AMERICAN JOURNAL of PHYSICS

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W. Elberfeld and M. Kleber

Citation: American Journal of Physics **56**, 154 (1988); doi: 10.1119/1.15695
View online: http://dx.doi.org/10.1119/1.15695
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/56/2?ver=pdfcov
Published by the American Association of Physics Teachers

Time-dependent tunneling through thin barriers: A simple analytical solution

W. Elberfeld
Physik-Department, Technische Universität München, 8046 Garching, West Germany
M. Kleber[a)]
Center for Advanced Studies, University of New Mexico, Albuquerque, New Mexico 87131

(Received 1 December 1986; accepted for publication 12 April 1987)

Analytical solutions for the time-dependent tunneling of wave trains and wave packets through delta-function barriers are presented. Tunneling currents and density distributions are calculated with no approximations. A standby mechanism is demonstrated: The particle "waits" some time in front of the barrier before it tunnels through the barrier.

I. INTRODUCTION

One of the most difficult tasks in expounding on quantum processes is the phenomenon of potential-barrier penetration. This article presents exact analytical solutions of the tunneling problem that are new and can be used in a course on quantum mechanics. The level of presentation is fairly elementary: Apart from a basic knowledge of Laplace transforms (for deriving the results), only some experience with error functions (for evaluating the results) is needed.

Rigorous analytical solutions of time-dependent problems can lead to a better understanding of the complicated quantum dynamics. In the context of our problem we should mention two examples of a quantal time evolution: tunneling in a double-well potential[1] and the delta-shell model for alpha decay.[2] In this article we add a third example: tunneling of wave packets and semi-infinite plane waves through delta-function barriers. If one is not interested in the long-time behavior, then the motion of any particular wave packet can be determined by integrating the Schrödinger equation numerically. In many articles[3,4] and books[5–7] computer calculations for the motion of wave packets are described in detail. (It is impossible to give reference to all publications relevant for the tunneling problem.) We should, however, keep in mind that analytical and numerical methods do not compete with each other, rather they complement each other.

The dynamics of the tunneling process depends on the initial preparation of the particle's wavefunction. Another parameter is the shape of the barrier. For thick barriers one has to worry[8,9] about the time a particle spends in the classically forbidden tunneling region. For thin barriers the traversal time is negligible. Therefore, one would expect a negligible time delay when a particle tunnels through such a barrier. We shall, however, demonstrate that a particle with a well-defined energy must "wait" in front of the barrier.

In Sec. II the delta-function propagator is derived. Section III clarifies the propagator's mathematical structure by relating it to the time-evolution problem of a free semi-infinite wave. Such a wave packet contains one parameter, the momentum of the particle, and it describes a particle that is initially confined to one of the two half-spaces $x \lessgtr 0$. In Sec. III we also derive the time evolution of a long wave train passing through a delta-function barrier.

For a more general class of localized wave packets, analytic expressions for the tunneling process are derived in Sec. IV. Some examples of the theory are presented in Sec. V. An attempt to explain the results follows in Sec. VI.

II. DELTA-FUNCTION PROPAGATOR

In this section we derive a simple expression for the time-dependent propagator of a particle moving in a one-dimensional delta-function potential, located at $x = \xi$. We want to solve the time-dependent Schrödinger equation

$$[i\hbar\partial_t + (\hbar^2/2m)\partial_x^2]\psi(x,t) = V(x)\psi(x,t), \tag{1}$$

with

$$V(x) = V_0\delta(x-\xi). \tag{2}$$

In our problem $V_0 > 0$ is assumed, but the results are also valid for $V_0 < 0$. If, at some initial time t_0, the particle's wavefunction is $\psi(x,t_0)$, then its wavefunction at a later time t is given[10,11] by

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, K(x,t\,|x',t_0)\psi(x',t_0), \tag{3}$$

where $K(x,t\,|x',t')$ is the retarded propagator that obeys the integral equation[12]

$$K(x,t\,|x',t') = K_0(x,t\,|x',t') - \frac{i}{\hbar}\int_{t'}^{t} dt''\int_{-\infty}^{\infty} dx''\, K_0(x,t\,|x'',t'') \times V(x'')K(x'',t''|x',t'). \tag{4}$$

In Eq. (4) we introduced the free-particle propagator

$$K_0(x,t\,|x',t') = \left(\frac{m}{2\pi i\hbar(t-t')}\right)^{1/2} \times \exp\left(\frac{im(x-x')^2}{2\hbar(t-t')}\right). \tag{5}$$

Since the particle is acted on by a time-independent potential, the propagator K depends on $t-t'$ only. Without loss of generality we choose $t_0 = t' = 0$, and we introduce the shorthand notation

$$U_0(x-x';t) = K_0(x,t\,|x',0),$$
$$U(x,x';t) = K(x,t\,|x',0). \tag{6}$$

Combining Eqs. (2), (4), and (6) we obtain the integral equation

$$U(x,x';t) = U_0(x-x';t) - \frac{i}{\hbar}V_0\int_0^t dt'' \times U_0(x-\xi;t-t'')U(\xi,x';t''). \tag{7}$$

In order to solve for $U(x,x';t)$ we shall Laplace transform Eq. (7) with respect to t. Defining

$$\tilde{f}(s) = \mathscr{L}\{f(t)\} = \int_0^\infty dt\, e^{-st}f(t), \tag{8}$$

and using the convolution (faltung) theorem:

$$\mathscr{L}\left\{\int_0^t d\tau f_1(t-\tau)f_2(\tau)\right\} = \tilde{f}_1(s)\tilde{f}_2(s), \qquad (9)$$

we readily calculate the Laplace transform of U at $x=\xi$:

$$\tilde{U}(\xi,x';s) = \frac{\tilde{U}_0(\xi - x';s)}{1+(iV_0/\hbar)\tilde{U}_0(0;s)}. \qquad (10)$$

The Laplace transform of the free-particle propagator can be determined from Eqs. (5), (6), and (8). The result is

$$\tilde{U}_0(z;s) = \left(\frac{m}{2i\hbar s}\right)^{1/2} \exp\left[-\left(\frac{2ms}{i\hbar}\right)^{1/2}|z|\right]. \qquad (11)$$

The Laplace transform of Eq. (7) can therefore be written in the form

$$\tilde{U}(x,x',s) = \tilde{U}_0(x-x';s) - \frac{i}{\hbar} \times V_0 \frac{\tilde{U}_0(x-\xi;s)\tilde{U}_0(\xi-x';s)}{1+(iV_0/\hbar)\tilde{U}(0;s)}. \qquad (12)$$

Now we determine the inverse Laplace transform. Using expression (11) for $\tilde{U}_0$ and a table of Laplace transforms, for instance Ref. 13, we obtain

$$U(x,x';t) = U_0(x-x';t) - \frac{mV_0}{\hbar^2} \times M\left(|x-\xi| + |\xi - x'|; -i\frac{mV_0}{\hbar^2};\frac{\hbar}{m}t\right). \qquad (13)$$

In Eq. (13) we introduced the Moshinsky function,[14,15] which is defined in terms of the complementary error function[13]

$$M(x;k;t) = \tfrac{1}{2} e^{i(kx - k^2t/2)} \operatorname{erfc}\left[(x-kt)/\sqrt{2it}\right], \quad (14)$$

with

$$\sqrt{i} = \exp(i\pi/4) \quad \text{and} \quad 1/\sqrt{i} = \exp(-i\pi/4).$$

Before we discuss the physical meaning of the Moshinsky function in the next section, we should mention that there are other, less direct methods,[16] to derive the propagator U. Also, in Ref. 16 the semiclassical approximation ($\hbar \to 0$) is discussed in great detail.

III. THE MOSHINSKY SHUTTER

In 1952 Moshinsky[15] investigated the following problem: "A monochromatic beam of noninteracting particles of mass m and energy $\hbar^2k^2/2m$ moves parallel to the x axis from the left to the right. At $x=0$ the beam is stopped by a shutter perpendicular to the beam. If at $t=0$ the shutter is opened, what will be the transient particle current observed at a distance x from the shutter?"

For simplicity we assume that the shutter acts as a perfect absorber. Then, the wavefunction that initially represents a particle of the beam is given by

$$\psi(x,t=0) = \Theta(-x)e^{ikx}, \qquad (15)$$

with $\Theta(y)=0$ for $y<0$ and $\Theta(y)=1$ for $y>0$. The time evolution of ψ is obtained from Eqs. (3) and (5). The result is[15]

$$\psi(x,t\geqslant 0) = M(x;k;(\hbar/m)t). \qquad (16)$$

The Moshinsky function, can, therefore, be interpreted as the wavefunction of a monochromatic particle, which at $t=0$ is confined to the left half-space $x \leqslant 0$. Some important properties of $M(x;k;t)$ are given in the Appendix. The close relationship of M with the theory of diffraction follows from evaluating $|M|^2$,

$$|M(x;k;(\hbar/m)t)|^2 = \tfrac{1}{2}\{[\tfrac{1}{2}+C(u)]^2 + [\tfrac{1}{2}+S(u)]^2\}, \qquad (17)$$

which has the familiar form of Fresnel scattering of light by a semiplane.[17] In Eq. (17), $u = (\hbar/m\pi t)^{1/2}(kt - x)$ is the argument of the Fresnel integrals[17,13] $C(u)$ and $S(u)$. From Eq. (17) we see that $|M|^2$ is a function of u only. Let us remark that $M(x;k;t)$ can be expressed in terms of two variables. For example, we could write $M(x;k;t) = \tilde{M}(x/\sqrt{t};k\sqrt{t})$. We shall, however, not use the reduced variables $x/\sqrt{t}$ and $k\sqrt{t}$, because they do not have a direct physical meaning.

We now modify the above problem by putting a delta-function barrier right behind the shutter. If at $t=0$ the absorbing shutter is opened, the beam will hit the barrier. Now the question is: How do the particles of the beam tunnel through the barrier? The key to this problem comes from solving the time-dependent Schrödinger equation

$$[i\hbar\partial_t + (\hbar^2/2m)\partial_x^2]\psi(x,t) = V_0\delta(x)\psi(x,t), \qquad (18)$$

with the initial condition specified above [Eq. (15)].

In order to simplify notation, we shall use from now on "atomic units" ($\hbar = m = 1$). This means that $x_0 = \hbar^2/(mV_0)$ is the unit of length, $p_0 = mV_0/\hbar$ the unit of momentum, $m\,V_0^2/\hbar^2$ the unit of energy, and $\hbar^3/(mV_0^2)$ the unit of time.

It is clear from Eqs. (3), (6), and (13) that, in the presence of the one-dimensional delta function, the time evolution of ψ follows for $t>0$ from

$$\psi(x,t) = \psi_0(x,t) - V_0\int_{-\infty}^{\infty} dx' \times M(|x| + |x'|; -iV_0;t)\psi(x',0), \qquad (19)$$

where $\psi_0(x,t)$ is the free-particle wavefunction:

$$\psi_0(x,t) = \int_{-\infty}^{\infty} K_0(x,t\,|x',0)\psi(x',0) \qquad (20)$$

with K_0 given by Eq. (5).

For a monochromatic beam of particles, having the initial condition (15), we know already the free-particle solution: $\psi_0(x,t)$ is the wavefunction (16) for the Moshinsky problem. We next calculate the integral on the right-hand side of Eq. (19). This can be done by using some mathematical properties of the Moshinsky function that are listed in the Appendix. From Eq. (A6′) we can read off the result:

$$\psi(x,t) = M(x;k;t) + [V_0/(V_0 - ik)] \times [M(|x|;-iV_0;t) - M(|x|;k;t)]. \qquad (19')$$

For the purpose of illustration we shall calculate the tunneling current. As usual, the (particle number) current is given by

$$j(x,t) = \operatorname{Im}[\psi^*(x,t)\partial_x\,\psi(x,t)]. \qquad (21)$$

By applying Eq. (A1) to Eq. (19′), the derivative with respect to x becomes

$$\partial_x\,\psi(x,t) = \partial_x\,\psi_0(x,t) + [2\Theta(x) - 1]V_0\psi(|x|,t). \qquad (22)$$

From Eq. (22) we see that the first derivative of the wavefunction jumps as in the static case. As we go from the left ($x = -\epsilon$) of the barrier to the right ($x = \epsilon$) of the barrier we find for $t > 0$

$$\partial_x \psi(x,t)|_{x=\epsilon} - \partial_x \psi(x,t)|_{x=-\epsilon} = 2V_0\psi(0,t). \quad (23)$$

In Eq. (23), ϵ is an infinitesimally small positive number. The jump in the derivative occurs instantaneously, even if the wavefunction is prepared in such a way that it does not satisfy Eq. (23) at $t = 0$. This result is interesting but not surprising, because the Schrödinger equation is a nonrelativistic one; hence information and signals may be transmitted at any speed. From Eq. (3), it follows that the propagator controls the properties of $\psi(x,t > t_0)$. The propagator satisfies the time-dependent Schrödinger equation, and its first derivative with respect to x jumps according to Eq. (23).

Making use of Eqs. (22) and (A1) we can calculate the current. In the vicinity of the barrier we obtain

$$j(\pm\epsilon,t) = \mathrm{Im}\{\psi^*(\pm\epsilon,t) \times [ikM(|\epsilon|;k;t) - U_0(\pm\epsilon;t)]\}. \quad (24)$$

For $t > 0$ one therefore has a continuous tunneling current

$$j(+\epsilon,t) = j(-\epsilon,t), \quad \epsilon \to 0. \quad (25)$$

Equation (25) must, of course, be true as long as V_0 is real. Having worked out the tunneling dynamics for a semi-infinite plane wave, we shall now study the time evolution of a certain class of wave packets in the presence of the delta-potential barrier.

IV. TUNNELING OF WAVE PACKETS

In the last section we calculated the time evolution of an infinite wave train which reaches the barrier at $t = 0$ from one side (in our case from the left). We now discuss what happens to a localized wave packet when it runs against the delta-function barrier. From Eq. (A6) we conclude that the integral in Eq. (19) can be done analytically if $\psi(x,0)$ is of the form

$$\psi(x,0) = \sqrt{\alpha}\, e^{-\alpha|x+x_0|}e^{ik(x+x_0)}. \quad (26)$$

Such a wave packet is centered around $x = -x_0$ and it moves with an average momentum k. The decay parameter α must be positive; the phase $\exp(ikx_0)$ has no physical meaning but has been added for convenience. Using Eq. (A6), it is straightforward to calculate $\psi(x,t)$ from Eq. (19). For $t > 0$, the result is

$$\begin{aligned}\psi(x,t) = {} & \sqrt{\alpha}[M(x+x_0;k-i\alpha;t) \\ & + M(-x-x_0;-k-i\alpha;t)] \\ & + V_0\sqrt{\alpha}\{J(x_0,\lambda^*) - J(x_0,-\lambda) \\ & + e^{-\lambda x_0}[J(0,-\lambda) + J(0,\lambda)]\} \end{aligned} \quad (19'')$$

with $\lambda = \alpha - ik$ and

$$\begin{aligned} J(\xi,\lambda) = {} & [1/(V_0-\lambda)][M(|x|+\xi;-iV_0;t) \\ & - M(|x|+\xi;-i\lambda;t)]. \end{aligned} \quad (27)$$

It should be clear that $J(\xi,\lambda)$ depends also on V_0, $|x|$, and t. The first bracket on the rhs of Eq. (19″) describes the time evolution of the free ($V_0 = 0$) wave packet.

If we want to study the tunneling current, we must know the first derivative of $\psi(x,t)$ with respect to x. Instead of calculating the derivative directly from Eq. (19″), we rather use Eq. (19), and we obtain for $x = \pm\epsilon$

$$\partial_x \psi(x,t) = \partial_x \psi_0(x,t) \pm V_0\psi(x,t), \quad (28)$$

which holds true for an arbitrary solution of Eqs. (1) and (2) if $t > t_0 = 0$. As before, ψ_0 denotes the free ($V_0 = 0$) wave packet. The second term on the rhs of Eq. (28) does not contribute to the tunneling current because the corresponding imaginary part vanishes. Hence we are left with the following expression for the current through the barrier:

$$j(\pm\epsilon,t) = \mathrm{Im}[\psi^*(x,t)\partial_x \psi_0(x,t)]_{x=\pm\epsilon}. \quad (29)$$

The current is again continuous, i.e., $j(\epsilon,t) = j(-\epsilon,t)$, as it must be for a nonabsorbing potential barrier. We are now in a position to present some examples for the tunneling problem.

V. WAVE PACKETS VERSUS INFINITE WAVE

Let us first discuss the semi-infinite wave train of Sec. III. The tunneling current is given by Eq. (24), and it can be calculated by evaluating expression (19′) for the wavefunction. In other words, all one has to do is to calculate a few Moshinsky functions. In view of Eq. (14) this amounts to calculating the corresponding complementary error functions. Figure 1 shows the particle number current (24) right behind a barrier of unit strength ($V_0 = 1$). The current is plotted as a function of time for a wavefunction (15) with $k = 2$. We are not surprised to see that the stationary current is reached only after a finite amount of time. More interesting, we observe a time delay of the tunneling current compared with the current of a free ($V_0 = 0$) semi-infinite wave train. The reason for such a time delay is a standby effect: The current being the same before and behind the delta-function barrier [Eq. (29)], we conclude that the particle spends some time in front of the barrier before it undergoes tunneling. It is, however, impossible to trace back the particle's motion in configuration space because we are confronted with an incoming wave of infinite extension.

Another piece of information about the tunneling process is the time evolution of the particle's density distribution $|\psi(x,t)|^2$. Figures 2 and 3 depict the dependence of $|\psi(x,t)|^2$ upon position and time for the semi-infinite wave train discussed so far. Superposition of the incoming and

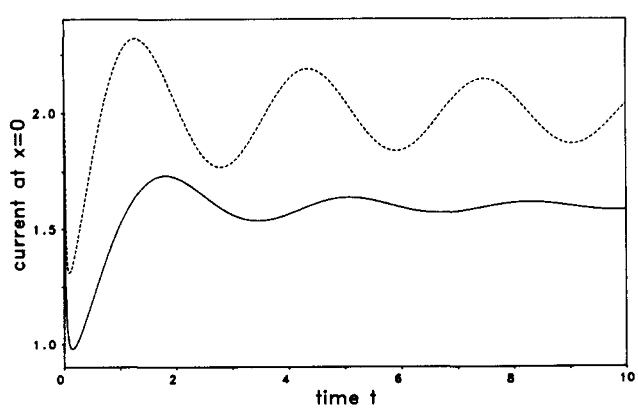

Fig. 1. Tunneling current (full curve, $V_0 = 1$) of a semi-infinite wave train, Eqs. (15) and (19′). Dotted line is the result for a free passage ($V_0 = 0$) of the wave train. Initial momentum of the particles: $k = 2$. For units see text following Eq. (18). Notice the time delay for tunneling.

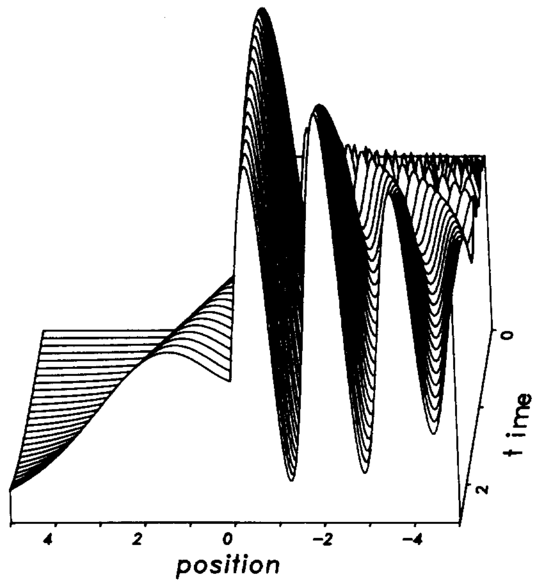

Fig. 2. "Three-dimensional" plot of the density distribution $|\psi(x,t)|^2$ (vertical direction) as a function of x and t. The density distribution illustrates the tunneling dynamics of the semi-infinite wave train of Fig. 1. At $t = 0$ the wave collides with the delta-barrier potential, located at $x = 0$.

reflected wave leads to an oscillatory profile for $x < 0$. For $x > 0$ the outgoing wave has a rather smoothly varying density profile.

Let us now study the tunneling dynamics of wave packets. For a compact presentation of the tunneling dynamics, we assume the wave packet (26) to start at $\langle x \rangle = -x_0 = -2$ with $\alpha = 1$. Since we want to compare the behavior of this wave packet with the behavior of the corresponding wave train, we choose $V_0 = 1$ and $\langle p \rangle = k = 2$ for $t = 0$. Figure 4 reveals that now the tunneling current is not delayed relative to a free passage of the wave packet. Furthermore, the current reaches its maxi-

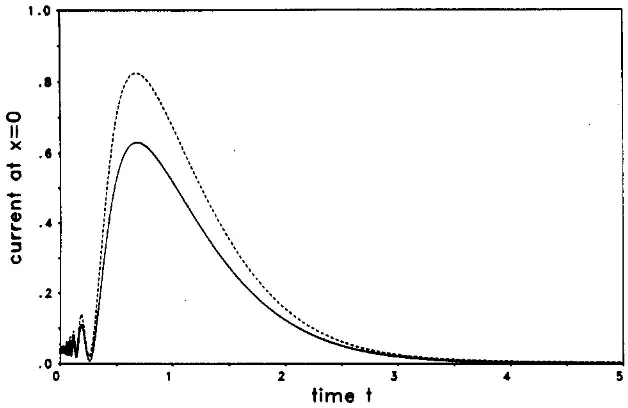

Fig. 4. Tunneling current (full curve, $V_0 = 1$) of the exponential wave packet, Eqs. (26) and (19″), with initial values $\langle x \rangle = -2$, $\langle p \rangle = 2$ and decay parameter $\alpha = 1$. Dotted line is the result for a free passage ($V_0 = 0$) of the wave packet.

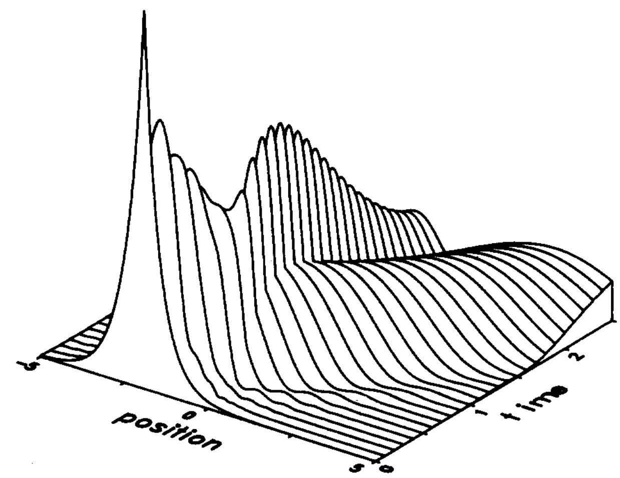

Fig. 5. "Three-dimensional" plot of the density distribution $|\psi(x,t)|^2$ (vertical direction) for the tunneling of the exponential wave packet of Fig. 4.

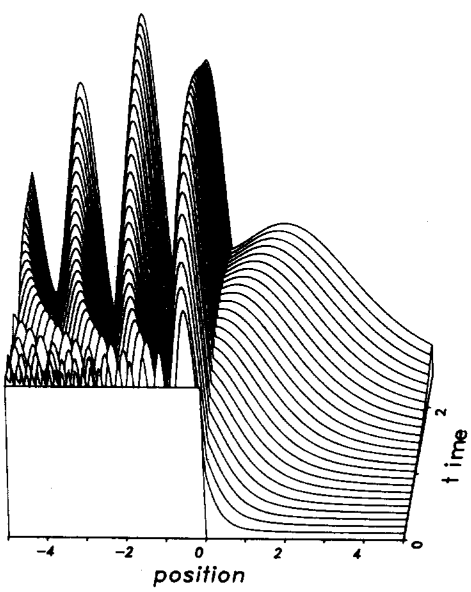

Fig. 3. Same plot as in Fig. 2, rear view.

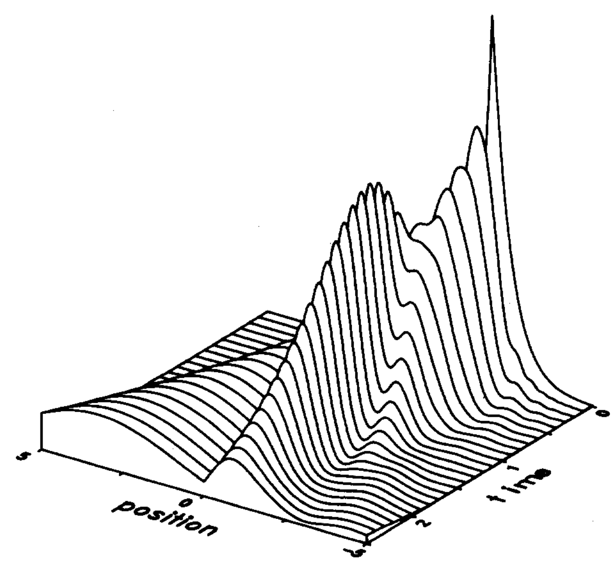

Fig. 6. Same plot as in Fig. 5, rear view.

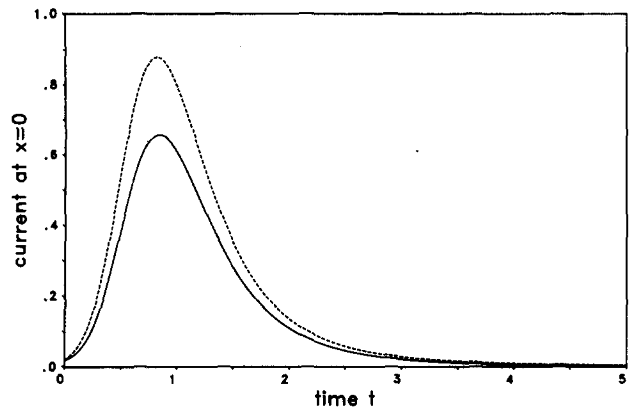

Fig. 7. Same as Fig. 4, but now for a Gaussian wave packet with initial values $\langle x\rangle = -2$, $\langle p\rangle = 2$, and $\Delta x^2 = \Delta p^2 = \frac{1}{2}$.

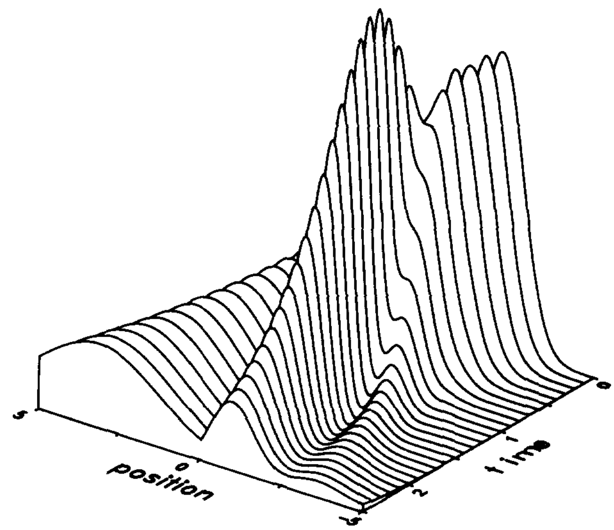

Fig. 9. Same plot as in Fig. 8, rear view.

mum somewhat before $t = 1$, the time at which the corresponding classical particle would arrive at $x = \xi = 0$. The oscillatory structure of the current for $t \ll 1$ is due to the initial kink in the particle's density distribution (see Figs. 5 and 6). This kink disappears immediately after the wave packet has started to move. It is not hard to understand this effect, if one bears in mind that the propagator (13) is an analytic function of x and x' except for $x = x' = \xi_0$, with ξ_0 being the position of the delta barrier.

We may wonder whether the vanishing time delay is a peculiarity of the exponential wavefunction (26) and (19″). Let us therefore take a Gaussian wave packet which has initially ($t = 0$) minimum uncertainty with $\Delta x^2 = \Delta p^2 = \frac{1}{2}$, and that has the same starting values as the exponential wave packet: $\langle x\rangle = -2$ and $\langle p\rangle = 2$. In order to obtain $\psi(x,t)$, we must numerically carry out the integration of the rhs of Eq. (19). The resulting tunneling current and the density profile are shown in Figs. 7–9. Again we find a smooth tunneling current with negligible time delay and a density distribution similar to that of Figs. 6 and 7.

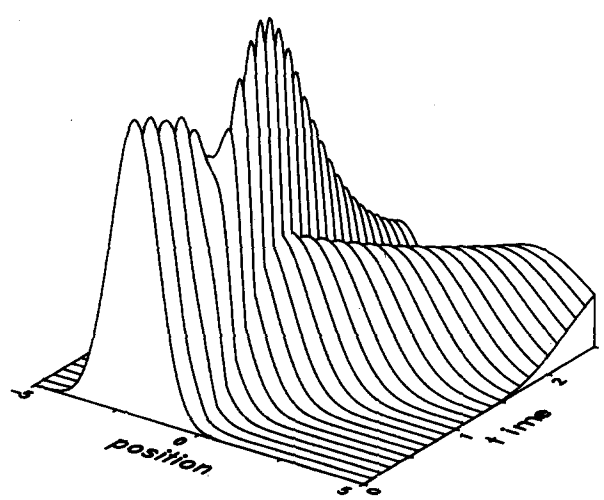

Fig. 8. "Three-dimensional" plot of the density distribution $|\psi(x,t)|^2$ (vertical direction) for the Gaussian wave packet of Fig. 7.

VI. INTERPRETATION OF THE RESULTS

We have found analytical tunneling solutions for delta-function barriers. It is well known that such a potential is the limit of a square-barrier potential of height $U \to \infty$ and thickness $a \to 0$. The area $a \cdot U$ remains fixed and finite and equal to V_0. Let us, for a moment, assume that both a and U are finite. Then, a classical particle with enough energy to pass over the barrier will be delayed as compared to a motion where the barrier is absent. Such a time delay has an obvious reason: While traversing the barrier, the particle loses momentum and, therefore, slows down. In a quantal treatment of over- and underbarrier passage problems one again finds time delay. For example, the slowing down of neutrons in a gravitational field gives rise to observable interference effects.[18] As an important example of underbarrier passage we should mention the "time delay line" concept in modern circuit theory.[19] The opposite is also true: There is a time advance[20] for particles that are scattered by an attractive potential. In all three examples mentioned here, the particle has reasonably sharp energy. In this article we have shown that particles with rather well-defined energy (represented by semi-infinite wave trains) will suffer time delay even when they pass through a zero-range barrier. From Eq. (19′) it follows immediately that as the momentum k of the particle increases, the scattering contribution to $\psi(x,t)$ becomes less important. Therefore, the time delay shrinks when the particle's energy is increased. Instead of varying the kinetic energy E of the particle one can also think of keeping E fixed and changing the area V_0 of the barrier. It is clear that large values of V_0 give rise to long delays.

From Sec. III it follows that for electrons the unit of time is given by

$$\tau = \frac{\hbar^3}{mV_0^2} = \frac{5.015}{V_0^2}\,10^{-15}\ \text{s}, \qquad (30)$$

if V_0 is measured in units of eV Å. Although typical tunneling barriers in microstructures[19] cannot be considered as delta-function barriers, we nevertheless estimate $V_0 = a \cdot U$ by taking $U = 0.2$ eV and $a = 50$ Å. In this case the unit of time is 5×10^{-17} s, which is also a typical value for a time delay (except for very slow electrons where the delay can easily be larger by a factor of 100). Such a short time could only be measured if the particle would tunnel back and forth many times. In this context we should mention the time delay of reflected light (Goos–Hänchen shift[21]) which can, in fact, be measured by multiple reflection experiments.

It is well known that localized (finite) wave packets can be viewed as a superposition of many different momenta states. In other words, wave packets localized in configuration space contain a more or less broad momentum distribution, depending on the degree of localization. The transmission probability[11]

$$T(E) = \frac{E}{E + mV_0^2/2\hbar^2} \tag{31}$$

favors the high-momentum components. These components determine the tunneling dynamics; but their time delay is small or negligible, a fact that is borne out by the results shown in Figs. 4 and 7. In this context we would like to remark that a better understanding of the tunneling dynamics can also be achieved by studying time-dependent tunneling in the presence of an external field,[22] i.e., tunneling through time-dependent barriers.

We have tried to understand "*a posteriori*" the rigorous results presented in Secs. II–V. The reader is, of course, welcome to modify the statements of this last section according to his own taste.

ACKNOWLEDGMENTS

We would like to thank Professor F. Koch for making us think about the problem of tunneling times. We are also grateful to Professor E. Merzbacher and Professor M. O. Scully for encouraging us to find analytical solutions. One of us (W. E.) appreciates financial support from the Deutsche Forschungsgemeinschaft and from the Bundesministerium für Forschung und Technologie.

APPENDIX

We want to summarize some properties of the Moshinsky function defined by Eq. (14). From this definition and from the definition[13] of the complementary error function we easily obtain

$$\partial_x M(x;k;t) = ikM(x;k;t) - U_0(x;t); \tag{A1}$$

$$\partial_k M(x;k;t) = ixM(x;k;t) - t\,\partial_x M(x;k;t); \tag{A2}$$

$$\partial_t M(x;k;t) = (x/2t)U_0(x;t) - (k/2)\partial_x M(x;k;t), \tag{A3}$$

where U_0 denotes the free propagator. Since M is a solution of the free Schrödinger equation we have

$$i\,\partial_t M(x;k;t) = -\tfrac{1}{2}\partial_x^2 M(x;k;t). \tag{A4}$$

Noting that $\mathrm{erfc}(z) + \mathrm{erfc}(-z) = 2$, we find

$$M(x;k;t) + M(-x;-k,t) = \exp(ikx - ik^2t/2)\,. \tag{A5}$$

The following indefinite integral of the Moshinsky function,

$$\int^x dx'\, e^{ikx'} M(ax' + b;c;t) = \frac{e^{ikx}}{i(k+ca)}[M(ax+b;c;t) - M(ax+b;-k/a;t)], \tag{A6}$$

can be verified through partial integration, using Eq. (A1). In Eq. (A6) the constants $a,b,c,t,$ and k are arbitrary but l'Hopital's rule must be applied for $k \to -ac$. From Eq. (A6) the definite integrals used in this article are readily obtained: In order to verify the rhs of Eq. (19') we need to know the definite integral

$$\int_{-\infty}^{0} dx'\, e^{ikx'} M(|x'| + |x|; -iV_0;t) = \frac{1}{V_0 - ik}[M(|x|;k;t) - M(|x|;-iV_0;t)], \tag{A6'}$$

which is a consequence of Eq. (A6). Also, the rhs of Eq. (19") can be verified by use of Eq. (A6).

[a] Permanent address: Physik-Department, Technische Universität München, 8046 Garching, West Germany.

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