Quantizing the damped harmonic oscillator

Bernard Yurke

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of mass was at radius 0.14 m, and radius of curvature = 1.15 m.

Finally, the anisotropy field H_A in our model is proportional to $\cos\alpha$ and therefore its effect is reduced as α is increased. This means that the applied field and the anisotropy field in our model are inversely coupled. When $\alpha = \pi/2$, H_A disappears. That is, when the model is tipped up to vertical, the curvature completely loses its effect on the arms.

Since we could think of no reasonable way to include temperature, this model represents only a single vertical line of the phase diagram in Fig. 1. There is, as expected for a first-order phase transition, a significant hysteresis at the SF to AF transition, which in our model is due entirely to friction. The model is fun to operate and very clearly demonstrates the difference between a first-order and second-order phase transition and gives considerable insight to the common but not easily understood easy-axis Heisenberg antiferromagnet.

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Quantizing the damped harmonic oscillator

Bernard Yurke
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

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A damped harmonic oscillator can be modeled as a manifestly conservative system by replacing the dissipative element with a string or transmission line of infinite extent. This conservative system can be quantized in a straightforward manner using the standard techniques of canonical quantization. The system may be used to illustrate various aspects of the quantum mechanics of a particle interacting with a field.

I. INTRODUCTION

In a previous paper[1] in this Journal a conservative model for the damped harmonic oscillator was presented in which the dissipative element or dashpot was modeled by a string of infinite extent. This model was used to illustrate a number of concepts in classical and statistical mechanics such as radiation damping, the Langevin equation, and Brownian motion.

In this paper the system is quantized using the standard techniques of canonical quantization.[2,3] Since the system is linear the calculations are simple and straightforward and are carried out without making any approximations. This conservative model of the damped harmonic oscillator should have pedagogical value as an example illustrating the quantum behavior of a discrete mechanical system coupled to a field. In this paper attention will be drawn to how the field modifies the ground state position probability distribution of the harmonic oscillator.

The approach to quantizing damped harmonic oscillators taken here is in the spirit of Senitzky[4] and Ford, Kac, and Mazur[5] in that the damping is provided by a heat bath of harmonic oscillators. A string or transmission line, satisfying a massless scaler Klein–Gordon equation, is a particularly simple heat bath to quantize. For a critical review of various other approaches to quantizing damped harmonic oscillators the reader is directed to Ref. 6 by Dekker. See also Ray's article in this Journal.[7]

The system to be quantized is depicted in Fig. 1. In Fig. 1(a) a string of infinite extent is attached to the harmonic oscillator's mass. The oscillator is constrained to move along the y axis via the guide G. As the oscillator's mass moves up and down waves are launched along the string. In this manner the oscillator loses energy and its motion is damped. The electrical analog, in which the charge stored in the capacitor plays the role of the position coordinate of the oscillator, is depicted in Fig. 1(b).

Since canonical quantization relies heavily on Lagrangians to provide the equations of motion and the momenta canonically conjugate to the position variables, the Lagrangian mechanics of the damped harmonic oscillator will be worked out in detail. The oscillator with a string of finite length will then be quantized by solving for the normal modes[8,9] of the system and then introducing the creation and annihilation operators associated with these modes. The quantum mechanics of the damped harmonic oscillator is then obtained by taking the thermodynamic limit, i.e., letting the length of the string become infinite.

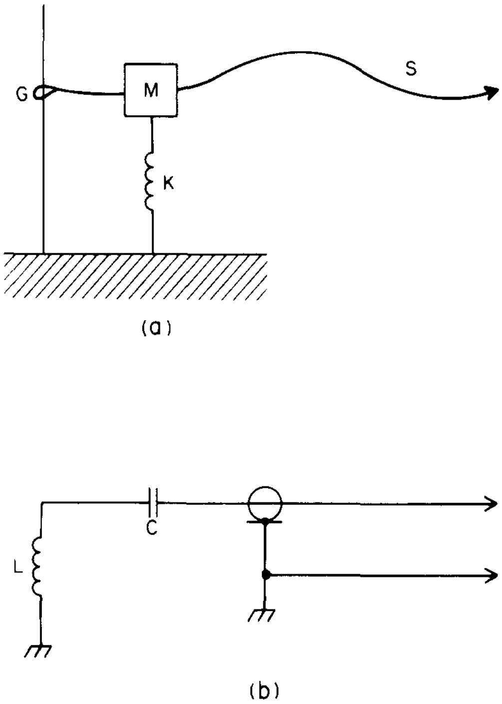

Fig. 1. (a) The damped harmonic oscillator in which the dashpot is modeled by a string extending to infinity. The guide G restricts the oscillator's motion along the vertical direction. (b) The electrical equivalent where a transmission line of infinite length simulates a resistor.

As an illustration, the ground state position probability distribution for the damped harmonic oscillator of Fig. 1(a) will be obtained. This probability distribution becomes narrower as damping is increased.

Although the calculations here will be carried out with a string of finite length, the limit of infinite length being taken only in the final stages of the analysis, one can in fact work directly in the infinite length limit as shown in Ref. 10.

II. LAGRANGIAN MECHANICS

In this section the Lagrangian mechanics for the damped harmonic oscillator of Fig. 1(a) is reviewed. Since the system under consideration has a distributed component, the string, it is appropriate to work with a Lagrangian density. Let $y(x,t)$ denote the height of the string above the x axis at position x and time t. Since the string is attached to the oscillator's mass, $y(0,t)$ is the position of the oscillator. The Lagrangian will have the form

$$L = \int_{-\infty}^{\infty} dx \mathscr{L}\left(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}\right), \tag{1}$$

where $\mathscr{L}$ denotes the Lagrangian density. The equations of motion are obtained from the variational principle

$$\delta L = 0, \tag{2}$$

which leads to the Euler–Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\partial y/\partial t)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial(\partial y/\partial x)}\right) - \frac{\partial \mathscr{L}}{\partial y} = 0. \tag{3}$$

The Lagrangian density for the mechanical system depicted in Fig. 1(a) is given by

$$\mathscr{L} = \delta(x)\left[\frac{M}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{K}{2}y^2\right] + h_+(x)\left[\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\sigma}{2}\left(\frac{\partial y}{\partial x}\right)^2\right], \tag{4}$$

where M and K denote the oscillator's mass and spring constant, respectively, ρ and σ denote the string's mass-density and tension, respectively, $\delta(x)$ is the Dirac delta function, and $h_+(x)$ is the Heavyside function

$$h_+(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}. \tag{5}$$

That Eq. (4) is the correct Lagrangian density is now verified by showing that the Euler–Lagrange equation (3) generates the correct equations of motion. Substituting (4) into (3) one obtains

$$\delta(x)\left[M\frac{\partial^2 y}{\partial t^2} + Ky\right] + h_+(x)\,\rho\frac{\partial^2 y}{dt^2} - \sigma\frac{\partial}{\partial x}h_+(x)\frac{\partial y}{\partial x} = 0. \tag{6}$$

By integrating this equation from $x = -\epsilon$ to $x = \epsilon$ and taking the limit as $\epsilon \to 0$ one obtains at $x = 0$

$$M\frac{d^2 y(t)}{dt^2} + Ky(t) - \sigma\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0^+} = 0, \tag{7}$$

where $y(t) \equiv y(0,t)$. The quantity $\sigma[\partial y(x,t)/\partial x]_{x=0^+}$ can be recognized as the y component of the force exerted by the string on the harmonic oscillator's mass.[1] Equation (7) can thus be recognized as Newton's equation of motion for the mass. For $x > 0$ Eq. (6) reduces to

$$\rho\frac{\partial^2 y}{\partial t^2} - \sigma\frac{\partial^2 y}{\partial x^2} = 0, \tag{8}$$

which is immediately recognized as the wave equation for the string, a massless scaler Klein–Gordon equation. Hence, the Lagrangian density (4) generates the correct classical equations of motion. The propagation velocity for waves governed by the Eq. (8) is given by

$$v = (\sigma/\rho)^{1/2} \tag{9}$$

and the wave impedance is

$$\Gamma = \sigma/v = (\rho\sigma)^{1/2}. \tag{10}$$

It has been shown in Ref. 1 that by breaking the field $y(x,t)$ into a part y_{in} and a part y_{out} propagating toward and away from the oscillator, respectively,

$$y(x,t) = y_{\text{in}}\left(\frac{x}{v} + t\right) + y_{\text{out}}\left(-\frac{x}{v} + t\right). \tag{11}$$

Equation (7) can be put into the form

$$M\frac{d^2 y}{dt^2} + \Gamma\frac{dy}{dt} + Ky = 2\Gamma\frac{dy_{\text{in}}}{dt}, \tag{12}$$

which is the equation of motion for a damped driven harmonic oscillator with a damping coefficient Γ. If the string is of infinite length and no waves are propagating toward the oscillator, the oscillator's motion will decay exponentially.

Having verified that the physical system under consideration gives rise to damped harmonic motion, the momentum $\pi(x,t)$ canonically conjugate to the field $y(x,t)$ will now be determined. The momentum conjugate to $y(x,t)$ is obtained from the Lagrangian via

$$\pi(x,t) = \frac{\partial \mathscr{L}}{\partial\{[\partial y(x,t)]/\partial t\}}. \tag{13}$$

Substituting (4) into (13) one finds

$$\pi(x,t) = [\delta(x)M + h_+(x)\,\rho][\partial y(x,t)]/\partial t. \tag{14}$$

The Hamiltonian density $\mathscr{H}$ is given by

$$\mathscr{H} = \pi(x,t)\{[\partial y(x,t)]/\partial t\} - \mathscr{L}, \tag{15}$$

which when using (14) and (4) becomes

$$\mathscr{H} = \delta(x)\left[\frac{M}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{K}{2}y^2\right] + h_+(x)\left[\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\sigma}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]. \tag{16}$$

The Hamiltonian

$$H = \int_{-\infty}^{\infty} dx\,\mathscr{H} \tag{17}$$

can then be written as

$$H = \frac{M}{2}\left(\frac{dy(x,t)}{dt}\right)^2 + \frac{K}{2}y^2(x,t) + \int_0^{\infty}\left[\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\sigma}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]dx. \tag{18}$$

The first two terms are recognized as the energy of the oscillator located at $x = 0$. The integral is recognized as the energy in the string.

The Lagrangian, the Hamiltonian, and the momentum canonically conjugate to the field $y(x, t)$ have now been constructed.

III. NORMAL MODES

Here the normal modes for an harmonic oscillator whose mass is coupled to a string of finite length l are obtained. The results of this section will be very useful when quantizing this system since quantization consists essentially of constructing the creation and annihilation operators for these modes. For convenience the boundary condition

$$y(l,t) = 0 \tag{19}$$

will be chosen for the far end of the string. The normal modes will consist of standing waves on the string. Hence, the normal modes will have the form

$$y_n(x,t) = y_n(x)e^{-i\omega_n t}, \tag{20}$$

where

$$y_n(x) = a_n \sin\left(\frac{\omega_n x}{v} + \theta_n\right). \tag{21}$$

That (20) together with (21) satisfy the wave equation (8) can be verified directly by substitution. The complex conjugate of (20) is also a normal mode of the system. Equations (20) and (21) must also satisfy (7) which can be regarded as a boundary condition[11] for one end of the string. Substituting (20) and (21) into (7) one obtains

$$(-\omega_n^2 M + K)\sin\theta_n - \Gamma\omega_n\cos\theta_n = 0. \tag{22}$$

This can be regarded as an equation for θ_n and can be reexpressed as

$$\theta_n = \cot^{-1}\left(\frac{M}{\Gamma}\frac{\omega_0^2 - \omega_n^2}{\omega_n}\right), \tag{23}$$

where

$$\omega_0 = \left(\frac{K}{M}\right)^{1/2} \tag{24}$$

is the natural resonant frequency of the harmonic oscillator. To build an intuition about θ_n consider a string of sufficient length that the lowest mode has a frequency much less than ω_0. From (23) one sees that as $\omega_n \to 0$ one has $\theta_n \to \cot^{-1}(\infty) = 0$. Hence, for frequencies much less than the oscillator's resonant frequency there is no phase shift, that is, the oscillator does not respond at low frequencies and hence acts like the boundary condition $y(0, t) = 0$. As $\omega_n \to \omega_0$ one has $\theta_n \to \cot^{-1}(0) = \pi/2$. Hence, modes close to the oscillator's resonant frequency suffer a $\pi/2$ phase shift, that is, the oscillator responds strongly near its resonant frequency and hence acts like the boundary condition

$$\frac{\partial y(0,t)}{\partial t} = 0.$$

As $\omega_n \to \infty$ one has $\theta_n \to \cot^{-1}(-\infty) = \pi$. At high frequencies the oscillator can no longer respond, a π phase shift is equivalent to the boundary condition $y(0,t) = 0$.

The eigenfrequencies of the normal modes are now determined by the boundary condition (19). Hence, one requires $y_n(l) = 0$, or

$$\omega_n = (v/l)(n\pi - \theta_n). \tag{25}$$

For a string with the boundary conditions $y(0,t) = 0$ and $y(l,t) = 0$ the modes are spaced according to

$$\omega_n = (v/l)n\pi. \tag{26}$$

Hence, from (25) and the discussion of the behavior of θ_n, one sees that θ_n has the effect of introducing an extra mode into the normal mode spectrum when compared with (26). This is depicted in Fig. 2. This extra mode is of course that associated with the degree of freedom belonging to the harmonic oscillator.

The orthogonality relations satisfied by these normal

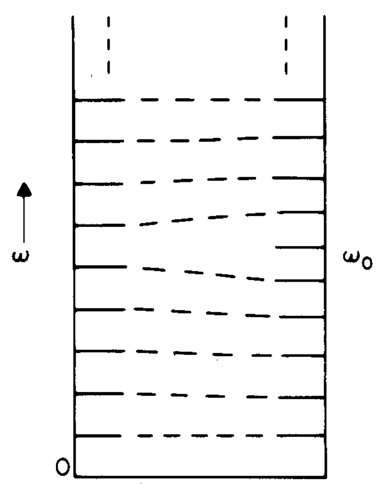

Fig. 2. A comparison of the normal modes of a string rigidly terminated at both ends (left side of diagram) with the normal modes of the same string when one end is terminated by an harmonic oscillator with a natural oscillation frequency ω_0 (right side of diagram). The harmonic oscillator inserts an extra mode, its own mode, in the normal mode spectrum.

modes will now be derived. To this end we first evaluate

$$\int_0^l y_n(x)y_{n'}(x)dx$$
$$= a_n a_{n'} \int_0^l \sin\left(\frac{\omega_n x}{v} + \theta_n\right) \sin\left(\frac{\omega_{n'} x}{v} + \theta_{n'}\right) dx\,. \quad (27)$$

When $n = n'$ the integral on the right-hand side is readily evaluated. The result is

$$\int_0^l y_n(x)y_n(x)dx$$
$$= a_n^2 \left[\frac{l}{2} + \frac{v\sin 2\theta_n}{\omega_n} - \frac{v\sin 2\left(\frac{\omega_n l}{v} + \theta_n\right)}{\omega_n}\right]. \quad (28)$$

The case when $n \neq n'$ is less straightforward. Solving (25) for θ_n and substituting this into Eq. (27) one obtains

$$\int_0^l y_n(x)y_{n'}(x)dx$$
$$= a_n a'_n \int_0^l \sin\left(\frac{\omega_n(x-l)}{v} + n\pi\right)$$
$$\times \sin\left(\frac{\omega_{n'}(x-l)}{v} + n'\pi\right)dx\,. \quad (29)$$

Making use of the relation

$$\sin(\alpha + n\pi) = (-1)^n \sin\alpha \quad (30)$$

and performing the change of variables $\eta = x - l$ the integral is put into standard form and when evaluated gives

$$\int_0^l y_n(x)y'_n(x)dx$$
$$= (-1)^{n+n'} a_n a_{n'} \frac{v}{2}$$
$$\times \left(\frac{\sin(\omega_n - \omega_{n'})\frac{l}{v}}{\omega_n - \omega_{n'}} - \frac{\sin(\omega_n + \omega_{n'})\frac{l}{v}}{\omega_n + \omega_{n'}}\right). \quad (31)$$

By substituting Eq. (26) into this equation to eliminate the ω_n one can show, using standard trigonometric identities and Eq. (22), that Eq. (31) reduces to

$$\int_0^l y_n(x)y_{n'}(x)dx = -a_n a_{n'} \frac{vM}{\Gamma} \sin\theta_n \sin\theta_{n'}\,. \quad (32)$$

If one chooses the normalization constant a_n such that

$$a_n^2 = \frac{1}{\left(\frac{\rho l}{2} + \frac{\Gamma}{4\omega_n}\sin\theta_n + M\sin^2\theta_n\right)}, \quad (33)$$

then from Eq. (28) and (32) one can show that the $y_n(x)$ satisfy the orthogonality relation[12]

$$\int_{-\infty}^l [M\delta(x) + \rho h_+(x)] y_n(x) y_{n'}(x) = \delta_{nn'}\,. \quad (34)$$

Having obtained the normal modes and their orthogonality relation one is now ready for quantization.

IV. QUANTIZATION

Quantization of the oscillator coupled to a string of finite length is readily accomplished in the Heisenberg picture following the usual prescription for canonical quantization. First, the equations of motion, Eqs. (6)–(8), are the correct Heisenberg equations of motion provided $y(x, t)$ is regarded as an operator. Since the equations are linear one does not have to normal order the operators in these equations. Canonical quantization requires that $y(x, t)$ and its canonically conjugate momentum $\pi(x, t)$ satisfy the equal time Boson commutation relations:

$$[y(x,t),y(x',t)] = [\pi(x,t),\pi(x',t)] = 0 \quad (35)$$

and

$$[y(x,t),\pi(x',t)] = i\hbar\delta(x - x')\,. \quad (36)$$

The operator $y(x, t)$ can be expanded in terms of the normal modes $y_n(x,t)$:

$$y(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{\hbar}{2\omega_n}} \left[y_n(x,t)A_n + y_n^*(x,t)A_n^\dagger\right]. \quad (37)$$

Using the orthogonality relation (34) one can derive expressions for the A_n in terms of $y(x, t)$:

$$A_n = \frac{i}{\sqrt{2\hbar\omega_n}} \int_{-\infty}^l dx[M\delta(x) + \rho h_+(x)]$$
$$\times \left[y_n^*(x,t)\frac{\partial y(x,t)}{\partial t} - \frac{\partial y_n^*(x,t)}{\partial t} y(x,t)\right]. \quad (38)$$

A similar expression can be obtained for $A_n^\dagger$.

From the commutation relations Eq. (35) and (36) and the expression for the conjugate momentum (13) one can show that A_n and $A_n^\dagger$ satisfy the usual commutation relations for Boson creation and annihilation operators:

$$[A_n,A_{n'}] = [A_n^\dagger,A_{n'}^\dagger] = 0\,, \quad (39)$$
$$[A_n,A_{n'}^\dagger] = \delta_{nn'}\,. \quad (40)$$

From (16) and (17) the Hamiltonian is

$$H = \int_{-\infty}^l dx\left\{\delta(x)\left[\frac{M}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{K}{2}y^2\right]\right.$$
$$\left. + h_+(x)\left[\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\sigma}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]\right\}. \quad (41)$$

By substituting (37) into (41) one can express H in terms of A_n and $A_n^\dagger$. To simplify the algebra it is useful to rewrite (41) in a somewhat different form. By integrating the term $(\sigma/2)(\partial y/\partial x)^2$ in equation (41) by parts using the boundary condition equation (19) and making use of Eq. (6) the Hamiltonian can be put into the form

$$H = \int_{-\infty}^l dx\,\tfrac{1}{2}[\delta(x)M + h_+(x)\rho]\left[\left(\frac{\partial y}{\partial t}\right)^2 - y\frac{\partial^2 y}{\partial t^2}\right]. \quad (42)$$

Now, the orthogonality relation (34) can readily be applied and H reduces to

$$H = \sum_{n=1}^{\infty} \hbar\omega_n A_n^\dagger A_n\,, \quad (43)$$

where, as is usually done, the zero point energy has been thrown away.[3] From (40) and (43) it is evident that the A_n and $A_n^\dagger$ are ladder operators for the Hamiltonian. Canonical quantization is completed by postulating the existence of a vacuum state $|0\rangle$ satisfying

$$\langle 0|0\rangle = 1 \quad (44)$$

and

$$A_n|0\rangle = 0 \quad \text{for all } n\,. \tag{45}$$

Any state of the system can be constructed by applying a suitable polynomial in the operators $A_n^\dagger$ to the vacuum state.

V. THE GROUND STATE POSITION PROBABILITY DISTRIBUTION

In this section the ground state position probability distribution for the damped harmonic oscillator will be evaluated. It will be shown that the ground state position probability distribution will become narrower or more localized as the damping constant Γ is increased.

The ground state position probability distribution will be obtained by evaluating the moments

$$\langle 0|y^k(x,t)|0\rangle\,, \tag{46}$$

where k is an integer. To simplify the analysis it is useful to decompose $y(x,t)$ into its positive and negative frequency components $y^{(+)}(x,t)$ and $y^{(-)}(x,t)$, respectively,

$$y(x,t) = y^{(+)}(x,t) + y^{(-)}(x,t)\,, \tag{47}$$

where

$$y^{(+)}(x,t) = \sum_{n=1}^{\infty}\sqrt{\frac{\hbar}{2\omega_n}}\,y_n(x)A_n e^{-i\omega_n t}\,, \tag{48}$$

$$y^{(-)}(x,t) = \sum_{n=1}^{\infty}\sqrt{\frac{\hbar}{2\omega_n}}\,y_n(x)A_n^\dagger e^{i\omega_n t}\,. \tag{49}$$

$y^{(+)}(x,t)$ satisfies the property

$$y^{(+)}(x,t)|0\rangle = 0\,. \tag{50}$$

$y^{(+)}$ and $y^{(-)}$ satisfy the commutation relations

$$[y^{(+)}(x,t),y^{(-)}(x,t)] = \sum_{n=1}^{\infty}\frac{\hbar}{2\omega_n}y_n^2(x) \tag{51}$$

and furthermore

$$[y(x,t),y^{(-)}(x,t)] = \sum_{n=1}^{\infty}\frac{\hbar}{2\omega_n}y_n^2(x)\,. \tag{52}$$

From (50) one can immediately show that

$$\langle 0|y(x,t)|0\rangle = 0 \tag{53}$$

and from (52) one can show

$$\langle 0|y^2(x,t)|0\rangle = \sum_{n=1}^{\infty}\frac{\hbar}{\omega_n}y_n^2(x)\,. \tag{54}$$

Hence,

$$(\Delta y)^2 \equiv \langle 0|y^2(x,t)|0\rangle - \langle 0|y(x,t)|0\rangle^2 = \sum_{n=1}^{\infty}\frac{\hbar}{\omega_n}y_n^2(x)\,. \tag{55}$$

In order to compute the higher order moments of $y(x,t)$ it is worth noting that if the commutator $C = [A,B]$ of two operators A and B is a complex number, then

$$[A^n,B] = nCA^{n-1}\,. \tag{56}$$

Keeping this and equation (52) in mind and using (50) one has

$$\begin{aligned}\langle 0|y^k(x,t)|0\rangle &= \langle 0|y^{k-1}(x,t)y^{(-)}(x,t)|0\rangle \\ &= \langle 0|y^{k-2}(x,t)|0\rangle(k-1)\sum_{n=1}^{\infty}\frac{\hbar}{\omega_n}y_n^2(x)\,.\end{aligned} \tag{57}$$

Using the relationship with Eq. (53) and (54) one can show inductively that

$$\langle 0|y^{2k+1}(x,t)|0\rangle = 0 \tag{58}$$

and

$$\langle 0|y^{2k}(x,t)|0\rangle = (2n-1)!!(\Delta y)^{2n}\,. \tag{59}$$

From the moments Eq. (58) and Eq. (59) one can deduce that the probability distribution $P(y)$ of finding the oscillator's mass at position y is a Gaussian[13] of the form

$$P(y) = \frac{1}{\sqrt{2\pi}\Delta y}\exp\left(-\frac{y^2}{4(\Delta y)^2}\right), \tag{60}$$

where

$$(\Delta y)^2 = \sum_{n=1}^{\infty}\frac{\hbar}{\omega_n}y_n^2(0)\,. \tag{61}$$

Hence, the ground state probability distribution for the harmonic oscillator coupled to a string is a Gaussian.

The quantity Δy will now be evaluated for the case when $l\to\infty$, that is, when one takes the thermodynamic limit. In this limit the harmonic oscillator becomes a truly damped harmonic oscillator, since the energy it radiates along the string never returns to the oscillator.

From Eq. (21) and (33) one has

$$(\Delta y)^2 = \sum_{n=1}^{\infty}\frac{\hbar}{2\omega_n}\frac{\sin^2\theta_n}{\left(\dfrac{\rho l}{2}+\dfrac{\Gamma}{4\omega_n}\sin 2\theta_n + M\sin^2\theta_n\right)}\,. \tag{62}$$

From Eq. (22) one can show

$$\sin^2\theta_n = \frac{1}{1+\left(\dfrac{M}{\Gamma}\right)^2\left(\dfrac{\omega_0^2-\omega_n^2}{\omega_n}\right)^2} \tag{63}$$

and from (25) one has

$$\Delta\omega \equiv \omega_{n+1}-\omega_n = (v/l)(\pi+\theta_n-\theta_{n+1})\,. \tag{64}$$

With Eq. (63) and Eq. (64) one can show that in the limit $l\to\infty$ the sum (62) reduces to the integral

$$(\Delta y)^2 = \frac{\hbar}{\pi\Gamma}\int_0^\infty \frac{d\omega}{\omega\left[1+\left(\dfrac{M}{\Gamma}\right)^2\left(\dfrac{\omega_0^2-\omega^2}{\omega}\right)^2\right]}\,. \tag{65}$$

The integral can be put into standard form with the change of variables

$$\zeta = \frac{\omega^2-\omega_0^2}{\omega_0^2}\,. \tag{66}$$

Then

$$(\Delta y)^2 = \frac{\hbar}{2\pi\Gamma}\int_{-1}^{\infty}\frac{d\zeta}{1+\zeta+Q^2\zeta^2}\,, \tag{67}$$

where

$$Q = (M/\Gamma)\omega_0 \tag{68}$$

is the quality factor for the damped harmonic oscillator. Upon evaluation, the integral (67) gives

$$(\Delta y)^2 = \begin{cases} \dfrac{\hbar}{\pi\sqrt{MK}} \dfrac{Q}{\sqrt{4Q^2-1}} \cot^{-1}\left(-\dfrac{2Q^2-1}{\sqrt{4Q^2-1}}\right) & Q>\frac{1}{2} \\ \dfrac{\hbar}{\pi\sqrt{MK}} & Q=\frac{1}{2}. \\ \dfrac{1}{2\pi\sqrt{MK}} \dfrac{Q}{\sqrt{1-4Q^2}} \log\left(\dfrac{1-2Q^2+\sqrt{1-4Q^2}}{1-2Q^2-\sqrt{1-4Q^2}}\right) & Q<\frac{1}{2} \end{cases} \tag{69}$$

The three cases $Q>\frac{1}{2}$, $Q=\frac{1}{2}$, and $Q<\frac{1}{2}$ correspond, respectively, to the underdamped, critically damped, and overdamped harmonic oscillator. Equations (69) have been independently derived by the author[8,10] and by Caldeira and Leggett.[14] As the quality factor $Q\to\infty$ the probability distribution should become that of the undamped harmonic oscillator. In fact in the limit $Q\to\infty$ equation (69) gives

$$(\Delta x)^2 = \left(\frac{\hbar^2}{4MK}\right)^{1/2}, \tag{70}$$

which is the textbook result for the undamped harmonic oscillator. For finite Q, Δy is smaller than that for the undamped case [compare Eq. (70) with (69) for $Q=\frac{1}{2}$, for example] and approaches 0 as Q goes to 0. This shrinking of the ground state position probability distribution has observable consequences in the tunneling rates of Josephson junctions.[10,14]

Having evaluated $(\Delta y)^2$ and determined the ground state probability distribution $P(y)$ for the position of the damped harmonic oscillator's mass, one might next try to evaluate $(\Delta p)^2$ for the harmonic oscillator's momentum:

$$p = M\frac{\partial y(0,t)}{\partial t}. \tag{71}$$

Substituting (47) into this equation and following the procedure that led to (65) one obtains

$$(\Delta p)^2 = \frac{M^2\hbar}{\pi\Gamma}\int_0^\infty \frac{\omega d\omega}{\left[1+\left(\frac{M}{\Gamma}\right)^2\left(\frac{\omega_0^2-\omega^2}{\omega}\right)^2\right]}. \tag{72}$$

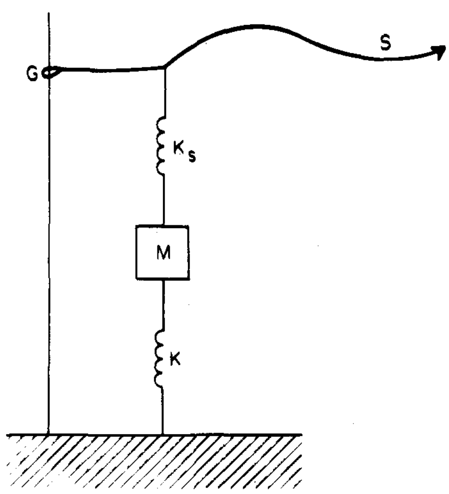

Fig. 3. An harmonic oscillator in which the string is connected to the mass via a stiff spring K_s. At high frequencies this spring decouples the mass from the string.

At high frequencies the integrand goes as $1/\omega$, hence this integral has a logarithmic untraviolet divergence. This divergence is due to the response of the harmonic oscillator to the zero point fluctuations in the high frequency modes of the string.

For real physical systems, a lumped circuit description will break down at high frequencies due to various nonidealities. A real spring is not massless, but has a finite mass distributed over its length. Also the mass of a real oscillator will not reside in a rigid object, but an object that will flex a small amount as the string tugs on it. Such nonidealities can be expected to introduce high frequency cutoffs in the integral (72). As an example, one might simulate the nonrigidity of the object in which the mass resides with the lumped component model depicted in Fig. 3. The stiff spring K_s between the string and the lumped component mass simulates the nonrigidity of the object in which the mass resides. For this system equation (72) becomes

$$(\Delta p)^2 = \frac{M^2\hbar}{\pi\Gamma} \times\int_0^l \frac{\omega d\omega}{\left(1+\frac{K}{K_s}-\omega^2\frac{M}{K_s}\right)^2+\left(\frac{M}{\Gamma}\right)^2\left(\frac{\omega_0^2-\omega^2}{\omega}\right)^2}. \tag{73}$$

In the limit $K_s\to\infty$ this reduces to (72). But for finite K_s, for frequencies above

$$\omega_c = \sqrt{K_s/M}, \tag{74}$$

the integrand goes as $1/\omega^3$ and the integral is finite. Thus, the spring K_s decouples the oscillator from the string at frequencies above the cutoff frequency ω_c.

VI. CONCLUSIONS

A conservative model of a damped harmonic oscillator in which the dissipative element is modeled by a string of infinite extent has been quantized using the standard techniques of canonical quantization. Since the Heisenberg equations of motion are linear operator equations, the calculations are straightforward to carry out. The system presented here can be used to illustrate various aspects of the quantum mechanics of a particle interacting with a field. It was shown that for the oscillator of Fig. 1 the ground state position probability distribution becomes narrower as damping is increased. The second moment of the momentum distribution has a logarithmic ultraviolet divergence. It was pointed out that this divergence can be eliminated by coupling the string to the mass via a spring as in Fig. 3. This spring decouples the mass from the string at high frequencies and hence provides a high frequency cut off.

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On the distribution of the nearest neighbor

M. Berberan Santos
Secção de Química-Física, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal

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The classical derivation of the distance distribution function of the nearest neighbor is discussed, and its limitations outlined. A new derivation, more general, is presented and applied to a random distribution of particles in a sphere.

I. INTRODUCTION

The approximation to consider, only, the interaction between a particle and its nearest neighbor is sometimes made in many-particle systems. For example, this has been done in electronic energy transfer[1,2] and stellar dynamics.[3] Using the distance distribution function of the nearest neighbor (DFN) the mean distance between the particles can also be obtained (Appendix A).

The classical DFN[4] is valid whenever two conditions are fulfilled: (i) uniform distribution of the particles and (ii) mean distance between particles much smaller than the dimensions of the volume containing the particles.

A case where the first condition is not met is that of a liquid: its molecules are crowded, and therefore the volume occupied by each molecule must be taken into account. Besides this *excluded volume effect*, a *short range order* exists. Both effects are mathematically expressed by the well-known radial distribution function, $g(r)$.[5] This function is defined as the ratio of the actual number density (number of particles per unit volume) at distance r from the particle, $n(r)$, to the bulk number density, n,

$$g(r)=n(r)/n. \tag{1}$$

Two typical cases are shown in Fig. 1: in Fig. 1(a) the radial distribution function of a monoatomic liquid and in Fig. 1(b) the radial distribution of a dilute monoatomic gas. In both cases r is the center-to-center distance. The probability of two molecules having a very small separation is low because of the repulsive forces ("excluded volume"). Several progressively decreasing peaks occur in $g(r)$ of the liquid, reflecting a sort of multilayer disposition of the molecules around the central molecule. Only for large distances is the distribution uniform, with $g(r)=1$.

In order to have a mean distance between particles similar to the dimensions of the vessel, it is clear that the particles can only be a few, say, less than one thousand. But this is really a very small number. Does it have any physical meaning? The answer is yes, and systems with this peculiarity are not unknown. They may be called compartmentalized systems. Examples are gases in porous media and molecules dissolved in micelles. In both cases a large number of molecules is distributed by an equally large number of compartments. In this way, each compartment contains only a few molecules.

It is the purpose of this paper to derive DFN's for the two above-mentioned cases, where the classical DFN is not valid. For the sake of completeness, we start with the derivation of the classical DFN.

II. CLASSICAL DISTRIBUTION

Consider a large volume V, containing N particles, $N\gg 1$. The number density is then $n=N/V$. Let $w(r)$ be the sought-for distribution function of the nearest neighbor. If we choose a particle at random, and define a sphere of radius r centered in that particle, and if the particles are uniformly distributed, the probability for a particle to occur inside the sphere is simply v/V, where $v=4\pi r^3/3$.

Since the particles are considered dimensionless, they can occur in any number (up to N) interior to r. Then the probability of having K particles in the sphere (plus the central particle) is given by the binomial law

$$p(K)=\binom{N}{K}\left(\frac{v}{V}\right)^K\left(1-\frac{v}{V}\right)^{N-K}. \tag{2}$$

The probability that no particles occur interior to r is of course $P(0)$, but is also equal to one minus the probability that the nearest neighbor occurs between zero and r, that is,

$$1-\int_o^r w(r)dr=\left(1-\frac{v}{V}\right)^N. \tag{3}$$

By taking the limit $N\rightarrow\infty$, while fixing $n=N/V$, we get

$$1-\int_o^r w(r)dr=\exp\left(-\frac{4\pi r^3 n}{3}\right) \tag{4}$$

thus

$$w(r)=4\pi r^2 n\exp(-4\pi r^3 n/3). \tag{5}$$

Now it is clear that the derivation of the classical DFN involves two assumptions, as referred in the introduction: (i) uniform distribution of particles, that is, dimensionless and noninteracting entities and (ii) infinite volume, valid

