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The hydrogen atom in one dimension

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The Schrödinger, Klein–Gordon, and Dirac equations for the hydrogen atom are solved in D dimensions, where D may be noninteger. For $D = 1$ the nondegenerate ground state with infinite binding energy is obtained as a limit as D tends to 1 for the Schrödinger equation. The corresponding solution of the Klein–Gordon equation is shown to be unacceptable, contradicting previous work. Unexpectedly, no bound state solutions of the Dirac equation with $D = 1$ are found at all. A few comments are made on the $D = 3$ and 2 cases.

I. INTRODUCTION

Discussion of the hydrogen atom in one, two, and three dimensions is a regular feature of this Journal. Justification for this continuing interest has been given recently by Spector and Lee[1]; their paper gives references through which much of the earlier literature may be traced.

Attention has been directed in particular to the case of the hydrogen atom in one dimension, since problems have emerged. One concerns the existence of even as well as odd wavefunctions and the consequent double degeneracy of all energy levels, with the possible exception of the ground state. Another is the question of whether this nondegenerate ground state exists, since it would have infinite binding energy and a wavefunction located entirely at the proton. Since the potential is singular, no condition is imposed on the derivative of the wavefunction at the proton; only the wavefunction itself must be continuous there so that, provided the wavefunction vanishes at the proton, both even and odd wavefunctions are allowed (see, for example, Ref. 2).

The existence of the nondegenerate ground state is the main concern of this work. It has been approached[3] through the use of rounded or truncated potentials, for which this state does exist. Recently the influence of special relativity has been considered[1] by solving the Klein–Gordon equation for the hydrogen atom in one dimension and showing that there exists a state which has finite binding energy but which, in the nonrelativistic limit, corresponds to the state of interest. It was recognized[1] that it is the Dirac equation that is the appropriate relativistic equation for the electron, but it was anticipated that similar conclusions would have been obtained if the Dirac equation had been used.

In this article the problem is approached in yet another way by solving the wave equation in D dimensions, where D is arbitrary. This has been done before, for example by Nieto,[4] but here D is regarded as a continuous parameter so that the one-dimensional problem may be approached by taking the limit as D tends to unity; indeed the symbol D, rather than N or n, has been chosen for the number of dimensions to emphasize the possibility of noninteger dimensionality. Justification for the use of fractional dimensions has been given by Stillinger[5] and the possibility that the number of space-time dimensions deviate slightly from 4 is topical.[6] Another reason for using arbitrary D is that the $D = 3$ and 2 cases may be recovered; some new results for these are mentioned in passing. It should be noted that only the radial part of the problem is of concern here, with the results[5,7] for the angular part adopted without comment.

In Sec. II the techniques used in this article are introduced. Section III includes an account of the new approach to the one-dimensional case of the Schrödinger equation by taking the limit as D tends to unity. In Sec. IV the Klein–Gordon equation is solved; previous work is critcized and, in particular, the ground state solution found by Spector and Lee[1] is shown to be unacceptable. The Dirac equation is considered in Sec. V, where for $D = 1$ the unexpected result, that there are no bound state solutions at all, obtained.

II. PRELIMINARIES

Atomic units are employed. Accordingly the Coulomb potential is taken to be $V = -Z/r$ for all values of D; for $D = 1$ the radial coordinate r is to be interpreted as $|r|$. A convenient source for the main results concerning D dimensions has been presented by Louck.[8] The volume element is $d\tau = r^{D-1}\,dr$, while the Laplacian is given by

$$\nabla^2 = \frac{d^2}{dr^2} + \left(\frac{D-1}{r}\right)\frac{d}{dr} - \frac{l(l+D-2)}{r^2}. \tag{1}$$

The Dirac equation contains first derivatives with respect to spatial coordinates, rather than second, but consideration of the relevant changes will be postponed until Sec. V since they are not relevant to the other wave equations.

To solve the differential equations that occur it is sufficient for our purposes to refer to Eq. 22.6.17 of Ref. 9:

$$y'' + [(2q+\gamma+1)/2x + (1-\gamma^2)/4x^2 - \tfrac{1}{4}]y = 0, \tag{2}$$

where

$$y = \exp(-x/2)x^{(\gamma+1)/2}L_q^{(\gamma)}(x), \tag{3}$$

being $L_q^{(\gamma)}(x)$ a generalized Laguerre polynomial; $q = 0,1,2...$ is used here (following Schiff[10]), since n is reserved for the principal quantum number, and γ is employed instead of α. These solutions of Eq. (2) have been chosen to be normalizable so that their behavior at infinity has already been taken into account. It remains for us to consider the small distance behavior of the solutions. For $D = 1$ it is only necessary for the solution to vanish at the nucleus for odd wavefunctions; even wavefunctions may be finite there. Normalizability for bound states requires no comment except that it is more restrictive to insist that the expectation value of the kinetic energy be finite (see, for example, Ref. 11); the expectation value of the potential energy is also expected to be finite. Finally, the behavior of the solution at the origin must be such that the operator ∇^2 is Hermitian.

In investigating whether integrals involving products of generalized Laguerre polynomials are finite it is usually sufficient to take the small x behavior to be

$$L_q^{(\gamma)}(x) \simeq 1 + O(x), \tag{4}$$

but, if γ is a negative integer $(-m)$, this is not so. It appears that

$$L_q^{(-m)}(x) = (-1)^m[(q-m)!/q!]x^m L_{q-m}^{(m)}(x), \tag{5}$$

provided that $q \geqslant m$, so that for small x

$$L_q^{(-m)}(x) \simeq x^m + O(x^{m+1}). \tag{6}$$

We have been unable to find relation (5) in the literature, except for the case $m = q$ (for example, 8.973.4 of Ref. 12). Also, some of the definitions and relationships for generalized Laguerre polynomials are not applicable to negative γ, but such polynomials do satisfy Eq. (2). Relation (5) may readily be proved using the definition (see Eq. 8.970.1 of Ref. 12)

$$L_q^{(\gamma)}(x) = (q!)^{-1}\exp(x)x^{-\gamma}d^q[\exp(-x)x^{q+\gamma}]/dx^q. \tag{7}$$

Finally, there seems to be some variation in the use of the principal quantum number for the $D = 1$ case. Here the $D = 3$ conventions are followed. This means that $n = 1,2,3...$ is the principal quantum number.

III. THE SCHRÖDINGER EQUATION

In atomic units the radial part of the Schrödinger equation in D dimensions is

$$\psi'' + [(D-1)/r]\psi' + [2Z/r - l(l+D-2)/r^2 + 2E]\psi = 0. \tag{8}$$

This may be transformed to the form (2) by changing the variable to x, where $x^2 = -8Er^2$, and by letting $\psi = r^{-(D-1)/2}f$ so that

$$f'' + \{[Z/(-2E)^{1/2}]/x - [(2l+D-2)^2 - 1]/4x^2 - \tfrac{1}{4}\}f = 0. \tag{9}$$

Comparison with Eq. (2) then leads to

$$\gamma = \pm(2l + D - 2) \tag{10}$$

and, introducing the principal quantum number

$$n = q + l + 1, \tag{11}$$

the energy

$$E = -2Z^2/(2n - 2l + \gamma - 1)^2. \tag{12}$$

The wavefunction is

$$\psi = \exp(-x/2)x^{(\gamma - D + 2)/2}L_{n-l-1}^{(\gamma)}(x), \tag{13}$$

where $x = 4Zr/(2n - 2l + \gamma - 1)$.

Recalling the small distance behavior (4) of the generalized Laguerre polynominals, it can be seen that the lowest power of r (or x) in the integrand of the normalization integral is $(\gamma + 1)$ so that for normalization to be possible $\gamma > -2$. However, for the expectation value of the potential energy to be finite $\gamma > -1$, while for the kinetic energy the same condition must apply since the kinetic energy is related to the potential energy through the Schrödinger equation. For the operator ∇^2 to be Hermitian the behavior of the wavefunction at the origin is restricted such that $\gamma > 0$. It will be seen shortly that in a sense these considerations are not relevant because of relation (5).

For $D = 1$ the only value that l can take is 0, so that $(2l + D - 2)$ is negative and any equations developed for general D, such as those given by Nieto,[4] should be used with care. Accordingly, $\gamma = \pm(-1) = \pm 1$ and for $\gamma = +1$ familiar results[3] are obtained:

$$E = -Z^2/2n^2,\ \psi = \exp(-x/2)xL_{n-1}^{(1)}(x),\ x = 2Zr/n, \tag{14}$$

where, as throughout this article, $n = 1,2,3...$. All these wavefunctions vanish at $x = 0$ so both odd and even solutions are acceptable with the associated double degeneracy. For $\gamma = -1$ the solutions are

$$E = -Z^2/2(n-1)^2,\ \psi = \exp(-x/2)L_{n-1}^{(-1)}(x),\ x = 2Zr/(n-1). \tag{15}$$

In this case relation (5) with $m = 1$ ensures that for $n > 1$ these solutions just reproduce the others with principal quantum number $(n - 1)$. However, for $n = 1$ in (15) the old problem of an infinitely bound state located entirely at the nucleus reappears.

A novel approach to this question is to let $D = 1 + \delta$ and to take the limit as δ tends to 0 when appropriate. Then

$$E = -2Z^2/\delta^2,\ \psi = \exp(-x/2)L_0^{(-1+\delta)}(x) = \exp(-x/2),\ x = 4Zr/\delta. \tag{16}$$

Using $r^\delta\, dr$ as the volume element, normalization of (16) is straightforward, the normalization constant is $N = [(4Z/\delta)^{1+\delta}/\Gamma(1+\delta)]^{1/2}$. The wavefunction may thus be regarded as a representation of a Dirac delta function in the limit when δ tends to 0. However, since it does not vanish at $x = 0$, an odd wavefunction cannot be constructed from it and the state is nondegenerate. The expectation values of the potential and kinetic energies are readily determined to be $-4Z^2/\delta^2$ and $2Z^2/\delta^2$, respectively, so that the virial theorem is satisfied. Of course, when δ is allowed to tend to 0, the potential, kinetic, and total energies all diverge. Thus, as for the limit of rounded or truncated potentials,[3] the nondegenerate ground state may be regarded as existing as a limit, in the case of (16) as $\delta \to 0$.

Relation (5) also means that some negative values of γ may be permitted for $D = 3$ and 2, although no new solutions are obtained. It is readily shown that, provided $n \geqslant (3l + D - 1)$, there is a duplication of some results obtained for positive γ. For example, if $D = 3$ and $l = 0$, the solution for $n = 3$ with γ chosen to be negative corresponds to the solution for $n = 2$ with the positive value of γ. These consequences of relation (5) do not appear to be well known.

IV. THE KLEIN–GORDON EQUATION

The Klein–Gordon equation for an hydrogenic atom is

$$(\epsilon + Z\alpha^2/r)^2\psi = (1 - \alpha^2\nabla^2)\psi, \tag{17}$$

where the energy is given by $E = mc^2\epsilon$ and α is the fine structure constant. Changing the variable to x, where $x^2 = 4(1-\epsilon^2)r^2/\alpha^2$, letting $\psi = x^{-(D-1)/2}f$, and introducing $\lambda = Z\alpha\epsilon/(1-\epsilon^2)^{1/2}$, so that

$$E = mc^2[1 + (Z\alpha/\lambda)^2]^{-1/2}, \tag{18}$$

Eq. (17) becomes

$$f'' + \{\lambda/x + [Z^2\alpha^2 - l(l+D-2) - (D-1)(D-3)/4]/x^2 - \tfrac{1}{4}\}f = 0. \qquad (19)$$

Comparison with Eq. (2) shows that

$$\gamma = \pm [(2l+D-2)^2 - 4Z^2\alpha^2]^{1/2} \qquad (20)$$

and

$$\lambda = n - l + \gamma/2 - \tfrac{1}{2}, \qquad (21)$$

where (11) has been used to introduce the principal quantum number. The wavefunction is given by the same expression (13) as for the Schrödinger equation, but the relationship between x and r is more complicated. The major difference from Sec. III is that γ is no longer an integer and relation (5) will not be relevant to the discussion. As earlier, the condition for normalizability is $\gamma > -2$ and for finite potential energy $\gamma > -1$. However, the Klein–Gordon equation contains the square of the potential energy and also the operator ∇^2. Although the kinetic energy in a relativistic theory is a more complicated function of the momentum than in a nonrelativistic theory, it seems reasonable that the expectation values of $1/r^2$ and ∇^2 should both be finite, as they both occur in the Klein–Gordon equation. Of course, these expectation values could contain compensating divergent contributions, but the condition $\gamma > 0$ is imposed here since this is the same restriction needed for Hermiticity of ∇^2 to be ensured.

Since l may only be 0 for $D = 1$,

$$\gamma = \pm (1 - 4Z^2\alpha^2)^{1/2}. \qquad (22)$$

The wavefunction is

$$\psi = \exp(-x/2)x^{(\gamma+1)/2}L_{n-1}^{(\gamma)}(x). \qquad (23)$$

For the positive value of γ

$$E = mc^2\{1 + 4Z^2\alpha^2[2n - 1 + (1-4Z^2\alpha^2)^{1/2}]^{-2}\}^{-1/2} \qquad (24)$$

$$= mc^2 + mc^2\alpha^2\{-(Z^2/2n^2) + (Z^4\alpha^2/2n^3)[(3/4n) - 2] + O(\alpha^4)\}. \qquad (25)$$

Equation (24) may be obtained from Nieto,[4] but his power series expansion should not be used since it assumes that $(2l + D - 2) > 0$. Expression (24) also agrees with Spector and Lee[1]; note that their $n = 0,1,2...$ is the same as q here and so corresponds to our $n = 1,2,3...$. Equation (25) also agrees with Ref. 1 provided allowance is made for typographical errors and for the fact that Spector and Lee change their definition of n during their expansion to that used throughout here. Note also that the $n = 0$ case quoted in Ref. 1 does not arise from (24). In the nonrelativistic limit the wavefunction (23) is the same (14) as for the Schrödinger equation. In addition, the nonrelativistic limit of the energy is the same as Eq. (14) when allowance is made for the electron rest energy and it is noted that $mc^2\alpha^2$ is just the atomic unit of energy.

If it is supposed that the negative value of γ leads to acceptable solutions, the energy is given by

$$E = mc^2\{1 + 4Z^2\alpha^2[2n - 1 - (1-4Z^2\alpha^2)^{1/2}]^{-2}\}^{-1/2}. \qquad (26)$$

For $n > 1$ this may be expanded as

$$E = mc^2 + mc^2\alpha^2[-Z^2/2(n-1)^2] + [Z^4\alpha^2/2(n-1)^3][3/4(n-1) + 2] + O(\alpha^4)\}, \qquad (27)$$

but this possibility is ignored in Ref. 1. Since γ is noninteger, relation (5) does not hold and the solutions with negative γ do not reproduce solutions with positive γ. Indeed, Eqs. (26) and (27) do not give quite the same energies as Eqs. (24) and (25) when values of n differing by unity are substituted. Thus there would be a small relativistic splitting of the energy levels if these solutions were acceptable.

Before considering this question, the case $n = 1$ should be mentioned. Although (27) is clearly not appropriate, the energy may nevertheless be expanded as a power series in α with the result

$$E = mc^2[Z\alpha + (\tfrac{1}{2})Z^3\alpha^3 + O(\alpha^5)]. \qquad (28)$$

The nonrelativistic limit of the wavefunction is $\psi = \exp(-x/2)$ with $x = 2r/\alpha$. This is the solution found by Spector and Lee[1] (although they label it $n = 0$). It would be the ground state of the one-dimensional hydrogen atom with finite binding energy and localized to the order of the Compton wavelength from the nucleus. It would correspond to the nonrelativistic state with infinite binding energy and a delta function wavefunction. However, if this solution is accepted, so should all others with negative values of γ.

It has already been indicated that the wavefunction (23) is normalizable and has a finite expectation value for the potential energy, even for the negative value of γ, since it is just greater than -1. However, the square of the potential energy and the operator ∇^2, both of which appear in the Klein–Gordon equation, have infinite expectation values for $\gamma \leqslant 0$; in addition, to retain the Hermitian property for ∇^2 the wavefunction must have $\gamma > 0$. This is true not only for $n = 1$ but for all n, since γ is noninteger and there is no special relation for the generalized Laguerre polynomials. It should be noted that the device of letting $D = 1 + \delta$ does not help here since the integrals diverge even for nonzero δ.

Either all or none of these solutions with negative γ should be accepted. It is true, in the nonrelativistic limit, these solutions become acceptable and the Schrödinger equation solutions discussed earlier are recovered, but only because (5) then comes into operation. It seems expedient to reject all the solutions with negative γ. Appealing to the Klein–Gordon equation does not provide any additional support for the questionable ground state of the Schrödinger equation for the one-dimensional hydrogen atom. This conclusion is in contradiction to that of Spector and Lee,[1] who find the negative γ solution with $n = 1$ (their $n = 0$) to be acceptable, but do not consider the negative γ solutions with higher n.

In passing we note that the $D = 3$ case has been studied in textbooks (see, for example, Ref. 11). Only positive values of γ lead to acceptable wavefunctions and the Schrödinger equation result is recovered in the nonrelativistic limit. We are not aware of the $D = 2$ case having been considered before; Nieto[4] treats general D but not the specific value of 2. Here $l = 0$ is not allowed, since γ is then complex; as pointed out in Ref. 11, when studying the $D = 3$ case, this would mean that the wavefunction had infinitely many oscillations. A consequence of this is that there is no $n = 1$ state and the ground state has $n = 2$ and $l = 1$.

V. THE DIRAC EQUATION

The correct description of the electron in an hydrogenic atom is, of course, given by the Dirac equation. However, since the results of this section are largely negative and the background fairly copious, only sufficient details for present purposes are given.

The Dirac equation may be written in polar coordinates:

$$mc^2\left[\beta - i\alpha\rho\Sigma\left(\frac{\partial}{\partial r} - \frac{\Lambda}{r}\right) - \frac{Z\alpha^2}{r}\right]\Psi = E\,\Psi = mc^2\epsilon\,\Psi \tag{29}$$

(see, for example, Refs. 13 and 14). Here β and ρ are Dirac matrices, while in three dimensions the operators Σ and Λ are given by $\Sigma = (\boldsymbol{\sigma}\cdot\mathbf{r})/r$ and $\Lambda = \boldsymbol{\sigma}\cdot\mathbf{l}$. Without going into detail, the solutions of the Dirac equation may be written as [7,14]

$$\Psi = \begin{pmatrix} f(r)\psi_l^{(a)} \\ ig(r)\psi_{l+1}^{(b)} \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} f(r)\psi_{l+1}^{(b)} \\ ig(r)\psi_l^{(a)} \end{pmatrix}, \tag{30}$$

where the behavior of the angular functions under the angular operators is given by

$$\Lambda\psi_l^{(a)} = l\psi_l^{(a)},\ \Lambda\psi_l^{(b)} = -(l+D-2)\psi_l^{(b)},$$
$$\Sigma\psi_l^{(a)} = -\psi_{l+1}^{(b)},\ \Sigma\psi_{l+1}^{(b)} = -\psi_l^{(a)}. \tag{31}$$

Substitution of (30) into (29) gives two pairs of first-order differential equations for the radial functions f and g. After much manipulation these functions may be expressed in terms of the solutions of just one second-order differential equation:

$$y'' + \{[1 + (2\epsilon Z\alpha/\lambda)]/2x + [4Z^2\alpha^2 - (2l+D)(2l+D-2)]/4x^2 - \tfrac{1}{4}\}y = 0, \tag{32}$$

where $\lambda = (1-\epsilon^2)^{1/2}$. Comparison with (2) shows that

$$\gamma = \pm[(2l+D-1)^2 - 4Z^2\alpha^2]^{1/2}; \tag{33}$$

this is to be contrasted with the corresponding Eq. (20) for the Klein–Gordon equation. Further comparison of (32) with (2) leads to the energy

$$E = mc^2[1 + 4Z^2\alpha^2(2n - 2l - 2 + \gamma)^{-2}]^{-1/2}. \tag{34}$$

It is rather more difficult to see whether soltutions are acceptable wavefunctions, since f and g are both linear combinations of products of exponentials, powers of r, and generalized Laguerre polynomials, so that only the results are stated.

For $D = 1$ it might be noted that the Dirac equation may be set up by following Lapidus,[15] who considered a delta function potential, but the same results emerge. Since $l = 0$, Eq. (33) gives $\gamma = \pm(-4Z^2\alpha^2)^{1/2}$. As for the $D = 2$, $l = 0$ case for the Klein–Gordon equation, imaginary values of γ are not acceptable[11] and it must be concluded that the Dirac equation for the electron in a one-dimensional hydrogenic atom has no bound state solutions. This appears to be a new result and shows that the expectation of Spector and Lee,[1] that results for the Dirac equation would differ only in terms of order $Z^4\alpha^4$ from results for the Klein–Gordon equation, is quite incorrect.

Finally, we note that for $D = 2$ the $l = 0$ case does not prove an embarrassment in contrast to the Klein–Gordon equation.

VI. CONCLUSIONS

In this article the Schrödinger, Klein–Gordon, and Dirac equations have been solved for the bound states of the electron in a hydrogenlike atom in D dimensions, where D is arbitrary and may be noninteger. A number of new results have been noted even for $D = 3$ and 2. The novel relation (5) concerning generalized Laguerre polynomials throws light on acceptable solutions of the Schrödinger equation and yet does not appear to have been noted before. For $D = 2$ it has been shown that there are no solutions of the Klein–Gordon equation if the angular momentum quantum number is 0, although this does not happen for the Dirac equation; the Klein–Gordon equation thus suggests that the ground state has $l = 1$ and principal quantum number $n = 2$.

For $D = 1$ the existence or otherwise of a nondegenerate ground state with infinite binding energy is a question that has attracted interest for at least 25 years. The solution of the Schrödinger equation has been identified as a limit as D tends to unity. The corresponding solution of the Klein–Gordon equation has been shown to be unacceptable, in disagreement with the conclusion of Ref. 1. Finally, no bound state solutions of the Dirac equation have been found at all, which is a quite unexpected result.

One should not conclude that the physical systems, which have revived interest in this problem,[1] do not exist or have solutions missing, but merely that the models employed to describe them may be unsuitable; for example, giving the nucleus finite size might be appropriate. Of course, this does not remove the puzzling problem that some relativistic solutions to the Klein–Gordon and Dirac equations do not reduce to the corresponding Schrödinger results in the nonrelativistic limit. However, this situation occurs in other cases. For example, the Dirac equation ($D = 3$) has no bound state electron solutions for $Z \geqslant 119$ (see, for example, Ref. 16). In addition, Plesset[17] showed for $D = 3$ that, of attractive polynomial power potentials that give bound state solutions for the Schrödinger equation, only the Coulombic potential has bound state solutions for the Dirac equation. This may be regarded as a manifestation of Klein's paradox (see, for example Ref. 18); a similar explanation is responsible for the results reported here.

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Cosmic strings: Gravitation without local curvature

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Cosmic strings are very long, thin structures which might stretch over vast reaches of the universe. If they exist, they would have been formed during phase transitions in the very early universe. The space-time surrounding a straight cosmic string is flat but nontrivial: A two-dimensional spatial section is a cone rather than a plane. This feature leads to unique gravitational effects. The flatness of the cone means that many of the gravitational effects can be understood with no mathematics beyond trigonometry. This includes the observational predictions of the double imaging of quasars and the truncation of the images of galaxies.

I. INTRODUCTION

The techniques and results of Einstein's general theory of relativity are on the whole far too complicated to present to beginning students of physics. We can describe qualitatively the beautiful picture of gravity as curved space-time, and some of the exotic features of black holes and relativistic cosmologies, but the details are tough. Without the physical and mathematical preparation required for metric tensors, stress-energy tensors, and the field equations, beginning students can normally be shown only a few handwaved results, which must seem to be surrounded by a certain aura of magic, since they can work out none of these results for themselves.

We concentrate instead on Newtonian gravitation, which is not only mathematically simpler and of enormous importance historically, but is usually such an excellent approximation that it would be unthinkable not to teach how to use it.

It is therefore astonishing to find that there is a simple but nontrivial solution of Einstein's equations which is potentially of great importance in observational astronomy, cosmology, and particle physics, which has been much in the news, yet which can also easily be understood by students who have had very little physics and no mathematics beyond trigonometry. In fact, the behavior of particles in this space-time is both very different and very much simpler than the analogous, but unimportant, problem in Newtonian physics. The space-time in question is that surrounding a straight cosmic string.[1–5]

II. COSMIC STRINGS

Recent attempts to marry the grand unified theories of particle physics with general relativistic models of the early evolution of the universe have predicted the possible existence of enormously long objects called cosmic strings.[3–8] Grand unified theories try to unite the electromagnetic, weak, and strong nuclear forces into a single theory, in which the forces merge together at very high energies. Below energies of about 10^{15} GeV, the strong force is separated from the electroweak force; below energies of about 100 GeV, the weak and electromagnetic forces separate, leaving the three familiar forces at low energies, each with a different strength. The gravitational force is not incorporated into this scheme; it is thought that if gravity is eventually united with the other forces, this must take place at even higher energies, perhaps above 10^{19} GeV.

Grand unified energies of 10^{15} GeV cannot be achieved in the laboratory, which explains the sudden interest of particle physicists in cosmology. The hot big bang scenario, for which there is impressive evidence, provides the needed energies in the very early universe. In this scenario, the universe was in thermal equilibrium and very hot at early times; it has subsequently cooled off as it expanded. The introduction of grand unified theories into the hot big bang models predicts phase transitions at the separation energies: That is, as the temperature of the universe falls to the point where typical particle energies kT correspond to one of the separation energies, the particles experience a phase transition. Perhaps the observed universe contains fossils

