Quantum mechanics of a chargeless spinning particle in a periodic magnetic field: A simple, soluble system

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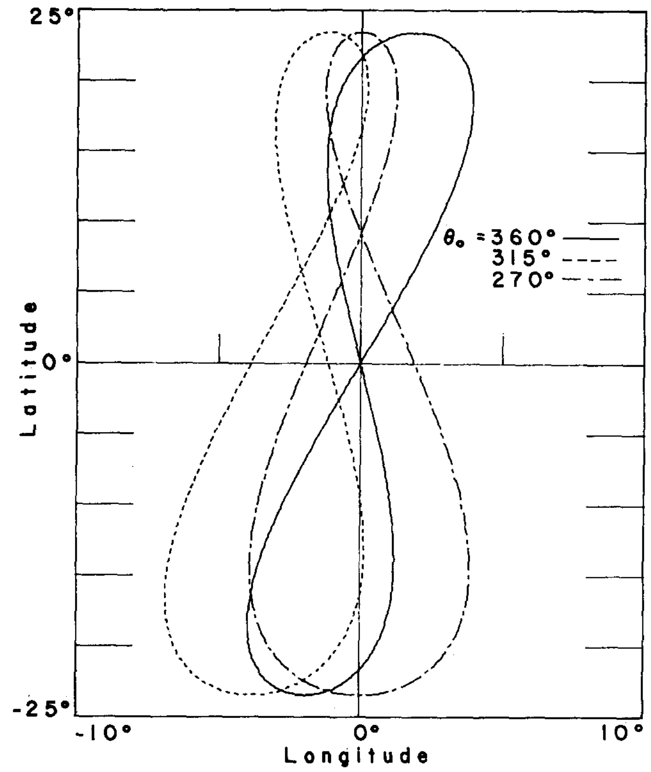

Fig. 7. Variation of the analemma due to precession of the Earth's polar axis. $\theta_0 = 270°, 315°, 360° (0°)$.

$\theta_0 = 315°$, which will next occur in the year 4305 A.D., we see the asymmetry increase. At $\theta_0 = 360° (0°)$ corresponding to the year 7530 A.D., perigee and the vernal equinox coincide and the analemma shows inversion symmetry. The appearance of the analemma at other epochs can be found by using the symmetries cited earlier.

V. CONCLUSION

In this work we have investigated the important practical problem of the trajectory of the subsatellite point for geosynchronous satellites. The problem is treated using vector geometry and orbital dynamics in a form familiar to those with a background in physics rather than spherical astronomy. The trajectories are obtained by a transformation from the inertial frame in which the Keplerian equations are cast to a rotating frame fixed in the Earth. Symmetry of the trajectories under reflections in the longitude of perigee are discussed and are seen to make all trajectories obtainable from a single quadrant of this parameter.

The formalism developed can be directly applied to account for the appearance of the analemma. The analysis shows the intimate connection between the appearance of the analemma and the parameters of the Sun's apparent path about the Earth. In particular, the formalism permits us to see the variation of the appearance of the analemma due to precession of the Earth's polar axis.

In addition to explaining these phenomena with mathematics accessible to undergraduates, this work also shows the connection between two not obviously related phenomena: geosynchronous satellite orbits and the analemma.

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[1]P. Bielkowicz, AIAA J. 4, 2190 (1966).

[2]R. N. Mayall and M. W. Mayall, *Sundials* (Sky Publishing Co., Cambridge, MA, 1973), 2nd ed.

[3]CRC Standard Mathematical Tables, 26th ed., W. H. Beyer, ed., pp. 316 and 319.

[4]The 1984 Astronomical Almanac, U. S. Government Printing Office, Washington, D. C.

[5]See also, B. M. Oliver, Sky and Telescope, July 1972, p. 20.

Quantum mechanics of a chargeless spinning particle in a periodic magnetic field: A simple, soluble system

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The Schrödinger equation for a spin-$\frac{1}{2}$ neutral particle with a magnetic dipole moment interacting with a helical periodic magnetic field of arbitrary strength and period is solved exactly. The solution is easily obtained by exploiting the symmetries of the Hamiltonian and it is expressed in terms of elementary functions. Several interesting physical aspects of the solution emerge. The behavior of the spin, the group velocity, and the effective mass tensor are obtained, yielding some novel and nontrivial results. Because of the mathematical simplicity, this problem is particularly suitable for an elementary graduate course in quantum mechanics.

I. INTRODUCTION

The most outstanding quantum mechanical behavior of matter is, even for the most simple system, somewhat obscured by the mathematics. Perhaps with the sole exception of a free particle in a square well, the most commonly studied systems such as the harmonic oscillator, the hydrogen atom, or charged particles in constant electromagnetic fields require, at some stage, the use of either special functions or very formal operator manipulations in order to extract the relevant quantum properties from the mathematical solutions. An even worse situation is found in some other physically interesting systems of greater complexity such as an electron in a periodic lattice. This problem usually defies an exact mathematical solution and, therefore, one must resort to elaborate approximation methods to exhibit the physical content of the theory. The concept of energy bands, group velocities, or effective masses can only be elucidated after several approximations and often intrincate calculations.

Among the great variety of mathematical approaches which have been developed to study such problems, a prominent role is played by the systematic use of the symmetries of the system. In fact, it has provided a very elegant and powerful technique which not only has greatly simplified the computations leading to the solution, but has also yielded great physical insight. Moreover, it has established the fundamental link between symmetries and conservation laws. In the particular case of a particle moving in a periodic potential, the Bloch theorem which results from the discrete translational symmetry of the interaction, introduces an important simplification in the energy eigenvalue problem and sets the stage for various mathematical approximations.

It is the purpose of this paper to study a system within this category. The important difference is that it can be solved exactly by elementary methods. We will consider an electrically neutral spin-$\frac{1}{2}$ particle with a nonvanishing magnetic moment and interacting with an external helical magnetic field. Because of the simplicity of the resulting eigenvalue problem, we will be able to derive some of the physical concepts mentioned above in an exact and simple way, thus avoiding the use of complex mathematics. Moreover, the great power of the symmetry principles and the corresponding conservation laws will be corroborated once again. It should be mentioned at this point that this problem is not only of academic interest. In fact, several real systems lead to an eigenvalue equation of this form. In particular the motion of slow neutron interacting with the magnetic field produced by some rare-earth elements can be described, to a good approximation, in this form.[1]

It should be mentioned, however, that this problem represents a somewhat exceptional case of a particle interacting with a periodic field. In fact, its solution does not exhibit the typical feature of periodic potentials, which is the appearance of energy bands separated by gaps. As will be shown below, this fact is a consequence of the special symmetries which characterize this problem but which, on the other hand, make the mathematical solution so simple.

II. THE EIGENVALUE EQUATION

Let us consider a chargeless spin-$\frac{1}{2}$ particle of mass m and magnetic moment μ_0 interacting with an external helical magnetic field given by

$$\mathbf{B}(y) = B_0 \begin{pmatrix} \cos \kappa y \\ 0 \\ \sin \kappa y \end{pmatrix}, \tag{1}$$

where B_0 is the constant field strength and $\kappa = 2\pi/\Lambda$ with Λ being the constant helix pitch (see Fig. 1). Some examples of materials which produce these kind of fields are discussed in Ref. 1. The corresponding eigenvalue equation

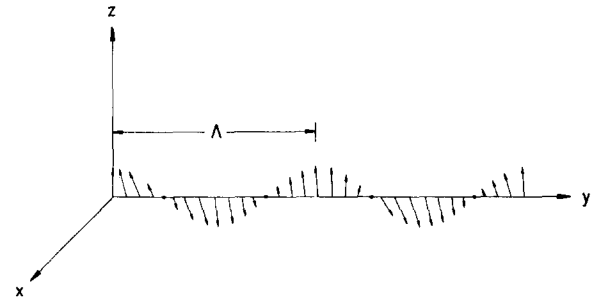

Fig. 1. The helix represents the magnetic field **B** at each (x,z) plane as a function of y. Λ is the period of the field.

is

$$\hat{H}\psi(\mathbf{x}) = E\psi(\mathbf{x}), \tag{2}$$

where

$$\hat{H} = (\mathbf{p}^2/2m) - \mu_0 \boldsymbol{\sigma}\cdot\mathbf{B} \tag{3}$$

and

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi^{(1)}(\mathbf{x}) \\ \psi^{(2)}(\mathbf{x}) \end{pmatrix}, \tag{4}$$

$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. In order to obtain the eigenenergies and eigenfunctions of Eq. (2) we note that the Hamiltonian is independent of the x and z coordinates and, consequently, the canonical momenta conjugate to them are conserved

$$[\hat{p}_x, \hat{H}\,] = [\hat{p}_z, \hat{H}\,] = 0. \tag{5}$$

This implies that the eigenfunctions $\psi(\mathbf{x})$ can also be eigenfunctions of $\hat{p}_x$ and $\hat{p}_z$

$$\psi(\mathbf{x}) = e^{ip_x x/\hbar} e^{ip_z z/\hbar} \phi(y) = e^{i\mathbf{p}_t \cdot \mathbf{x}} \phi(y), \tag{6}$$

where $\mathbf{p}_t = (p_x, 0, p_z)$ are the corresponding eigenvalues and $\phi(y)$ is an unknown spinor wavefunction. Substituting Eq. (6) into Eq. (2) one easily derives that

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} - \mu_0 B_0 \begin{bmatrix} \sin \kappa y & \cos \kappa y \\ \cos \kappa y & -\sin \kappa y \end{bmatrix} \right) \phi(y) = \left(E - \frac{\mathbf{p}_t^2}{2m} \right) \phi(y). \tag{7}$$

At first sight it would seem that this system of coupled eigenvalue differential equations of Mathieu type[2] do not admit solutions which can be expressed in terms of simple functions. Let us show that this is not the case. In fact the symmetries of $\hat{H}$ will allow us to obtain the spectrum in a very simple way. It is obvious from Eq. (1) that the system is not invariant under arbitrary displacements along the y axis except by Λ or multiple periods of it. There is, however, a combined operation consisting of an arbitrary displacement followed by an appropriate rotation around the y axis which constitutes a continuous symmetry of $\hat{H}$. This is equivalent to "screwing" the helix along the y direction. It is then easily shown that this transformation is generated by an operator $\hat{\rho}$ which

$$[\hat{\rho}, \hat{H}] = 0 \tag{8}$$

with

$$\hat{\rho} = \hat{p}_y + (\hbar\kappa/2)\sigma_y. \tag{9}$$

Furthermore, it is obvious that

$$[\hat{\rho}, \hat{p}_x] = [\hat{\rho}, \hat{p}_z] = 0 \tag{10}$$

and, therefore, we can also diagonalize

$$\hat{\rho}\phi(y) = p'\phi(y), \tag{11}$$

where p' is the eigenvalue. Let's write

$$\phi(y) = e^{ip'y/\hbar}\tilde{\phi}_{p'}(y). \tag{12}$$

Substituting Eq. (12) into Eq. (11) we obtain

$$i\hbar \frac{d}{dy}\tilde{\phi}_{p'}(y) = \frac{\hbar\kappa}{2}\sigma_y \tilde{\phi}_{p'}(y). \tag{13}$$

This equation has the same form as a time-dependent Schrödinger equation for a two-level system and is easily integrated

$$\tilde{\phi}_{p'}(y) = e^{-i(\kappa y/2)\sigma_y}\bar{\phi}_0(p'), \tag{14}$$

where $\bar{\phi}_0(p')$ is a constant but undetermined spinor. Thus combining Eqs. (6), (12), and (14) we obtain

$$\psi(\mathbf{x}) = e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{(i\kappa y/2)\sigma_y}\bar{\phi}_0(p'), \tag{15}$$

where $\mathbf{p} = (p_x, p', p_z)$. In order to determine $\bar{\phi}_0(p')$ in Eq. (15) we must substitute Eq. (15) for $\psi(x)$ in Eq. (2) yielding

$$\left[\mathbf{p}^2 + \left(\frac{\hbar\kappa}{2}\right)^2 - 2mE + \hbar\kappa p'\sigma_y - 2m\,\mu_0 B_0 \sigma_x\right] \times \begin{pmatrix}\bar{\phi}_0^{(1)}(p') \\ \bar{\phi}_0^{(2)}(p')\end{pmatrix} = 0. \tag{16}$$

This equation admits nontrivial solutions provided its secular determinant vanishes. This requires that

$$E(p,\pm) = \frac{1}{2m}\left\{p^2 + \left(\frac{\hbar\kappa}{2}\right)^2 \pm [(\hbar\kappa p')^2 + (2m\,\mu_0 B_0)^2]^{1/2}\right\}, \tag{17}$$

and the corresponding eigenvectors can be written as

$$\bar{\phi}_0^{(\pm)}(p') = \exp\left(i\frac{\theta(p')}{2}\sigma_x\right)\eta_\pm, \tag{18}$$

where

$$\eta_+ = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \eta_- = \begin{pmatrix}0\\1\end{pmatrix}. \tag{19}$$

The angular parameter $\theta(p')$ is given by

$$\tan\frac{\theta(p')}{2} = \frac{\hbar\kappa p'}{[(\hbar\kappa p')^2 + (2m\,\mu_0 B_0)^2]^{1/2} + 2m\,\mu_0 B_0}. \tag{20}$$

This function is depicted in Fig. 2. Finally the normalized energy eigenfunctions can be written as

$$\psi_{\mathbf{p},\pm}(\mathbf{x}) = \frac{1}{\sqrt{N}}e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}D_y(\kappa y)D_x[\theta(p')]\eta_\pm, \tag{21}$$

with N a proper normalization factor and where D_x and D_y are 2×2 rotation matrices around the x and y axes by angles κy and $\theta(p')$, respectively.

The physical behavior of the spin of these eigenstates is easily derived from Eq. (21). Clearly for all states the spin rotates as a function of y in the same way as the magnetic field. However, for a given $p' \neq 0$ the spin and the **B** field are not parallel. Actually the former is rotated by an angle $\theta(p')$ in the $(\mathbf{B},y)$ plane for the $+$ case and by $\pi - \theta(p')$ for the $-$ case.

It is instructive to study some limiting cases of the spectrum Eq. (17). Let us first consider the $\kappa\to 0$ limit. In this

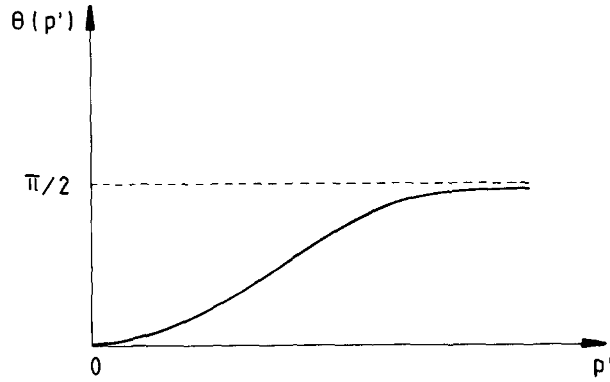

Fig. 2. The angle $\theta(p')$ as a function of p' is shown. This angle measures the relative orientation of the spin with respect to the **B** field. For $p' = 0$ the spin becomes parallel or antiparallel to **B**. For $p' \to \infty$ the spin is perpendicular to the magnetic field.

case we find

$$E(\mathbf{p},\pm) \to \frac{\mathbf{p}^2}{2m} \pm \mu_0 B_0 \tag{22}$$

in accordance with the constant field expression. On the other hand, the limit $B_0 \to 0$ and $\kappa \neq 0$ gives

$$E(\mathbf{p},\pm) = \frac{1}{2m}\left[p_x^2 + \left(p' \pm \frac{\hbar\kappa}{2}\right)^2 + p_z^2\right]. \tag{23}$$

This expression does not coincide with standard free particle result. This apparent discrepancy is related to the nonuniqueness in the choice of a complete set of commuting operators. Normally the operator $\hat{p}_y$ is chosen in this set. However, we may also choose $\hat{\rho}$ rather than $\hat{p}_y$. Then the free particle energy spectrum is parametrized differently.

One further aspect of the spectrum is worth noticing. It is well known that particles moving under the effect of an arbitrary periodic potential have energy bands separated by gaps. In the present case, we also have a periodic interaction, however, we have not obtained any discontinuities in $E_{\mathbf{p},\pm}$. The reason for the absence of gaps is related to the continuous symmetry of $\hat{H}$ generated by $\hat{\rho}$, which is absent in the general case. We can understand this as follows: In a periodic potential the origin or the bands arises from the fact that there are regions in which the wavefunction is being strongly reflected. This occurs at those points in which the forces are stronger (namely, where the potential has a maximum derivative). These regions will occur with the same periodicity as the potential. Therefore, when the wavelength of the particle is twice the distance between these points, the interference between transmitted and reflected waves will produce standing waves and this gives rise to the energy gap. This situation, however, does not arise for a helical magnetic field. In this case, the force acting on the particle will have a constant strength everywhere along the difference of motion and consequently no standing waves and no bands.

Let us next evaluate some physically important properties of the system. We will first consider the group velocity of the eigenstates Eq. (21). Clearly the velocity along the x and z axes coincides with that of a free particle. On the other hand, the velocity along the y axis shows a very different behavior

$$\langle v_y \rangle = \int \psi^+_{\mathbf{p},\pm}(\mathbf{x}) \frac{\hat{p}_y}{m} \psi_{\mathbf{p},\pm}(\mathbf{x}) d^3x$$

$$= \frac{p'}{m}\left(1 \pm \frac{(\hbar\kappa)^2}{2[(\hbar\kappa p')^2 + (2m\mu_0 B_0)^2]^{1/2}}\right). \quad (24)$$

This result could be also derived from the general formula $\langle v_y \rangle = \partial E / \partial p'$. The most interesting feature of Eq. (24) is that, for a given p', the velocity depends on the spin orientation relative to the magnetic field. In fact, we note that for the + state the resulting velocity is always larger than that of the state with the same p' but in a constant field. For the − state, the situation is just the opposite. Moreover, we also find that if $(\hbar\kappa)^2/2m > 2\mu_0 B_0$ there is a range of p' for which particles with opposite spin orientations move in opposite directions.

Let us finally consider the effective mass. This is defined as the tensor

$$M_{ij} = \left[\frac{\partial^2 E}{\partial p_i \, \partial p_j}\right]^{-1} = \begin{bmatrix} m & 0 & 0 \\ 0 & m^*(p') & 0 \\ 0 & 0 & m \end{bmatrix}, \quad (25)$$

where $i,j = x,y,z$ (with $p_y = p'$), and

$$m^*_\pm(p') = \left[\frac{\partial^2 E}{\partial p'^2}\right]^{-1}$$

$$= m \Bigg/ \left(1 \pm \frac{(2m\mu_0 B_0)^2}{2\hbar\kappa\left[p'^2 + \left(\frac{2m\mu_0 B_0}{\hbar\kappa}\right)^2\right]^{3/2}}\right). \quad (26)$$

Clearly m^*_+ is always smaller than m but is, nevertheless, always positive. In contrast, m^*_- is always larger than m if $2\mu_0 B_0 > \hbar^2\kappa^2/2m$. If $\hbar^2\kappa^2/2m > 2\mu_0 B_0$, m^*_- will be infinite for some p'_0; for $p' < p'_0$ the effective mass m^*_- is negative. This concept is particularly useful in studying the response of the particle to the influence of an external electric field provided the validity of the semiclassical approximation[3] is justified. Obviously a neutral particle is not affected by an electric field. Nevertheless it can be easily shown that the results derived in this section are equally valid for charged particles provided we limit ourselves to the case $\mathbf{p}_t = 0$ (see Refs. 1 and 3).

Let us conclude by noting that this eigenvalue problem arises in other physical contexts.[4–6] Moreover, the discussion for higher spins follows along the same lines.[7]

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[1] M. Calvo, Phy. Rev. B **18**, 5073 (1978).
[2] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. I, p. 556.
[3] N. Ashcroft and D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York, 1976), p. 213.
[4] R. Elliot and F. Wedgewood, Proc. Phys. Soc. **84**, 63 (1964).
[5] J. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967).
[6] M. Calvo, Phys. Rev. D **17**, 1567 (1978).
[7] M. Calvo, Phys. Rev. A **21**, 1469 (1980).

Light scattering from fibers: An extension of a single-slit diffraction experiment

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The similarities and differences between diffraction from single slits and scattering from single fibers can be dramatically demonstrated by some simple experiments. Geometrical properties such as fiber size, cross section, tilt, and nonuniformity all contribute to the intricate detail of the diffraction pattern. An examination of the role of the complex refractive index, polarized light, and geometrical effects in scattering phenomena will contribute to a deeper understanding of the interaction of light with small apertures and particles.

I. INTRODUCTION

Light scattering from fibers is an active research area in optics motivated by interest in asbestos fiber pollution, optical pipes, and fibers for communication and natural phenomena such as halos due to ice crystals. Although neglected in most traditional optics courses, the nature of light scattered from fibers is easy to demonstrate and strikingly similar in concept to the well-known phenomena of single-slit diffraction. Single-slit diffraction is a basic experiment which demonstrates the wave nature of light and the role of optical size and wavelength in forming geometrical images and diffraction patterns. The experimental variations are limited however since a "perfect slit" is completely characterized by a single parameter, its width W. More parameters are needed to characterize fibers and consequently