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An operator solution for the hydrogen atom with application to the momentum representation

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effect — a decrease in the effective mass of the system.

In solid-state theory, an electron's effective mass can be more or less than its vacuum value due to interaction with the crystal lattice.[5] An electric field interacts, not only with the electron, but also with the lattice to which the electron is coupled. Similarly, an effective mass for our cart system can be defined by the initial change in momentum after the yank by the suddenly taut dial cord:

$$M_{\text{eff}} = (M_B V_B + M_C V_C)/V_B. \quad (3)$$

This quantity is plotted against M_A in Fig. 6, using experimental values of the velocities in the first full 0.1-s intervals after contact. In the threshold region, the effective mass of the cart system is within 5% of 400 g, the sum of the cart masses. As M_A increases well beyond the value needed for bond breaking, the curve shows M_{eff} approaching M_B.

A consideration of the limiting behavior of the system provides an intuitive feel for the shape of this curve with different intermagnetic bond strength. The weaker the bond, the more rapid the decline to the asymptote (curve labeled *weak*); in the limit of an infinitely strong bond, there would be no decline at all and the effective mass would remain equal to the sum of the two cart masses, no matter how hard the yank at contact.

The experimental study described here is time efficient and dynamic, simple of technique, and rewarding in its potential harvest of physics principles.

[1]A. P. French, *Newtonian Mechanics* (Norton, New York, 1971), p. 339. See also Kittel *et al.*, *Mechanics* (Berkeley Physics Course) (McGraw-Hill, New York, 1973), 2nd ed., 197, prob. 7.

[2]Magnets available from Edmund Scientific, 101 E. Gloucester Tpke, Barrington, NJ 08007. Catalog # 603.47.

[3]Hooke's law spring available from Central Scientific Co., 11222 Melrose Ave., Franklin Park, IL 60131-1364 (1985–86 catalog #73955).

[4]W. R. Smythe, *Static and Dynamic Electricity*, (McGraw-Hill, New York, 1950) 2nd ed. p. 435. The relevant equations in the second edition of this graduate text (the author being aware of no later edition) have several typographical errors. The power of r in Eq. (4) should be 4, not 3. More confusing is the constant appearance of the permeability μ in the denominators of Eq. (2)–(4). Dimensionally, μ must be in the numerator.

[5]See, for example, R. A. Smith, *Wave Mechanics of Crystalline Solids* (Chapman & Hull, London, 1961), p. 126.

An operator solution for the hydrogen atom with application to the momentum representation

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The radial form of Hylleraas' equation for the hydrogen atom, $\Lambda_l|El\rangle = 4\hbar^4 a^{-2}|El\rangle$ (a = Bohr radius), is considered and it is shown that the operator Λ_l can be factorized. Hence ladder operators $P_l^{\pm}$ are derived that are linear in the position operator $\mathbf{r}$ and are nonlinear functions of the momentum operator $\mathbf{p}$. It is proven that $P_l^{\pm}|El\rangle = 2\hbar^2 a^{-1}[1 + (l + \frac{1}{2} \pm \frac{1}{2})^2 \times (2M\hbar^{-2}a^2E)]^{1/2}|E,l \pm 1\rangle$. In the momentum representation of wave mechanics the solutions to these equations are the radial momentum-space wavefunctions for the hydrogen atom. Thus a simple method of calculating these wavefunctions is obtained. The results complement the familiar operator solution for the hydrogen atom that is based on factorization of the radial Hamiltonian and yields operators that are linear in $\mathbf{p}$ and are nonlinear functions of $\mathbf{r}$.

I. INTRODUCTION

The abstract ladder operator method is a standard technique for solving the simpler problems of quantum mechanics: for example, angular momentum, the linear oscillator, and the hydrogen atom. It can be used to determine the eigenvalues of an operator such as the Hamiltonian by constructing operators that transform eigenkets into each other. An attractive feature of this method is that it is based on commutation relations, such as the canonical commutation relations for the position and momentum operators, and does not depend on any particular realization of the operators. In fact, it is usually less cumbersome to apply than the more frequently used techniques of wave mechanics.

A particularly clear application of the abstract operator method to the isotropic three-dimensional oscillator and the hydrogen atom has been given by Newmarch and Golding,[1] who derive ladder operators by factorizing the radial Hamiltonian.[2] The operators obtained in this way are linear in the momentum operator $\mathbf{p}$ and they are also nonlinear functions of the position operator $\mathbf{r}$. When expressed as wave-mechanical operators in the coordinate representation these ladder operators are the differential operators derived by Schrödinger[3] and also by Infeld and Hull[4] in their study of factorization of the Sturm–Liouville

equation. The corresponding ladder operator equations are first-order differential equations whose solution yields the radial part of the coordinate-space wavefunctions.[5]

It is often emphasized in texts on quantum mechanics that the symmetry between $\mathbf{p}$ and $\mathbf{r}$, which is evident from a comparison of wave mechanics in the coordinate and momentum representations, arises because these are two particular representations of quantum mechanics.[6] Thus it is natural to inquire whether for the isotropic three-dimensional harmonic oscillator (hereinafter referred to as the oscillator) and the hydrogen atom, one can derive abstract ladder operators analogous to those described above, but which are linear in $\mathbf{r}$ and are also nonlinear functions of $\mathbf{p}$. Of course, for the oscillator the answer is fairly obvious; it is considered here because it is instructive (Sec. II).

For the hydrogen atom the answer is less obvious and forms the main purpose of this article. Our starting point is an alternative form of Schrödinger's equation for the hydrogen atom, namely, that derived by Hylleraas.[7] We show that the radial Hylleraas operator [Eq. (15)] can be factorized, and in this way we derive ladder operators that are linear in $\mathbf{r}$ and are nonlinear functions of $\mathbf{p}$ (Sec. III). The coefficients for the transformations effected by the ladder operators are calculated, and in the usual way they yield the energy eigenvalues. The wave-mechanical forms of these operators in the momentum representation are ladder operators for the orbital angular momentum quantum number l in the radial part of the momentum-space wavefunctions. Thus we obtain a simple method for calculating these wavefunctions (Sec. IV). The usual methods of calculation are often described as difficult[8–11] and they are not given in any of the standard texts on quantum mechanics. The earliest method used Fourier transformation of the coordinate-space wavefunctions[12]: this requires some difficult integrations.[8] Momentum-space wavefunctions for the hydrogen atom have also been derived by solving Hylleraas' equation in spherical coordinates,[7] and in toroidal coordinates,[10] and by solving an integral equation.[13]

By contrast, the calculations presented in this article are analogous to the familiar ladder operator method that is based on factorization of the radial Hamiltonian: they would be suitable for inclusion in an introductory or intermediate quantum mechanics course. To increase the pedagogical value of this article we give a brief summary in Appendix A of the well-known factorizations of the radial Hamiltonian for the oscillator and the hydrogen atom. A comparison of these factorizations with those presented in Secs. II and III should be helpful and instructive to new students of this subject.

II. THE OSCILLATOR

The Hamiltonian is

$$H = (2M)^{-1}\mathbf{p}^2 + \tfrac{1}{2}M\omega^2\mathbf{r}^2. \tag{1}$$

In Appendix A we summarize the familiar factorization of a radial Hamiltonian for the oscillator that yields ladder operators linear in $\mathbf{p}$. From these calculations it is clear how to obtain ladder operators that are linear in $\mathbf{r}$.[14] We introduce the radial Hermitian operators

$$p = (\mathbf{p}\cdot\mathbf{p})^{1/2} \tag{2}$$

and

$$r_p = \tfrac{1}{2}(\hat{\mathbf{p}}\cdot\mathbf{r} + \mathbf{r}\cdot\hat{\mathbf{p}}) = p^{-1}(\mathbf{p}\cdot\mathbf{r} + i\hbar). \tag{3}$$

These satisfy the commutation relation

$$[p, r_p] = -i\hbar \tag{4}$$

and the operator identity

$$\mathbf{L}^2 = \mathbf{p}^2(\mathbf{r}^2 - r_p^2), \tag{5}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. If we use Eq. (5) to eliminate $\mathbf{r}^2$ in favor of r_p^2 in Eq. (1) and then replace $\mathbf{L}^2$ with $\hbar^2 l(l+1)$ we obtain the radial Hamiltonian

$$H_l = \tfrac{1}{2}M\omega^2[r_p^2 + \hbar^2 l(l+1)p^{-2}] + (1/2M)\mathbf{p}^2. \tag{6}$$

The factorization of Eq. (6) is similar to that of Eq. (A2) in Appendix A, and we merely state the results:

$$M\omega^2 P_{l\pm1}^{\mp} P_l^{\pm} = 2H_l - 2\hbar\omega(l + \tfrac{1}{2} \pm 1), \tag{7}$$

$$P_l^{\pm} = r_p \mp i\hbar(l + \tfrac{1}{2} \pm \tfrac{1}{2})p^{-1} \pm i(M\omega)^{-1}p, \tag{8}$$

$$P_{l\pm1}^{\mp} = (P_l^{\pm})^{\dagger}, \tag{9}$$

$$P_l^{\pm}|El\rangle = \{2M^{-1}\omega^{-2}[E - (l + \tfrac{1}{2} \pm 1)]\hbar\omega\}^{1/2} \times |E \mp \hbar\omega, l \pm 1\rangle, \tag{10}$$

where † denotes the adjoint operator and $|El\rangle$ is a normalized eigenket of H_l with eigenvalue E. In Eq. (10) the phase factor has been set equal to unity. If $\omega > 0$ then from Eq. (10), $E = (n + \frac{3}{2})\hbar\omega$, where $n = 0,1,2,\ldots$.

The operators $P_l^{\pm}$ of Eq. (8) are linear in $\mathbf{r}$. In the momentum representation of wave mechanics $\mathbf{r} = i\hbar\nabla_p$, and Eq. (3) becomes

$$r_p = i\hbar\left(\frac{\partial}{\partial p} + \frac{1}{p}\right). \tag{11}$$

Equations (8), (10), and (11) yield a pair of first-order differential equations whose solutions are the radial momentum-space wavefunctions for the oscillator.[15]

III. THE HYDROGEN ATOM

The Hamiltonian is

$$H = (2M)^{-1}\mathbf{p}^2 - \hbar^2(Ma)^{-1}r^{-1}, \tag{12}$$

where $a = 4\pi\epsilon_0\hbar^2(Me^2)^{-1}$ is the Bohr radius. We have summarized in Appendix A the familiar factorization of the radial Hamiltonian, which yields ladder operators linear in $\mathbf{p}$.

We now consider the topic that is the main purpose of this article: to obtain ladder operators $P_l^{\pm}$ for the hydrogen atom that are analogous to those presented above for the oscillator; that is, operators that are linear in $\mathbf{r}$. Comparison with the results given above for the oscillator suggest how this may be done: if we can write Schrödinger's equation for the hydrogen atom in a form involving an operator that is quadratic in $\mathbf{r}$ and if we use the identity Eq. (5) to obtain the radial form of this operator, factorization of this radial form may yield the desired operators.

Actually, for the hydrogen atom a Schrödinger equation that is quadratic in $\mathbf{r}$ was first presented long ago by Hylleraas.[7] It can be derived as follows. The action of the operator $\mathbf{r}^2(\mathbf{p}^2 - 2MH)^2$ on a ket $|Elm\rangle$ is the same as that of the operator

$$\Lambda = \mathbf{r}^2(\mathbf{p}^2 - 2ME)^2 - 2i\hbar\mathbf{p}\cdot\mathbf{r}(\mathbf{p}^2 - 2ME) + 4\hbar^2(\mathbf{p}^2 - 2ME), \tag{13}$$

as is evident by expanding the former operator and using $\mathbf{r}^2[H,\mathbf{p}^2] = 2i\hbar^3(Ma)^{-1}(\mathbf{p}\cdot\mathbf{r} + 2i\hbar)r^{-1}$. Thus noting Eq.

(12), we have

$$\Lambda|Elm\rangle = 4\hbar^4 a^{-2}|Elm\rangle, \tag{14}$$

which is Hylleraas' equation for the hydrogen atom.[7] (For convenience we omit a subscript E on Λ.)

We use Eq. (5) to eliminate $\mathbf{r}^2$ in favor of r_p^2 in the first term of Eq. (13), and then, for the purposes of operating on $|Elm\rangle$, we replace $\mathbf{L}^2$ with $\hbar^2 l(l+1)$. In the second term of Eq. (13) we use Eq. (3). This yields the radial operator

$$\Lambda_l = r_p^2(\mathbf{p}^2 - 2ME)^2 - 2i\hbar r_p p(\mathbf{p}^2 - 2ME) + \hbar^2 l(l+1)p^{-2}(\mathbf{p}^2 - 2ME)^2. \tag{15}$$

To obtain the factorization

$$P_{l\pm1}^{\mp} P_l^{\pm} = \Lambda_l + F_l^{\pm}, \tag{16}$$

where $F_l^{\pm}$ is independent of $\mathbf{r}$ and $\mathbf{p}$, inspection of Eq. (15) suggests that we try

$$P_l^{\pm} = r_p(\mathbf{p}^2 - 2ME) \pm f_l^{\pm}(p). \tag{17}$$

Then

$$P_{l\pm1}^{\mp} P_l^{\pm} = r_p^2(\mathbf{p}^2 - 2ME)^2 - 2i\hbar r_p p(\mathbf{p}^2 - 2ME) \pm (f_l^{\pm} - f_{l\pm1}^{\mp}) r_p(\mathbf{p}^2 - 2ME) \pm i\hbar \frac{df_l^{\pm}}{dp}(\mathbf{p}^2 - 2ME) - f_{l\pm1}^{\mp} f_l^{\pm}, \tag{18}$$

where we have used $[r_p, f(p)] = i\hbar(df/dp)$. From Eqs. (15), (16), and (18)

$$\pm (f_l^{\pm} - f_{l\pm1}^{\mp}) r_p(\mathbf{p}^2 - 2ME) \pm i\hbar \frac{df_l^{\pm}}{dp}(\mathbf{p}^2 - 2ME) - f_{l\pm1}^{\mp} f_l^{\pm} = \hbar^2 l(l+1)p^{-2}(\mathbf{p}^2 - 2ME)^2 + F_l^{\pm}. \tag{19}$$

Equations (19) can be solved for $F_l^{\pm}$ and the functions $f_l^{\pm}(p)$. First we note that $\mathbf{r}$ appears only in the first term in Eqs. (19). Therefore,

$$f_{l+1}^{-} = f_l^{+} \tag{20}$$

and

$$i\hbar \frac{df_l^{+}}{dp}(\mathbf{p}^2 - 2ME) - (f_l^{+})^2 = \hbar^2 l(l+1)p^{-2}(\mathbf{p}^2 - 2ME)^2 + F_l^{+}, \tag{21}$$

$$-i\hbar \frac{df_l^{+}}{dp}(\mathbf{p}^2 - 2ME) - (f_l^{+})^2 = \hbar^2 (l+1)(l+2)p^{-2}(\mathbf{p}^2 - 2ME)^2 + F_{l+1}^{-}. \tag{22}$$

Addition and subtraction of Eqs. (21) and (22) yield

$$(f_l^{+})^2 = -\hbar^2(l+1)^2 p^{-2}(\mathbf{p}^2 - 2ME)^2 - \tfrac{1}{2}(F_l^{+} + F_{l+1}^{-}) \tag{23}$$

and

$$i\hbar \frac{df_l^{+}}{dp} = -\hbar^2(l+1)p^{-2}(\mathbf{p}^2 - 2ME) + \tfrac{1}{2}(F_l^{+} - F_{l+1}^{-})(\mathbf{p}^2 - 2ME)^{-1}. \tag{24}$$

Because $\int (\mathbf{p}^2 - 2ME)^{-1}\, dp$ is a function that does not occur in Eq. (23), we must have

$$F_{l+1}^{-} = F_l^{+}. \tag{25}$$

Then the solution to Eq. (24) is

$$f_l^{+} = i\hbar(l+1)p^{-1}(\mathbf{p}^2 + 2ME) + C_l, \tag{26}$$

where C_l is independent of $\mathbf{p}$. From Eqs. (23), (25), and (26)

$$F_l^{+} = 4\hbar^2(l+1)^2(2ME) - 2i\hbar(l+1) \times C_l p^{-1}(\mathbf{p}^2 + 2ME) - C_l^2.$$

But F_l^{+} is independent of p: therefore, $C_l = 0$.

Thus we have derived the factorization Eq. (16), where

$$P_l^{\pm} = r_p(\mathbf{p}^2 - 2ME) \pm i\hbar(l + \tfrac{1}{2} \pm \tfrac{1}{2})p^{-1}(\mathbf{p}^2 + 2ME) \tag{27}$$

and

$$F_l^{\pm} = 4\hbar^2(l + \tfrac{1}{2} \pm \tfrac{1}{2})^2(2ME). \tag{28}$$

The adjoint of Eq. (27) is

$$(P_l^{\pm})^{\dagger} = P_{l\pm1}^{\mp} - 2i\hbar p, \tag{29}$$

which is different from the corresponding result for the oscillator, Eq. (9).

Let $|El\rangle$ denote a normalized eigenket of Λ_l with eigenvalue $4\hbar^4 a^{-2}$ [see Eq. (14)]. Then from Eqs. (16) and (25)

$$\Lambda_{l\pm1}(P_l^{\pm}|El\rangle) = 4\hbar^4 a^{-2}(P_l^{\pm}|E,l\rangle).$$

Thus

$$P_l^{\pm}|El\rangle = \beta_{El}^{\pm}|E,l\pm1\rangle. \tag{30}$$

The coefficient $\beta_{El}^{\pm}$ is given by

$$|\beta_{El}^{\pm}|^2 = \langle (P_l^{\pm})^{\dagger} P_l^{\pm}\rangle = \langle P_{l\pm1}^{\mp} P_l^{\pm} - 2i\hbar p P_l^{\pm}\rangle, \tag{31}$$

where $\langle\Omega\rangle = \langle El|\Omega|El\rangle$. It is shown in Appendix B that $\langle pP_l^{\pm}\rangle = 0$. Substituting Eqs. (16) and (28) in Eq. (31) yields

$$|\beta_{El}^{\pm}|^2 = 4\hbar^4 a^{-2}[1 + (l + \tfrac{1}{2} \pm \tfrac{1}{2})^2(2M\hbar^{-2}a^2E)]. \tag{32}$$

Thus for negative energy states Eq. (32) yields the Bohr formula

$$E = -\hbar^2(2Ma^2n^2)^{-1}, \quad n = 1,2,..., \tag{33}$$

and we can write

$$P_l^{\pm}|nl\rangle = \beta_{nl}^{\pm}|n,l\pm1\rangle, \tag{34}$$

$$|\beta_{nl}^{\pm}|^2 = 4\hbar^4 a^{-2}[1 - (l + \tfrac{1}{2} \pm \tfrac{1}{2})^2 n^{-2}]. \tag{35}$$

IV. APPLICATION TO THE MOMENTUM REPRESENTATION

In the momentum representation of wave mechanics r_p is given by Eq. (11). From Eqs. (2), (11), (27), (34), and (35) we obtain the first-order differential equations

$$\left[(p^2 + \hbar^2 a^{-2} n^{-2})\frac{d}{dp} \pm \left(l \pm \frac{7}{2} + \frac{1}{2}\right)p \mp \left(l + \frac{1}{2} \mp \frac{1}{2}\right)(\hbar^2 a^{-2} n^{-2})p^{-1}\right]\phi_{nl}(p) = 2\hbar a^{-1}\left[1 - \left(l + \frac{1}{2} \pm \frac{1}{2}\right)^2 n^{-2}\right]^{1/2}\phi_{n,l\pm1}(p), \tag{36}$$

where we have set the phase factors in $\beta_{nl}^{\pm}$ equal to i. From Eq. (36) we can calculate the radial momentum-space wavefunctions for the bound states of the hydrogen atom.

Starting with $l=n-1$, we have

$$\left((p^2+\hbar^2a^{-2}n^{-2})\frac{d}{dp}+(n+3)p\right.$$
$$\left.-(n-1)(\hbar^2a^{-2}n^{-2})p^{-1}\right)\phi_{n,n-1}(p)=0.$$

Thus

$$\phi_{n,n-1}(p)=N\frac{p^{n-1}}{(p^2+\hbar^2a^{-2}n^{-2})^{n+1}}, \tag{37}$$

where N is a constant. If we impose the normalization

$$\int_0^\infty|\phi_{n,n-1}|^2p^2\,dp=1,$$

it follows that

$$|N|^2=\frac{2^{4n+2}(n!)^2}{\pi(2n)!}\left(\frac{\hbar}{an}\right)^{2n+3}. \tag{38}$$

In this calculation we have used the integral[16]

$$\int_0^\infty u^{\alpha-1}(u+1)^{-(\alpha+\beta)}\,du=\Gamma(\alpha)\Gamma(\beta)[\Gamma(\alpha+\beta)]^{-1}.$$

The normalized wavefunctions with $l=n-2,\ n-3,\ldots,$ 1,0 can be obtained by successive application of the lowering operator in Eq. (36) to the wavefunction $\phi_{n,n-1}(p)$ of Eq. (37).

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APPENDIX A

We summarize the factorizations of the radial Hamiltonian for the oscillator and Coulomb potentials, which yield ladder operators linear in $\mathbf{p}$.[17] The radial Hermitian operators $r=(\mathbf{r}\cdot\mathbf{r})^{1/2}$ and $p_r=\frac{1}{2}(\hat{\mathbf{r}}\cdot\mathbf{p}+\mathbf{p}\cdot\hat{\mathbf{r}})=r^{-1}(\mathbf{r}\cdot\mathbf{p}-i\hbar)$ satisfy the commutation relation $[r,p_r]=i\hbar$ and the operator identity

$$\mathbf{L}^2=\mathbf{r}^2(\mathbf{p}^2-p_r^2). \tag{A1}$$

For the oscillator Eq. (A1) can be used to eliminate $\mathbf{p}^2$ in favor of p_r^2 in the Hamiltonian Eq. (1). Replacing $\mathbf{L}^2$ with $\hbar^2l(l+1)$ yields the radial Hamiltonian

$$H_l=(2M)^{-1}[p_r^2+\hbar^2l(l+1)r^{-2}]+\tfrac{1}{2}M\omega^2\mathbf{r}^2. \tag{A2}$$

This can be factorized:[17]

$$R_{l\pm1}^{\mp}R_l^{\pm}=2MH_l-2(l+\tfrac{1}{2}\pm1)M\hbar\omega, \tag{A3}$$

where

$$R_l^{\pm}=p_r\pm i\hbar(l+\tfrac{1}{2}\pm\tfrac{1}{2})r^{-1}\mp iM\omega r, \tag{A4}$$

$$R_{l\pm1}^{\mp}=(R_l^{\pm})^{\dagger}. \tag{A5}$$

Let $|El\rangle$ be a normalized eigenket of H_l with eigenvalue E. From Eq. (A3)

$$H_{l\pm1}(R_l^{\pm}|El\rangle)=(E\mp\hbar\omega)(R_l^{\pm}|El\rangle).$$

Therefore,

$$R_l^{\pm}|El\rangle=\alpha_{El}^{\pm}|E\mp\hbar\omega,l\pm1\rangle. \tag{A6}$$

From the expectation value of Eq. (A3)

$$|\alpha_{El}^{\pm}|^2=2M\,[E-(l+\tfrac{1}{2}\pm1)\hbar\omega].$$

If $\omega>0$ it follows that

$$E=(n+\tfrac{3}{2})\hbar\omega,\quad n=0,1,\ldots.$$

Similarly, for the hydrogen atom

$$H_l=(2M)^{-1}[p_r^2+\hbar^2l(l+1)r^{-2}-2\hbar^2a^{-1}r^{-1}] \tag{A7}$$

and[17]

$$R_{l\pm1}^{\mp}R_l^{\pm}=2MH_l+\hbar^2a^{-2}(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2}, \tag{A8}$$

where

$$R_l^{\pm}=p_r\pm i\hbar(l+\tfrac{1}{2}\pm\tfrac{1}{2})r^{-1}\mp i\hbar\alpha^{-1}(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}, \tag{A9}$$

$$R_{l\pm1}^{\mp}=(R_l^{\pm})^{\dagger}. \tag{A10}$$

Let $|El\rangle$ denote a normalized eigenket of H_l with energy E. From Eq. (A8)

$$H_{l\pm1}(R_l^{\pm}|El\rangle)=E(R_l^{\pm}|El\rangle).$$

Therefore,

$$R_l^{\pm}|El\rangle=\alpha_{El}^{\pm}|E,l\pm1\rangle. \tag{A11}$$

The expectation value of Eq. (A8) yields

$$|\alpha_{El}^{\pm}|^2=2ME+\hbar^2a^{-2}(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2}. \tag{A12}$$

Thus for negative energy states E is given by Eq. (33).

In the coordinate representation of wave mechanics $p_r=-i\hbar(r^{-1}+\partial/\partial r)$, and the solutions to Eqs. (A6) and (A11) are the radial coordinate-space wavefunctions for the oscillator and the hydrogen atom, respectively.

APPENDIX B

From Eq. (27)

$$pP_l^{\pm}=pr_p(\mathbf{p}^2-2ME)\pm i\hbar(l+\tfrac{1}{2}\pm\tfrac{1}{2})(\mathbf{p}^2+2ME). \tag{B1}$$

Now

$$\begin{aligned}\langle pr_p(\mathbf{p}^2-2ME)\rangle&=2\hbar^2a^{-1}\langle pr_pr^{-1}\rangle\\&=2\hbar^2a^{-1}\langle(\mathbf{p}\cdot\mathbf{r}+i\hbar)r^{-1}\rangle\\&=2\hbar^2a^{-1}\langle r^{-1}(\mathbf{r}\cdot\mathbf{p}-i\hbar)\rangle\\&=0,\end{aligned} \tag{B2}$$

because

$$[H,r]=-i\hbar M^{-1}r^{-1}(\mathbf{r}\cdot\mathbf{p}-i\hbar)$$

and

$$\langle[H,r]\rangle=0.$$

Also

$$\langle\mathbf{p}^2+2ME\rangle=2\langle2MH+\hbar^2a^{-1}r^{-1}\rangle=0, \tag{B3}$$

because $[H,\ \mathbf{r}\cdot\mathbf{p}]=-i\hbar M^{-1}(2MH+\hbar^2a^{-1}r^{-1})$ and $\langle[H,\mathbf{r}\cdot\mathbf{p}]\rangle=0$. Equations (B1)–(B3) yield $\langle pP_l^{\pm}\rangle=0$.

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A spherical mirror Fabry–Perot interferometer for microwave demonstrations

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Although Fabry–Perot interferometers have found widest application in the visible portion of the spectrum, their properties are conveniently observed in the microwave region, where high-quality spherical mirrors are easily fabricated on a lathe, and partially transmitting (high reflectance) flat mesh is commercially available at modest cost. The scaling properties of paraxial beams of such interferometers, as the frequency is varied from optical to microwave for a fixed resonator geometry, are well known. This allows studies that are relevant to contemporary optics research to be carried out in an upper division undergraduate laboratory without the need for sophisticated equipment.

I. INTRODUCTION

Microwave optics experiments using low-cost klystron sources are commonplace in elementary physics laboratories for demonstration of wave phenomena such as interference, diffraction, and polarization. This article describes a natural follow-up to these familiar experiments: a method for study of the eigenfrequencies and intensity profiles of the Gaussian-beam modes supported by a spherical-mirror Fabry–Perot resonator is presented. These quasioptical investigations, suitable for upper-division laboratories, lead the student directly into contemporary research concerns. For example, in the field of high-precision measurements, by servo-locking a laser frequency to a transmission or reflection fringe of an isolated optical resonator, linewidth below 100 mHz is obtained[1] and it has been conjectured[2] that laser frequency can be measured to one part in 10^{12}–10^{14} by locking the free spectral range of a Fabry–Perot interferometer to a rf frequency standard.

The Fabry–Perot interferometer, as commonly defined, consists of two reflectors which face each other and are separated from one another by a distance d. The separation may be variable, serving to tune the interferometer. With the development of lasers, our knowledge[3] of the properties of curved-mirror Fabry–Perot resonators has greatly increased.

We have devised a simple method for students to observe features of the Gaussian beam modes supported by a Fabry–Perot interferometer. The key to our approach is that the radiation source is a microwave sweep oscillator rather than the usual tunable laser. Since the beam spot size parameter scales as $\sqrt{\lambda}$ for fixed resonator dimensions, a spot size of 0.2 mm at $\lambda = 600$ nm increases to 5 cm at $\lambda = 3$ cm. The microwave range has several significant advantages over the optical range for the experiments.

(1) A spherical aluminum mirror that is figured to $\lambda/300$ or better can rather easily be turned on a lathe. It is readily polished to sufficient smoothness that scattering loss is completely negligible.

(2) By using a metal grid as a flat partial reflector, a freely propagating Gaussian beam is coupled out of the cavity. The intensity profile of the generated beam can be probed external to the cavity.

(3) In contrast to the optical range, easily operated monochromatic tunable rf oscillators are commercially available. We have borrowed an X-band plug-in unit of a Hewlett–Packard model 8690A BWO system from the Department of Electrical Engineering at our school, but