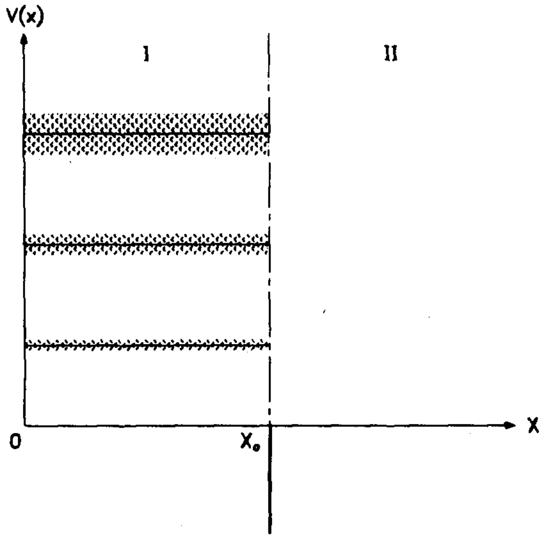

Fig. 1. Diagram of the potential $V(x) = -|W_0|\delta(x - x_0)$. The attractive delta-function potential at $x = x_0$ acts as a barrier that, together with the impenetrable wall at $x = 0$, creates a well that supports scattering resonant energy states.

to match the solutions for regions I and II at the boundary $x = x_0$ gives

$$\psi_k(x) = \left(\frac{2}{\pi}\right)^{1/2} e^{i\delta_k} \begin{cases} \sin(kx + \delta_k), & x \geqslant x_0, \\ A_k \sin(kx), & 0 \leqslant x \leqslant x_0, \end{cases} \quad (3)$$

where $k = (2\mu E/\hbar^2)^{1/2}$ and the phase shift δ_k and amplitude A_k are

$$\delta_k = -kx_0 + \arctan\left(\frac{\tan(kx_0)}{1 + (\alpha/kx_0)\tan(kx_0)}\right) \quad (4)$$

and

$$A_k = \{\sin^2(kx_0) + \cos^2(kx_0) \times [1 + (\alpha/kx_0)\tan(kx_0)]^2\}^{-1/2} \times \text{sign}[1 + (\alpha/kx_0)\tan(kx_0)]. \quad (5)$$

The dimensionless constant α is defined by

$$\alpha = 2\mu x_0 W_0/\hbar^2 \quad (6)$$

and takes on the sign of W_0:α is negative (positive) for an attractive (repulsive) delta-function potential. These expressions are just those derived by Massmann, except that A_k will now be negative when $(\alpha/kx_0)\tan(kx_0) < -1$.

The phase shift δ_k is plotted as a function of kx_0 in Fig. 2 for various values of α. The steplike behavior arises from the requirement that the phase function be continuous when the energy of the scattering particle passes through a resonance. The positions and widths of the scattering resonances due to a potential with a particular α value can then be estimated qualitatively by inspection, the peaks of the resonances corresponding to the points of greatest slope on the curve and their widths to the value of the slope at those points. It is evident from the figure that the resonances become sharper (narrower) and approach the bound state energies for a particle in an infinite square well of width $x_0(k_n x_0 = n\pi$ and $E_n = \hbar^2 k_n^2/2\mu)$ as $|\alpha| \to \infty$, although that limit is approached from above for the case $\alpha \to -\infty$ and from below for the case $\alpha \to +\infty$. Not surprisingly, the phase shift δ_k is negative for $\alpha > 0$, which reflects the simple fact that an incident particle sees a repulsive potential, while δ_k is positive for $\alpha < 0$, corresponding to scattering from an attractive potential.

The behavior of the phase shift in the limit of vanishing particle energy is very surprising, however. Equation (4) and Fig. 2 give

$$\lim_{k \to 0} \delta_k = \begin{cases} 0, & \alpha > -1, \\ \pi/2, & \alpha = -1, \\ \pi, & \alpha < -1, \end{cases} \quad (7)$$

which, according to Levinson's Theorem,[3] indicates the existence of a zero-energy resonance belonging to the potential with $\alpha = -1$, and the existence of a single bound state (with $E < 0$) belonging to the potentials with $\alpha < -1$.

The occurrence of this resonant state and bound state may be demonstrated explicitly by determining the poles of the scattering matrix. Poles lying on the imaginary axis of the complex k plane correspond to bound or virtual states, while those lying off the imaginary axis correspond to resonant states. It is then a straightforward matter to calculate the energies of the bound and resonant states for all values of α.

The S matrix, written as a function of the phase shift δ_k, is

$$S_k = e^{2i\delta_k} = (1 + i\tan\delta_k)/(1 - i\tan\delta_k), \quad (8)$$

which, by substituting the expression for δ_k from Eq. (4), can be seen to have poles k that satisfy

$$1 + (\alpha/kx_0)\tan(kx_0) + [1 + (i\alpha/kx_0)]\tan^2(kx_0) = 0. \quad (9)$$

Replacing k by iK and rearranging terms gives

$$\alpha = -Kx_0[1 + \tanh(Kx_0)]/\tanh(Kx_0). \quad (10)$$

Evidently, the pole in the complex k plane which gives rise to the bound state starts out at $-\infty$ on the negative imaginary k axis for $\alpha = 0$ and reaches the origin at $\alpha = -1$. For potential strengths commensurate with $-1 < \alpha \leqslant 0$, then, this pole corresponds to a virtual state. For $\alpha < -1$, the pole is on the positive imaginary axis and therefore corresponds to a bound state with energy $E = -\hbar^2 K^2/2\mu$. The pole at the origin for $\alpha = -1$ represents a zero-energy

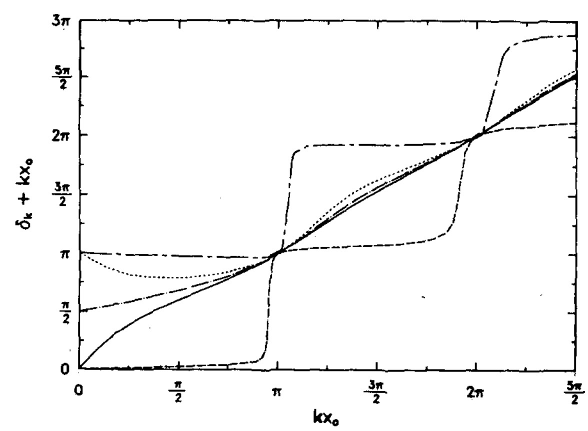

Fig. 2. Phase shift δ_k plotted as a function of kx_0 for $\alpha = -20$ (short dash–long dashed curve), -2 (dotted curve), -1 (dot-dashed curve), -0.5 (solid curve), and 20 (dashed curve). As the "strength" $|\alpha|$ of the delta-function potential increases, the resonance structures for the two cases $\alpha > 0$ and $\alpha < 0$ become essentially indistinguishable. The behavior of δ_k as $kx_0 \to 0$ signifies the existence of a bound state for potentials with $\alpha < -1$, a zero-energy resonance for potentials with $\alpha = -1$, a virtual state for potentials with $-1 < \alpha \leqslant 0$, and no bound or virtual state for potentials with $\alpha > 0$.

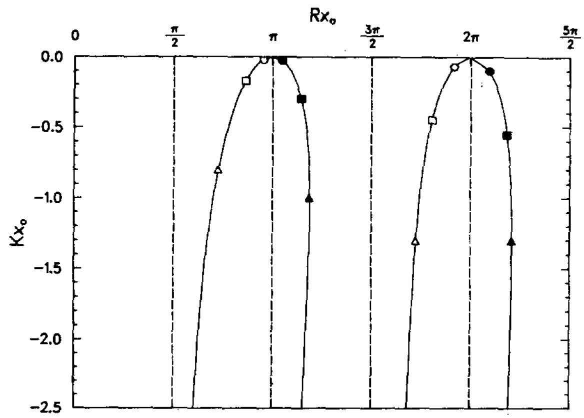

Fig. 3. Resonance poles of the scattering matrix, Eq. (8). The poles are constrained to lie on the curves depicted here and approach $(Rx_0,Kx_0) = (n\pi,0)$ as $|\alpha| \to \infty$ (as the "strength" of the delta-function potential increases). The curves for $\alpha < 0$ are bounded by $Rx_0 = n\pi$ and $(n + \frac{1}{2})\pi$, and asymptotically approach $Rx_0 = n\pi$ as $\alpha \to -0$; the curves for $\alpha > 0$ are bounded by $Rx_0 = (n - \frac{1}{2})\pi$ and $n\pi$, and asymptotically approach $Rx_0 = (n - \frac{1}{2})\pi$ as $\alpha \to +0$. Resonance poles for several values of α are plotted to illustrate these relationships: ▲ $\alpha = -1$, ■ $\alpha = -5$, ● $\alpha = -20$, △ $\alpha = 1$, □ $\alpha = 5$, ○ $\alpha = 20$.

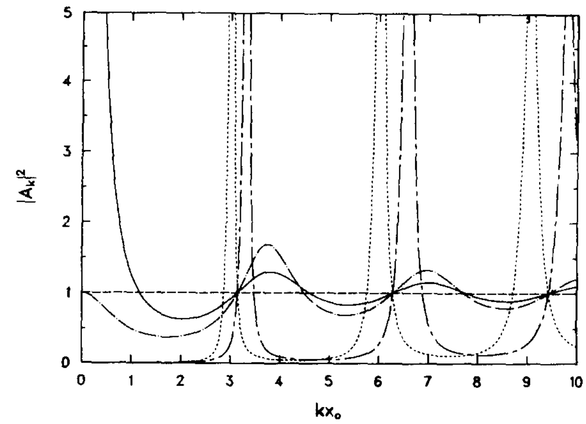

Fig. 4. The square modulus $|A_k|^2$ of the amplitude of the wavefunction in the well $0 \leqslant x \leqslant x_0$ plotted as a function of kx_0 for $\alpha = -20$ (short dash-long dashed curve), -2 (dot-dashed curve), -1 (solid curve), 0 (dashed curve), and 20 (dotted curve). The probability for finding a low-energy particle inside the well is greatly enhanced for potentials with α near -1, due to the existence of a nearby bound or virtual state. Potentials with $\alpha = -20$ and 20 give rise to standard resonance structures. As $|\alpha| \to \infty$, structures for $\alpha < 0$ and corresponding structures for $\alpha > 0$ increasingly overlap, the resonances in the former case approaching $kx_0 = n\pi$ from higher energies and the resonances in the latter case approaching $kx_0 = n\pi$ from lower energies.

resonance rather than a zero-energy bound state because the corresponding wavefunction is not normalizable while the zero-energy cross section diverges. The right-hand side of Eq. (10) can never be greater than zero, so no bound (or virtual) state is possible for $\alpha > 0$. (When the impenetrable wall at the origin is removed and a second attractive delta-function potential is placed at $x = -x_0$, two poles appear, one corresponding to an odd parity bound state and behaving identically to the pole discussed here as α is varied, and the other corresponding to an even parity bound state and starting at the origin for $\alpha = 0$ and moving to infinity as $\alpha \to -\infty$. There is thus one bound state for $-1 \leqslant \alpha < 0$ and two bound states for $\alpha < -1$.[4])

Resonant states are represented by the zeros of Eq. (9) when k is replaced by $R + iK$. The poles (R,K) must then satisfy the two equations

$$\alpha = -\frac{Rx_0[1 + \tan^2(Rx_0)][1 + \tanh(Kx_0)]}{\tan(Rx_0)[1 - \tanh(Kx_0)]} \quad (11a)$$

and

$$\alpha = -\frac{Kx_0[1 + \tan^2(Rx_0)][1 + \tanh(Kx_0)]}{\tanh(Kx_0) + \tan^2(Rx_0)}. \quad (11b)$$

Equating the right-hand sides gives the set of curves

$$Kx_0[1 - \tanh(Kx_0)] - [Rx_0/\tan(Rx_0)]\tanh(Kx_0) - Rx_0\tan(Rx_0) = 0, \quad (12)$$

upon which all the resonance poles must lie. Those curves closest to the imaginary axis are plotted in Fig. 3 together with several resonance poles calculated from Eqs. (11a) and (11b). The poles approach $(Rx_0,Kx_0) = (n\pi,0)$ from higher and lower energies as $\alpha \to -\infty$ and $+\infty$, respectively, in accord with the behavior of the phase shifts in Fig. 2, and asymptotically approach $(Rx_0,Kx_0) = (n\pi, -\infty)$ and $[(n - \frac{1}{2})\pi, -\infty]$ as $\alpha \to -0$ and $+0$. Each pole represents a resonance centered at energy $E = (\hbar^2/2\mu)(R^2 - K^2)$ with width $\Gamma/2 = \hbar^2 R |K|/\mu$.

It is clear from Figs. 2 and 3 that the main effect on scattering states of the replacement of the repulsive delta-function barrier by an attractive delta-function potential is simply to shift the positions of the resonances. For a low-energy particle incident on a δ well with strength α near -1, however, the presence of the nearby bound or virtual state will lead to a greatly enhanced probability for finding the wavefunction within the well. This is demonstrated in Fig. 4, where the square of the amplitude A_k of the wavefunction in the well is plotted as a function of kx_0 for $\alpha = -20, -2, -1, 0$, and 20. The quantity A_k^2 can be regarded as the ratio of the probability for finding the particle with wavefunction ψ_k in the well to the probability for finding the particle in the interval $0 \leqslant x \leqslant x_0$ in the absence of the delta-function potential ($\alpha = 0$).

ACKNOWLEDGMENT

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[1]H. Massmann, Am. J. Phys. **53**, 679 (1985).

[2]In this formulation, the delta-function potential essentially represents a boundary, which provides a simplification in scattering problems that has been exploited for pedagogical purposes by various textbook authors, for example D. ter Haar, *Selected Problems in Quantum Mechanics* (Academic, New York, 1964); and S. Flügge, *Practical Quantum Mechanics* (Springer, New York, 1974).

[3]See, for example, F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic, New York, 1967), Chap. 22.

[4]C. L. Hammer, T. A. Weber, and V. S. Zidell, Am. J. Phys. **45**, 933 (1977).