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above, the photogate release detector ultimately suffers from this same fundamental problem.

The third scheme, listed above, utilized a mechanical suspension/release similar to the kind described by Blackburn and Koenig.[20] The suspended object itself is used to provide continuity in an electric circuit. At the instant of release, contact is broken, thus providing a release signal. Standard mechanical switches usually bounce over a period of about 1 ms. In this particular case the contacts open relatively slowly and so the problem is further exascerbated by exceptionally long bounces (sometimes up to 20 ms). Debouncing the switch is not a solution. One is left with the uncertainty of which bounce (during the bouncing period) corresponds to the release or the actual start of the free fall.

In summary, the object is to determine the time of release with an uncertainty of much less than 1 ms. The three methods discussed above have proven to be fundamentally unsuitable for such a measurement.

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[1]M. E. Brandon, M. Gutiérrez, R. Labbé, and A. Menchaca-Rocha, Am. J. Phys. **52**, 890 (1984).

[2]E. P. Manche, Am. J. Phys. **47**, 542 (1979).

[3]C. G. Adler and B. L. Coulter, Am. J. Phys. **46**, 199 (1978).

[4]J. A. Blackburn and R. Koenig, Am. J. Phys. **44**, 855 (1976).

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[9]*Physics Demonstration Experiments*, edited by H. F. Meiners (Ronald, New York, 1970), Vol. I, pp. 115–119.

[10]Reference 4.

[11]E. H. Hall, Phys. Rev. **17**, 245 (1903). R. V. Pound first brought this work to our attention. Yes, this is the same Hall that gave us the Hall effect in 1879.

[12]Hall performed his experiments in the tower of the Jefferson Physical Laboratory of Harvard University. As a historical note, this was not the last time that "objects" were dropped in the tower. R. V. Pound and G. A. Rebka, Jr. [Phys. Rev. Lett. **4**, 337 (1960)] and later with J. L. Snider [Phys. Rev. Lett. **13**, 539 (1964); Phys. Rev. **140**, B788 (1965)] used this same tower for their important and famous falling-photon experiment in which they determined the gravitational red shift.

[13]D. W. Latham, Smithsonian Astrophysical Observatory, Harvard University.

[14]J. W. Kane and M. M. Sternheim, *Life Science Physics* (Wiley, New York, 1978), pp. 256–258. This reference gives a good elementary discussion of the problem. Most texts on fluid mechanics contain a complete treatment. For example, the reader is referred to the classic texts of L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1959), pp. 168–172; and H. Schlichting, *Boundary Layer Theory* (McGraw-Hill, New York, 1960), pp. 11–18. The latter reference contains many interesting experimental results correlated with the theory.

[15]In the case of compressible fluids (when elastic forces are important) C_D is also a function of the Mach number ($M = v/c$). However, up to $M \ll 0.3$, the influence of the Mach number is negligible, as is the case here (maximum $M \approx 0.03$).

[16]G. D. Garland and K. Jung, *Handbuch der Physik*, edited by J. Bartels (Springer, Berlin, 1956). This calculated value turns out to be in excellent agreement with the experimental value of 980.38(6775) provided by the Charles Stark Draper Laboratory in Cambridge, MA. The significant digits in parentheses correspond to the local g value in *their* laboratory. Note that the number of significant figures in their measurement implies a sensitivity of ± 3 mm in altitude!

[17]The problem of determining the drag coefficient of various bodies and the dependence of the drag coefficient upon the various similarity parameters of the particular flow situation is a recurring problem in fluid mechanics. Specialized literature on this subject abounds. For example, the reader is referred to S. F. Hoerner, *Fluid-Dynamic Drag* (Midland Park, NJ, 1958), 2nd ed. for a useful summary of much data of this type.

[18]L. D. Landau and E. M. Lifschitz, Ref. 14, p. 170.

[19]B. L. Coulter and C. G. Adler, Am. J. Phys. **47**, 841 (1979).

[20]Reference 4.

Quantum strings

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The theory of the classical string and the method of canonical quantization are reviewed. The nonrelativistic string is then quantized using the canonical quantization procedure. A number of properties of quantized strings, such as quantized normal modes of vibration and ground state fluctuations, are discussed. The quantized nonrelativistic string is compared to quantized fields and to the strings in the recent superstring theories. The theory of relativistic strings is reviewed.

I. INTRODUCTION

Strings are physical objects which can be quantized by the canonical quantization procedure. The study of the quantization of strings is instructive for several reasons. The first is that strings, like harmonic oscillators, can be quantized exactly. Thus they serve as another example of an exactly solvable physical system in quantum mechanics and, as such, give insight into the mathematical structure of quantum theory. Second, unlike physical systems discussed in the standard textbook applications of quantum mechanics, strings are continuous. The quantization of strings therefore parallels the quantization of fields and may serve as an introduction to field quantization. Strings, however, are easier to conceptualize than fields. Insight into features of the quantization of continuous systems, such as fluctuations of the ground state, are more easily obtained in the study of quantized strings than in the study of quantized fields. However, much of the intuition obtained while working with quantized strings can be applied to the more mathematically technical study of quantized fields. In addition to the above, there has recently been a great deal of interest in the possibility of unifying gravity and matter within the framework of a quantum theory of relativistic strings.[1] Much of the basic mathematical framework of these theories is present in the theory of nonrelativistic quantized strings. This basic framework is easier to understand in the case of nonrelativistic strings due to the absence of complicating technical requirements such as relativistic invariance or supersymmetry. Thus the quantization of nonrelativistic strings is a good place to begin for a person wishing to obtain an understanding of the string theories presently being considered in elementary particle physics.

The quantization of nonrelativistic strings by the technique of canonical quantization is discussed in this article. Section II gives a review of the classical theory of strings (the adjective "nonrelativistic" will be assumed from this point on). In Sec. III the method of canonical quantization is discussed. As an example, this method is used to quantize the harmonic oscillator. The use of raising and lowering operators is also reviewed in Sec. III. In Sec. IV the canonical quantization procedure is applied to strings and in Sec. V the quantum ground state fluctuations of the string are discussed. Section VI reviews the main features of quantized nonrelativistic strings and gives a comparison to quantized fields and to quantized relativistic strings. An introduction to the theory of superstrings is also given in Sec. VI.

The more advanced discussions of quantum fields and quantum relativistic strings referenced at the end of this paper should be more easily accessible after reading the material in the present article.

II. THE CLASSICAL MECHANICS OF STRINGS

This section briefly reviews the classical mechanics of strings and sets the notation. For a more detailed discussion, refer to a textbook on classical mechanics.[2] Here it will be assumed that the string has tension T, mass per unit length, ρ, is tied between the points $x=0$ and $x=L$, and is restricted to vibrate in the x–y plane. The string at time t can then be described by the function $y(x,t)$ giving the displacement y from the x axis. Note that $y(0,t)=y(L,t)$. Here x is just a parameter labeling the points along the string and not a dynamical variable.

The momentum per unit length at point x and time t is

$$\pi(x,t)=\rho\,\frac{\partial y(x,t)}{\partial t}. \qquad (2.1)$$

The force on an infinitesimal segment of the string at point x is calculated by considering the difference between the forces due to the tension on each side of the segment. The result is $T(\partial^2 y/\partial x^2)$ for the force per unit length. Note that if the second derivative is positive (curvature is positive) then the force is in the $+y$ direction, and that if the second derivative is negative then the force is in the $-y$ direction.

The equation of motion, by Newton's second law, is

$$\rho\,\frac{\partial^2 y}{\partial t^2}-T\,\frac{\partial^2 y}{\partial x^2}=0. \qquad (2.2)$$

Note that Eq. (2.2) is a wave equation.

The kinetic energy of the string is the integral of the kinetic energy per unit length along the string:

$$\mathrm{KE}=\int_0^L\left[\frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2\right]dx. \qquad (2.3)$$

The potential energy of the string is

$$\mathrm{PE}=\int_0^L\left[\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right]dx. \qquad (2.4)$$

Thus the Hamiltonian H is

$$H=\int_0^L\left[\frac{1}{2\rho}\pi^2+\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right]dx. \qquad (2.5)$$

The function $y(x,t)$ vanishes at the endpoints and therefore can be expanded in a Fourier sine series:

$$y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin[n\pi(x/L)]. \qquad (2.6)$$

Note that for fixed time t, the state of the string is given either by the function $y(x,t)$ or by the countably infinite set of real numbers $b_n(t)$.

In the classical mechanical discussion of strings, the goal is to find the state of the string $y(x,t)$ for all positive t given the two functions $y(x,0)$ and $\dot{y}(x,0)$. (An overdot above a function will signify its time derivative.) The solution to the equation of motion is found by substituting the expansion (2.6) into (2.2):

$$y(x,t)=\sum_{n=1}^{\infty}\frac{1}{2}\,[\beta_n(t)+\beta_n^*(t)]\sin\left(\frac{n\pi x}{L}\right), \qquad (2.7)$$

where

$$\beta_n(t)=\beta_n(0)\exp(-in\omega t). \qquad (2.8)$$

In Eq. (2.8) $\omega=(\pi/L)\sqrt{T/\rho}$. The frequency ω is called the fundamental frequency of vibration of the string. The speed of propagation of waves in the string is $C=\sqrt{T/\rho}$, as can be seen from the wave equation, Eq. (2.2). The momentum per unit length is

$$\pi(x,t)=-i\rho\omega\sum_{n=1}^{\infty}\frac{n}{2}\,[\beta_n(t)-\beta_n^*(t)]\sin\left(\frac{n\pi x}{L}\right). \qquad (2.9)$$

The dynamical variables $b_n(t)$ are called normal variables or normal coordinates. Each $b_n(t)$ evolves in time independently. From (2.7) and (2.8) it can be seen that to each n corresponds a standing wave of frequency $n\cdot\omega$ and wavelength $\lambda=2L/n$. These are called the normal modes of vibration of the string.

Equations (2.6) and (2.9) can be used to rewrite the Hamiltonian (2.5) as

$$H=\frac{1}{4}\rho L\omega^2\sum_n n^2|\beta_n|^2. \qquad (2.10)$$

This gives the energy of a classical string in terms of the magnitudes of the normal coordinates, which are basically the amplitudes of the normal mode vibrations of the string.

The above formalism is only good for small oscillations of the classical string. An *ideal* string, however, is a string defined by the Hamiltonian of (2.5). In this article the quantization of ideal strings will be discussed. The relativistic superstrings presently under investigation are fundamental objects and are ideal strings.

III. CANONICAL QUANTIZATION

In this section the method of canonical quantization is reviewed and used to quantize the harmonic oscillator. A review is given of the properties of the raising and lowering operators which generate the spectrum of the quantum harmonic oscillator. Concepts and techniques important to the discussion of the quantization of strings are introduced.

The method of canonical quantization can be summarized in a rule: Take the coordinates and corresponding conjugate momenta describing a physical system and "promote" them to operators on a Hilbert space obeying canonical commutation relations. An equivalent way of defining canonical quantization involves changing the coordinates and momenta to operators and changing the Poisson bracket relations in classical mechanics to commutator relations (with factors of i and $\hbar$). Here the former definition will be used.

The best way to elaborate on this method is to use it in an example. The harmonic oscillator is one of the simplest and most instructive cases. Many of the results obtained here will be used later in the discussion of the quantization of strings.

A one-dimensional classical harmonic oscillator is described by the coordinate $q(t)$ giving the Cartesian position of the oscillator along an axis. The Hamiltonian for the harmonic oscillator is

$$H(q,p) = (1/2m)p^2 + \tfrac{1}{2} kq^2, \tag{3.1}$$

where the momentum $p(t)$ conjugate to $q(t)$ is $p(t) = m\dot{q}(t)$.

To quantize the harmonic oscillator the variables $q(t)$ and $p(t)$ will be replaced by operators $Q(t)$ and $P(t)$ satisfying the canonical commutation relation

$$[P(t),Q(t)] = -i\hbar, \tag{3.2}$$

where $P(t)$ and $Q(t)$ are operators in the Heisenberg picture of quantum mechanics. The quantum Hamiltonian is the operator obtained by replacing p by P and q by Q in (3.1). The eigenvectors of the Hamiltonian are the energy eigenstates. The problem now is to find the eigenvalues and eigenvectors of the operator H given the relations (3.2). In order to do this, introduce the operator $a(t)$ defined by

$$a(t) = \sqrt{\frac{m\omega_0}{2\hbar}}\, Q(t) + i\sqrt{\frac{1}{2m\omega_0\hbar}}\, P(t), \tag{3.3}$$

where $\omega_0^2 = k/m$. Note that the operators $Q(t)$ and $P(t)$ are Hermitian, whereas $a(t)$ is not. The Hermitian conjugate of $a(t)$ is denoted $a^\dagger(t)$. These new operators satisfy the commutation relation

$$[a(t),a^\dagger(t)] = 1, \tag{3.4}$$

which follows from Eq. (3.2).

The Hamiltonian is, in terms of these new operators,

$$H = \hbar\omega_0(a^\dagger a + \tfrac{1}{2}), \tag{3.5}$$

as can be seen by substituting (3.3) into (3.5) and using (3.2).

Let $|E\rangle$ be an eigenvector of H with eigenvalue E. $a^\dagger(t)|E\rangle$ is then an eigenvector of H with eigenvalue $E + \hbar\omega_0$. The operator $a^\dagger(t)$ is called a raising operator. Similarly, $a(t)|E\rangle$ is an eigenvector of H with eigenvalue $E - \hbar\omega_0$ and $a(t)$ is called a lowering operator.

The energy eigenvectors of H are vectors $|\nu\rangle$, where ν is a positive integer or 0. The eigenvector $|\nu\rangle$ has energy eigenvalue $\hbar\omega_0(\nu + \frac{1}{2})$. For details, see one of the standard textbooks on quantum mechanics.[3]

The time dependence of the raising and lowering operators is given by the Heisenberg picture equation of motion, e.g.,

$$i\hbar\dot{a}(t) = [a(t),H] = \hbar\omega_0 a(t). \tag{3.6}$$

The solution to this equation is

$$a(t) = a_0 \exp(-i\omega_0 t), \tag{3.7}$$

where a_0 is a time-independent operator.

The following useful relations can be obtained from the above:

$$a_0^\dagger|\nu\rangle = \sqrt{\nu+1}\,|\nu+1\rangle, \tag{3.8}$$

$$a_0|\nu\rangle = \sqrt{\nu}\,|\nu-1\rangle, \tag{3.9}$$

$$a_0^\dagger a_0|\nu\rangle = \nu|\nu\rangle, \tag{3.10}$$

$$|\nu\rangle = (1/\sqrt{\nu!})(a_0^\dagger)^\nu|0\rangle. \tag{3.11}$$

The state corresponding to the eigenvector $|0\rangle$ satisfies $a_0|0\rangle = 0$ and is called the ground state (state of lowest energy). Because of Eq. (3.10) the operator $N = a_0^\dagger a_0$ is called the number operator.

The generalization of the canonical quantization procedure to the case where there is more than one dimension or particle is straightforward. In this case the classical dynamical variables would be $q_i(t)$, where the index i labels the different Cartesian dimensions and/or the different particles. $p_i(t)$ is the canonical momentum corresponding to the coordinate $q_i(t)$. To quantize, promote the variables $q_i(t)$ and $p_i(t)$ to operators $Q_i(t)$ and $P_i(t)$ on a Hilbert space. These operators are required to satisfy the commutation relations

$$[P_i(t),Q_j(t)] = -i\hbar\delta_{ij}, \tag{3.12}$$

$$[Q_i(t),Q_j(t)] = 0, \tag{3.13}$$

$$[P_i(t),P_j(t)] = 0. \tag{3.14}$$

Note in particular that the momentum operator corresponding to the index i commutes with the position operator corresponding to the index j if $i \neq j$. This means, for instance, that the measurement of one particle's momentum does not interfere with the simultaneous measurement of a different particle's position.

IV. THE QUANTIZED STRING

In this section the string discussed in Sec. II is quantized[4] using the canonical quantization method outlined in Sec. III. The "coordinate" which describes the string is the function $y(x,t)$. It is important to keep in mind that x is not a dynamical variable, but is only a parameter labeling points on the string.

For the string of Sec. II, the canonical momentum conjugate to $y(x,t)$ is the physical momentum per unit length $\pi(x,t)$. To quantize the string the functions $y(x,t)$ and $\pi(x,t)$ are promoted to operators $Y(x,t)$ and $\Pi(x,t)$ on a Hilbert space. Note that there is an operator Y and an operator Π for each value of the parameter x in the range 0–L. The canonical commutation relations for these operators, generalizing from (3.12)–(3.14), are

$$[\Pi(x,t),Y(x',t)] = -i\hbar\delta(x-x'), \quad (4.1)$$

$$[Y(x,t),Y(x',t)] = 0, \quad (4.2)$$

$$[\Pi(x,t),\Pi(x',t)] = 0. \quad (4.3)$$

The parameter x in (4.1)–(4.3) plays the same role as the parameter i in (3.12)–(3.14). The dimensions on each side of (4.1) match since the Dirac delta function has dimensions of $(\text{length})^{-1}$. Note that the operator Π at x commutes with the operator Y at any x' not equal to x. To put this another way, a measurement of the momentum of an infinitesimal segment of the string at x should not interfere with a simultaneous measurement of the position of a segment at some *other* point x'.

In analogy to the operator $a(t)$ defined in (3.3) define the operator $A_n(t)$ by

$$A_n(t) = \sqrt{\frac{\rho\omega n}{\hbar L}} \int_0^L \left[Y(x,t) + \frac{i}{\rho n\omega} \Pi(x,t) \right] \times \sin\left(\frac{n\pi x}{L}\right) dx. \quad (4.4)$$

The Hermitian conjugate of $A_n(t)$ is denoted $A_n^\dagger(t)$. The equations for $A_n(t)$ and $A_n^\dagger(t)$ can be inverted using the completeness and orthogonality relations obeyed by the functions $\sin(n\pi x/L)$ to give

$$Y(x,t) = \sqrt{\frac{\hbar}{\rho\omega L}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [A_n(t) + A_n^\dagger(t)] \times \sin\left(\frac{n\pi x}{L}\right) \quad (4.5)$$

and

$$\Pi(x,t) = -i\rho\omega \sqrt{\frac{\hbar}{\rho\omega L}} \sum_{n=1}^{\infty} \sqrt{n} [A_n(t) - A_n^\dagger(t)] \times \sin\left(\frac{n\pi x}{L}\right). \quad (4.6)$$

Note that if the replacement

$$A_n(t) = \sqrt{\rho\omega Ln/4\hbar}\, B_n(t) \quad (4.7)$$

is made in (4.5) and (4.6) then these equations are formally identical to equations (2.7) and (2.9). Thus Eqs. (4.5) and (4.6) are the quantum versions of Eqs. (2.7) and (2.9).

The operators $A_n(t)$ and $A_m^\dagger(t)$ satisfy the commutation relation

$$[A_n(t), A_m^\dagger(t)] = \delta_{nm}. \quad (4.8)$$

This follows from (4.1)–(4.3). Equation (4.8), for any n, is the same as the commutation relation (3.4) between the raising and lowering operators of the harmonic oscillator.

The quantum Hamiltonian is [see (2.5)]

$$H = \int_0^L \left[\frac{1}{2\rho} (\Pi(x,t))^2 + \frac{1}{2} T \left(\frac{\partial}{\partial x} Y(x,t) \right)^2 \right] dx. \quad (4.9)$$

Substituting the expansions (4.5) and (4.6) for $Y(x,t)$ and $\Pi(x,t)$ into this, the Hamiltonian becomes

$$H = \sum_n \hbar\omega_n \left(A_n^\dagger A_n + \frac{1}{2} \right), \quad (4.10)$$

where $\omega_n = n\cdot\omega$. Note that this is similar to Eq. (2.10) when Eq. (4.7) is used. The reader should go through the details of the calculations leading to (4.8) and (4.10).

Thus as can be seen from (4.10), *the Hamiltonian for the string is just the sum of an infinite number of harmonic oscillator Hamiltonians.* Each of these harmonic oscillators corresponds to a normal mode of vibration of the classical string. The nth harmonic oscillator has frequency ω_n, which is n times the fundamental frequency ω of the string.

The time dependence of the lowering operator $A_n(t)$ is given by the Heisenberg equation of motion:

$$i\hbar\dot{A}_n = [A_n(t),H] = \hbar\omega_n A_n(t). \quad (4.11)$$

Equation (4.11) shows that the harmonic oscillators corresponding to the different values of n are not coupled; the solution is

$$A_n(t) = A_n \exp(-i\omega_n t), \quad (4.12)$$

where A_n is a time-independent operator.

Equation (4.10) can be rewritten as

$$H = \sum_{n=1}^{\infty} H_n, \quad (4.13)$$

where H_n is the Hamiltonian for a harmonic oscillator with frequency ω_n. The eigenvectors of H_n are normalized vectors $|\nu_n\rangle$, where ν_n is a positive integer or 0, and

$$H_n|\nu_n\rangle = \hbar\omega_n(\nu_n + \tfrac{1}{2})|\nu_n\rangle. \quad (4.14)$$

Relations like (3.8)–(3.11) will hold for the operators A_n and the state vectors $|\nu_n\rangle$.

The eigenvectors of the full Hamiltonian H given in (4.13) are normalized vectors $|\nu_1, \nu_2, \nu_3, \nu_4,...\rangle$, where each of the ν_n are non-negative integers giving the quantum state of the nth normal mode of the string. These eigenvectors also satisfy relations similar to (3.8)–(3.11). For example,

$$A_n^\dagger|\nu_1, \nu_2,..., \nu_n,...\rangle = \sqrt{\nu_n + 1}\, |\nu_1, \nu_2,..., \nu_n + 1,...\rangle \quad (4.15)$$

corresponds to (3.8). The eigenvalues of H will be denoted $E_{\nu_1,\nu_2,\nu_3,\dots}$.

The state with lowest energy, $|\text{gnd}\rangle$ (called the ground state), is

$$|\text{gnd}\rangle = |0,0,0,...\rangle. \quad (4.16)$$

$|\text{gnd}\rangle$ satisfies the relation $A_n|\text{gnd}\rangle = 0$ for all n. The energy of the ground state is found by computing the expectation value of the Hamiltonian (4.13) for this energy eigenstate:

$$E_{0,0,0,\dots} = \langle\text{gnd}|H|\text{gnd}\rangle = \sum_{n=1}^{\infty} \frac{1}{2}\hbar\omega_n, \quad (4.17)$$

which is a divergent series. The ground state of the string is the state where each of the infinite number of harmonic oscillators (normal modes) is in its ground state, with energy $\frac{1}{2}\hbar\omega_n$. This presents a problem: The ground state and, in fact, all excited states of the string have infinite energy. The solution to this problem can be obtained by recognizing that only energy differences between states can be measured. Redefine the Hamiltonian by requiring the energy of the ground state to be 0:

$$H \equiv \left(\sum_{n=1}^{\infty} H_n\right) - \langle\text{gnd}| \left(\sum_{n=1}^{\infty} H_n\right) |\text{gnd}\rangle = \sum_{n=1}^{\infty} \hbar\omega_n A_n^\dagger A_n. \quad (4.18)$$

From this point on the Hamiltonian in (4.18) instead of the one in (4.10) will be used. Since the Hamiltonians in (4.10) and (4.18) differ only by a constant, they give rise to exactly the same physics.

The energy eigenvalue $E_{\nu_1,\nu_2,\ldots}$, corresponding to the eigenvector $|\nu_1, \nu_2,\ldots\rangle$, is now

$$E_{\nu_1,\nu_2,\nu_3,\ldots} = \sum_{n=1}^{\infty} \hbar\omega_n \, \nu_n. \quad (4.19)$$

Note that this is infinite unless the sum $\Sigma_{n=1}^{\infty} \nu_n$ is finite. The energy difference between a *physical* state and the ground state cannot be infinite. Thus it will be assumed that, for physical states, the integers ν_n which specify the energy eigenvector are such that (4.19) is finite. In particular, only a finite number of the integers ν_n can be different from 0. This means that, in contrast with the classical string, only a finite number of the normal modes may be in an excited state. Note that whereas each normal mode of the classical string of Sec. II may have any positive energy, the energy of a normal mode of the quantized string is restricted to be a positive integer times $\hbar\omega_n$ above the ground state energy. There is a lower bound ($\hbar\omega_n$) to the energy contained in an excited normal mode. This comparison is similar to that between classical and quantum harmonic oscillators. The energy levels of a quantum harmonic oscillator are quantized: Allowed energy levels are a positive integer times $\hbar\omega_0$ above the ground state energy.

V. GROUND STATE FLUCTUATIONS

For the quantum harmonic oscillator of Sec. III, the ground state expectation value of position q is $\langle 0|Q|0\rangle = 0$. The oscillator spends as much time to the right as to the left of the origin. The ground state expectation value of q^2, however, is

$$\langle 0|Q^2|0\rangle = \hbar/2m\omega_0 \quad (5.1)$$

The ground state uncertainty in position, Δq, is therefore $\sqrt{\hbar/2m\omega_0}$. Thus even in the ground state it is possible to find the oscillator away from the origin. This feature is known as the ground state fluctuation of the position of the oscillator. Ground state fluctuations also occur in the quantized string. These fluctuations, however, are qualitatively different for strings or quantized fields. Extra subtleties arise due to the continuous nature of these physical systems. These features will be investigated in this section.

The ground state expectation value of $y(x,0)$, the displacement of the string from the point x on the x axis, is $\langle \text{gnd}|Y(x,0)|\text{gnd}\rangle = 0$. To derive this, we used the relations $A_n|0\rangle = 0$ and $\langle 0|A_n^\dagger = 0$. The ground state expectation value for $[y(x,0)]^2$ is, after a short calculation,

$$\langle \text{gnd}|[Y(x,0)]^2|\text{gnd}\rangle = \frac{\hbar}{\rho\omega L}\sum_n \frac{1}{n}\sin^2\left(\frac{n\pi x}{L}\right). \quad (5.2)$$

This is 0 for $x = 0$ and $x = L$, as one would expect. For $0 < x < L$, however, the sum in (5.2) diverges.

This result, that the uncertainty in $y(x,0)$ is infinite, seems very strange. The above result is not particular to $t = 0$ since $|\text{gnd}\rangle$ is a stationary state. This effect, however, is unphysical since the measurement of the displacement of an infinitely small section of the string is impossible. In a physical measurement, it is the displacement of a small *segment* of the string that is measured. In order to express this mathematically, define the operator $\overline{Y}(x,t;\epsilon)$ by

$$\overline{Y}(x,t^0,\epsilon) = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} Y(x',t)dx'. \quad (5.3)$$

In (5.3), ϵ is the size of the probe which is used to measure the displacement of the string at x. The operator $\overline{Y}(x;0;\epsilon)$ corresponds to the quantity $\bar{y}(x;0;\epsilon)$, which is just the average displacement of the segment of extension 2ϵ at x. This average displacement will depend on the probe size ϵ. The ground state expectation value of $\bar{y}(x,0;\epsilon)$ is still 0, but the expectation value of $[\bar{y}(x;0^\circ;\epsilon)]^2$ is now

$$\langle \text{gnd}|[\overline{Y}(x,0^0,\epsilon)]^2|\text{gnd}\rangle = \frac{\hbar L}{\rho\omega\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi x/L)\sin^2(n\pi\epsilon/L)}{n^3\epsilon^2}, \quad (5.4)$$

which is finite for ϵ not equal to 0. The expectation value in (5.4), however, is infinite for $\epsilon = 0$. Thus it seems that the smaller the scale of observation, the larger the ground state fluctuation. At infinitely small scales ($\epsilon = 0$) the ground state fluctuation is infinite. This is a general feature of quantized continuous systems.

The strings discussed above are idealized since normal modes of any wavelength are allowed in the expansion of $Y(x,0)$. It does not make sense to consider normal modes with wavelengths less than the distance between the atoms which make up the string. Thus the sum in the expansion (2.6) or (4.5) would not be from 1 to infinity, but rather would be from 1 to a maximum value $2L/D$, where D is the distance between the atoms in the string. The sum in (5.2), and therefore the ground state fluctuations, would then be finite since the sum consists of a finite number of terms. In this paper, however, the string will be idealized as an infinitely continuous object. This idealization is made in order to illustrate the features of quantized continuous systems (such as fields or relativistic strings).

To get an idea of the order of magnitude of these ground state fluctuations, define the operator S, corresponding to the area under the string, by

$$S = \int_0^L Y(x',0)dx'. \quad (5.5)$$

The ground state expectation of the area is $\langle \text{gnd}|S|\text{gnd}\rangle = 0$. The ground state expectation value of S^2 is nonzero but finite. The rms area under the string is $S_{\text{rms}} = (0.653)\sqrt{\hbar L/\rho\omega}$. For example, for a string with $L = 100$ cm, $\rho = 0.1$ g/cm, and $\omega = 500/\text{s}$, S_{rms} is $3\times 10^{-14}\,\text{cm}^2$. The corresponding classical string with this rms area and with only its fundamental ($n = 1$) mode vibrating would have amplitude $b_1 = 10^{-15}$ cm [see (2.6)], which is only 1% of the diameter of the proton. The average measurable fluctuations are small.

VI. DISCUSSION

The quantized string, like the classical string, is described in terms of the normal modes of its vibration, which are standing waves in the string. There are an infinite number of normal modes, one for each positive integer n. The nth normal mode vibrates with frequency $\omega_n = n\cdot\omega$. Each normal mode is dynamically independent of the others. For the case of the quantized string, there corresponds a quantum harmonic oscillator of frequency ω_n to each normal mode. The energy of a normal mode of a quantized string can only be $\nu\hbar\,\omega_n$ above the ground state energy, where ν is a positive integer. The energy eigenstates are the quantized vibrational modes of the string.

Quantized fields[5] have many features in common with quantized strings. A string is, in fact, a one-dimensional scalar field in a one-dimensional box. The dynamical variables for a field are, instead of the physical displacements from points along an axis, the value of the field (which would be, for example, a vector for a vector field or a number for a scalar field) at each point in space. The value of the field must be given for all points in space instead of, as in the case of the string, for points along some finite segment of the x axis. Because of this last feature the normal modes of fields are plane waves instead of standing waves and cannot be labeled using only positive integers, but must be labeled by real numbers or, in the case of three dimensions, by vectors. It is customary to label the normal modes (plane waves) of fields by the corresponding wave vector $\mathbf{k}$. The energy eigenstates of quantum fields are quantized vibrational modes of the field. These eigenstates carry energy and momentum and are identified as particles. The energy eigenvalues, like the energy eigenvalues of the string, are quantized: They may only take the values $\nu\hbar\omega_k$, where ν is an integer and ω_k is the frequency of the plane wave corresponding to the wave vector $\mathbf{k}$. The ground state in quantum field theories is called the "vacuum state." The ground state fluctuations discussed in Sec. V are present in quantum field theories and are dealt with by redefining the Hamiltonian in a manner similar to the redefinition described in Sec. V.

Relativistic quantum field theories are complicated due to the requirement that the theory be invariant under Lorentz transformations. The above features, however, are still present in the relativistic theories.

It will be necessary to make a few technical comments about the nonrelativistic strings discussed in this article before commenting on superstring theories. The string discussed above is restricted to movement in the x–y plane. If this restriction is removed, then the string can vibrate independently in the x–y plane or in the x–z plane. Thus there are two independent states with the same frequency for each allowed energy. There will also be, for each n, two types of raising and lowering operators: A_n^y and $(A_n^y)^\dagger$, and A_n^z and $(A_n^z)^\dagger$. In general, if space has N dimensions, there will be $N-1$ different raising and lowering operators for each value of n. For relativistic theories, the dimensionality of space-time D is used, and there are $D-2$ different raising and lowering operators for each value of n. These are denoted A_n^μ and $(A_n^\mu)^\dagger$. The index μ, which labels the different directions the string may vibrate, takes values from 1 to $D-2$.

The fundamental strings are not tied down at each end as are the nonrelativistic strings discussed in this article. Waves in these fundamental strings carry momentum. This momentum cannot be allowed to flow off the end of the string; it must be conserved. This requirement is met if the derivative of the string position [the generalization of $y(x,t)$] with respect to the length parameter (the generalization of x) at each end is 0. Thus the coordinate of the fundamental string may also be expanded in a Fourier series, but in this case it is a Fourier cosine series instead of a sine series.

The strings which are the fundamental objects of the recent string theories[1] are similar to the strings discussed in this article. The main difference is that the strings in the recent theories are relativistic. They are usually described in the Lagrangian formulation instead of in the Hamiltonian formulation. The string's path through space-time forms a "world sheet." The relativistic action for the string is proportional to the area of this world sheet. This action leads to the Hamiltonian (2.5) in the nonrelativistic limit.

These relativistic theories must be invariant under Lorentz transformations. The invariance of a quantum theory under a symmetry group such as the group of Lorentz transformations is expressed in terms of the generators of the symmetry transformations. It must be possible to represent these generators as operators on the Hilbert space of quantum states. A familiar example of this in quantum mechanics is the case of a particle moving under the influence of a spherically symmetric potential. Here the quantum theory is invariant under three-dimensional rotations. This invariance is expressed by the statement that the angular momentum operators, which are the generators of three-dimensional rotations, commute with the Hamiltonian. The angular momentum operators are given in terms of the position and momentum operators; for example, $L_z = XP_y - YP_x$. The angular momentum operators must also satisfy certain commutation relations among themselves such as $(L_x, L_y) = i\hbar L_z$. For relativistic strings the generators of the Lorentz transformations are given in terms of the operators $Y(x,t)$ and $\Pi(x,t)$. These generators must satisfy certain commutation relations among themselves (they must form the Lie algebra of the Lorentz group). It turns out that these commutation relations are only satisfied in 26-dimensional space-time. Thus a relativistic quantum theory of strings can only be formulated if $D = 26$.

Superstrings are extended objects described by anticommuting dynamical variables in addition to the $y(x,t)$ variables. The quantum operators corresponding to the anticommuting variables satisfy anticommutation relations instead of commutation relations. These theories are invariant under supersymmetry transformations which swap the anticommuting variables with the $y(x,t)$ variables. This new type of string is called a superstring, while the relativistic string discussed in the preceding paragraph is called a "bosonic string." The bosonic string theories contain a number of technical difficulties (such as the existance of tachyons) which are not present in the superstring theories. The critical dimension for superstring theories is ten. It is an intriguing feature of string theories that the requirement of consistency of the quantum theory determines the dimension of space-time.

The strings discussed above are free strings. The fundamental relativistic strings are interacting strings. Strings may interact by breaking at a point, by having two ends join, or by exchanging parts when a segment of one string touches a segment of another string. The theory of interacting relativistic strings seems to solve an old problem in theoretical physics. All attempts so far to find a quantum field theory of gravity have failed. It turns out, however, that relativistic quantum string theories automatically contain gravity in the limit where the length of the string is taken to 0. This is one of the main reasons people are excited about superstring theories.

The superstring theories recently being discussed propose that all fundamental particles, including photons and gravitons, are extended objects of length 10^{-33} cm (the Planck length) instead of pointlike objects. The first excited state of these strings have energy of 10^{19} GeV. Excited states of these strings are therefore not accessible to modern accelerators or, in fact, to any conceivable future accelerator. Thus all particles known today would be strings in their ground state.

To get from the 10-dimensional space-time required by superstrings to the usual four-dimensional space-time it is thought that six of the dimensions curl up into a compact manifold of size 10^{-33} cm. This is called compactification of the extra dimensions. The extra six dimensions would not be observable to us. To see why this is so consider a cylinder, which is a two-dimensional manifold. If the radius of the cylinder is very small then two-dimensional animals living on the surface of the cylinder would not be aware of the fact that they are living on a two-dimensional manifold. Since angular motion around the cylinder (which would give rise to angular momentum) is quantized, a large amount of energy would be required to excite these rotational modes. The smaller the radius of the cylinder, the larger the energy required to excite rotational motion. Thus the rotational motion would, in effect, be inaccessible.

Many details of superstring theory are not yet understood. For instance, no one knows how or whether these theories lead to the physics observed at present energies. It is believed, however, that the structure of superstring theories is rich enough to account for the present particle phenomenology. If this is the case then superstring theory would be a quantum theory which unifies the electroweak and strong interactions with gravity. There is still a great deal of work left to be done.

One last technical comment should be made for people interested in reading more detailed reviews of relativistic strings. The raising and lowering operators A_n and $A_n^\dagger$ are not used in the literature on relativistic strings and superstrings, but rather the operators (ignoring here any index μ labeling the different directions the string may vibrate) defined by

$$\alpha_n = \sqrt{n}\, A_n \tag{6.1}$$

and

$$\alpha_{-n} = \sqrt{n}\, A_n^\dagger \tag{6.2}$$

are employed. The index n is any positive or negative integer. Note that the value $n = 0$ is allowed here since the fundamental string is expanded in a cosine series instead of a sine series.

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Stellar sky as seen from the vicinity of a black hole

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The stellar sky—as seen from the vicinity of a black hole—is discussed as a method to visualize the effects of light deflection. For rotating black holes, spatial axes defining the spherical angles of arriving light rays were defined via the direction of the plumb lines. Some light rays circling a strongly rotating black hole (extreme Kerr black hole) are also shown.

The gravitational lens problem is a very old one (see e.g., Bourassa *et al.*[1] and references quoted therein) and has been thoroughly investigated with respect to image distortion and intensity questions for gravitational lenses that are at cosmological distances from the observer. Among the most spectacular examples for such lenses are large galaxies or clusters of galaxies which may multiply the images of a quasar lying on the prolongation of the line between an observer on the Earth and the galaxy acting as gravitational lense. As the light travel time for two different paths of the light is also different, intensity fluctuations of the quasar are first seen in one of its images and with a retardation proportional to the distances between the objects in the other image. So this effect might serve as a tool for determining the cosmic length scale needed to measure cosmic parameters like the Hubble constant.

On the other hand, the question of how the outer world is experienced by an observer located in a gravitational lens has drawn relatively little interest. This, however, is precisely the situation that is presently encountered in astro-