# Relativistic one-dimensional hydrogen atom

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derived from the average velocities associated with 1845 and 1855. The westward acceleration tabulated for 1850 means that the average westward velocity was greater in the decade 1850–1860 than it had been in the decade 1840–1850. Could this be a reflection of the California gold rush of 1849? The answer is "yes." A population of 92 597 was recorded for California at the census date of 1 June 1850, even though California was not admitted as a state until 9 September 1850. At the census date of 1 June 1860, the population of California was 379 994.[3] This quadrupling of California's population would contribute to the large westward velocity during the 1860's, which is seen as a reduced magnitude of $v(x)$ for 1865 and as positive $a(x)$ for 1860. The large westward acceleration (1940) for the period 1930–1950 probably reflects the effects of the rapid industrial development of the Pacific coast states during World War II, 1940–1950. The 1970 acceleration may reflect the growth of the high-technology industries in the west and southwest. The EW accelerations seem to alternate in sign while the EW velocities are always to the west. This would suggest some form of oscillation superimposed on the westward velocity. It would be interesting to see if a mechanism could be identified that was driving the oscillation.

The recent NS accelerations show only one interruption (1960) in an almost continuous acceleration of our population to the "sun belt" for several decades. This may be driven by people's desire to escape from the northern winters. The advent of central air conditioning could have been a factor in the large southward acceleration associated with 1950 while the civil rights turmoil may have been a factor in the reduction in the southward velocity associated with 1960. Unfortunately the data are coarse and do not permit fine-grained analysis.

## X. CONCLUSION

We often see physics experiments where students in the laboratory study the record of position marks made in equal time intervals by a body moving with constant acceleration. The students then use first and second differences to determine the average velocities and acceleration of the body as a function of time. Here is an example from an area outside of physics which uses these concepts plus the additional concepts of center-of-mass and of changing mass. This example could be the subject of a laboratory-type exercise. Students could be given three or more consecutive lines of the demographic data of Table II and could be asked to use them to calculate the values of interesting physical parameters that characterize our dynamic population.

[1]The tonne is the SI unit of mass equal to 1000 kg.

[2]"Superdigitation" is "the increasingly common practice of presenting numerical engineering data with many more digits than are warranted by the intrinsic certainty of the data." *Epsilonics* (Measurements Group, Inc., Raleigh, NC, 1982), p. 11.

[3]*The World Almanac and Book of Facts* (Newspaper Enterprise Assoc., Inc., New York, 1983.) This source gives population data for each state each time the state was included in the census, pp. 208–209.

# Relativistic one-dimensional hydrogen atom

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The problem of the one-dimensional hydrogen atom has evoked interest because of its relevance to the behavior of hydrogeniclike atoms in the presence of strong magnetic fields and of hydrogenic impurities confined in quantum-well wire structures. The binding energy of the one-dimensional nonrelativistic hydrogen atom has been found to be infinite in its ground state. We have solved the relativistic hydrogen atom problem for the one-dimensional case using the Klein–Gordon equation. We find that the binding energy in the ground state for the one-dimensional relativistic hydrogen atom is finite and is of order of the rest mass energy of the electron. Therefore a relativistic treatment removes the infinite binding energy found for the ground state for the one-dimensional nonrelativistic hydrogen atom.

## I. INTRODUCTION

The hydrogen-atom problem has always excited interest because of its relevance to understanding the behavior of one-electron atoms, hydrogenic impurities in semiconductors, positronium, and Wannier excitons in solids. The Schrödinger equation for the hydrogen atom has been solved in one,[1] two[2] and three[3] dimensions. The interest in the three-dimensional solution is obvious because of its relevance to the problem of the real hydrogen atom. In this case, the interaction between the negative and positive charges in the atom are via an attractive Coulomb potential. The solutions of the hydrogen-atom problem in one and two dimensions have become of interest because of the problem of electrons which are spatially confined to move in one or two dimensions.[1,4–7] With the recent development of molecular beam epitaxy (MBE) techniques,[8–10] there has been a growing interest in quantum-well structures in which the electrons are confined to move in two dimensions. This confinement of the electrons implies that if one is interested in the binding energies of hydrogenic impurities[11–15] or excitons[16,17] in these structures, one must solve the hydrogen-atom problem for a quasi-two-dimensional system. The interest in the solution of the hydrogen-atom problem in one dimension arises because of the problem of excitons in semiconductors in the presence of strong magnetic fields.[4] More recently there has been a growing interest in the problem of the hydrogen atom in the presence of the intense magnetic fields which can occur under astrophysical conditions, such as, for example, in the vicinity of a pulsar.[5,6] Under these conditions the motion of the electron-hole pair in the exciton or the electron in the hydrogen atom is spatially confined by the strong magnetic fields.

Loudon[1] solved the problem of the nonrelativistic one-dimensional hydrogen atom because of his interest in the behavior of excitons in semiconductors in the presence of strong magnetic fields.[4] Because he was interested in a situation where the bound electron-hole pair was spatially confined by the high magnetic field rather than a true one-dimensional hydrogen atom, he assumed that the electron-hole pair interacted via an attractive Coulomb potential which fell off inversely as the distance between the electron-hole pair. He found that each of the states of the one-dimensional hydrogen atom was doubly degenerate except for the ground state. For the ground state, he found that the binding energy was infinite and that the probability density arising from the ground-state wave function was a delta function localizing the electron at the origin of the attractive Coulomb potential. Andrews[18] has questioned the existence of this ground state of the one-dimensional hydrogen atom. However, Haines and Roberts[19] have shown that it does exist by investigating the solutions of the truncated Coulomb potential for the one-dimensional hydrogen atom. They also claimed that contrary to Loudon's conclusions, the energy spectrum of the one-dimensional hydrogen atom is not degenerate and that the solutions having even parity have a continuous energy spectrum. Gomes and Zimerman[20] have also studied the bound-state solutions of the one-dimensional hydrogen atom and questioned the validity of the even parity solutions proposed by Haines and Roberts.[19] However, none of the arguments proposed by these authors has eliminated the existence of the infinity binding energy of the nonrelativistic one-dimensional hydrogen atom with a ground-state wave function of even parity. Interest in the one-dimensional hydrogen-atom problem has been revived by the fabrication of quantum-well wire structures in which the carriers are confined to move in one dimension,[21] i.e., along the axis of the wire. Because of the fabrication of these quasi-one-dimensional structures, interest has arisen in the binding energy of hydrogenic impurities and excitons in these systems.[7]

The existence of the infinity binding energy for the ground state of the nonrelativistic one-dimensional hydrogen atom is a troubling feature. For systems in which the binding energy is greater than the rest mass energies of the particles in the systems, relativistic effects should be important. Therefore in this paper we solve the problem of the relativistic one-dimensional hydrogen atom to see whether the binding energy of the ground state is effected by taking relativistic effects into account. Lapidus[22] has considered the problem of the relativistic hydrogen atom but in his treatment he has taken the interaction potential to be a delta function rather than the one-dimensional Coulomb potential used by Loudon[1] and others.[18–20] This is in agreement with the approach he has taken in modeling the interaction potential in his treatment of the nonrelativistic one-dimensional nonrelativistic hydrogen atom.[23] However, it is not consistent with the case in which the Coulomb potential is three dimensional but the electron is confined to move in only one dimension[24] To find the eigenstates and eigenvalues for our relativistic one-dimensional hydrogen-atom problem, we solve the Klein–Gordon equation using an attractive Coulomb potential in one dimension. This approach would be valid for an electron without spin. To take spin effects into account, one would have to solve an analogous problem using the Dirac equation. However, since we are mainly interested in how in ground-state binding energy is effected by taking relativistic effects into account, we will solve the problem of the relativistic one-dimensional hydrogen atom using the Klein–Gordon equation rather than the Dirac equation.

## II. SOLUTION

The relativistic Schrödinger equation for the one-dimensional hydrogen atom is

$$\frac{d^2\Psi}{dx^2}+\frac{1}{\hbar^2c^2}\left(E^2+\frac{2Ze^2E}{|x|}+\frac{Z^2e^4}{x^2}-m^2c^4\right)\Psi=0, \quad (1)$$

where $Ze$ is the charge of the nucleus. This is the relativistic wave equation for a charged particle without spin, so this treatment using the Klein–Gordon equation will neglect any relativistic correlations to the energy of the one-dimensional hydrogen atom which arise because of the spin of the electron. The solution of Eq. (1) for the regions $x<0$ and $x>0$ is straightforward. However, because of the singularity of the Coulomb potential at $x=0$, it is not immediately obvious how the solution in the two regions should be joined at the origin. The wave function is continuous at the origin but its derivative with respect to $x$ at the origin need not be continuous because of the singularity of the potential at that point. This point has been treated at some length for the nonrelativistic case by Loudon[1] and others.[19,20] There is general agreement that there are solutions of odd parity which are continuous at the origin and whose first derivative is also continuous at the origin. There is some disagreement between the various authors on whether there is also a solution having even parity at the origin and if such a solution exists, whether or not it is degenerate in energy with the odd-parity solution. Some of the same questions arise as well in the relativistic case.

Introducing the dimensionless variables

$$\rho=2\epsilon x,\quad \alpha=(e^2/\hbar c),$$
$$\epsilon=(\hbar c)^{-1}(m^2c^4-E^2)^{1/2},\quad \lambda=Z\alpha E\,(m^2c^4-E^2)^{-1/2}$$

and using the ansatz

$$\psi(\rho)=\rho^S f(\rho)\exp-(\rho/2), \quad (2)$$

we obtain the following equation for $f(\rho)$:

$$\rho f''+(2S-\rho)f'+(\lambda-S)f=0, \quad (3)$$

where $S=1/2\pm 1/2(1-4Z^2\alpha^2)$. For bound states, $E<mc^2$, so that $\epsilon$ is real. This is the equation for the confluent hypergeometric function which has the general solu-

tions[25,26]

$$f(\rho) = c_1 F(S-\lambda, 2S, \rho) + c_2 F(1-S-\lambda, 2-2S, \rho)\rho^{1-2S}, \quad (4)$$

where $c_1$ and $c_2$ are arbitrary constants. For the wave function to remain finite at the $x = +\infty$, the power series expansion for the confluent hypergeometric function must terminate after a finite number of terms[3] yielding a finite polynominal in $\rho$ for $f(\rho)$. If the power series expansion for the confluent hypergeometric function does not terminate, the wave function $\psi$ will diverge at $x = +\infty$. The power series expansion will only terminate if $\lambda = n + S$ for the first solution in Eq. (4) or if $\lambda = n + 1 - S$ for the second solution in Eq. (4). Note that $S$ has two values, $1/2 \pm 1/2(1-4Z^2\alpha^2)$ and $n = 0,1,2,\cdots$. For our case, these two solutions are not linearly independent since the first solution for the smaller value of $S$ is identical to the second solution for the larger value of $S$ and vice versa. Therefore we can in complete generality set $c_2 = 0$ in Eq. (4) as the first solution for the two values of $S$ gives us our two linearly independent solutions, of the different equation given in Eq. (3). When we set $\lambda = n + S$, we get the following result for the energy of the relativistic one-dimensional hydrogen atom:

$$E = mc^2 \Big/ \left(1 + \frac{Z^2\alpha^2}{(n+S)^2}\right)^{1/2}. \quad (5)$$

The solution to the wave equation given by Eqs. (2)–(4) is valid in the region where $x > 0$. In the region $x < 0$, we get the same form of the solution if we make the change of variables $t = -\rho$. Therefore the solution for the wave function of the relativistic, (1D) hydrogen atom which is finite at both the origin and at $x = \pm\infty$ is

$$\psi(\rho) = a_1 \rho^S \exp(-\rho/2) F(-n, 2S, \rho) \quad \text{for } x > 0 \quad (6a)$$

and

$$\psi(\rho) = a_2(-\rho)^S \exp(\rho/2) F(-n, 2S, -\rho) \quad \text{for } x < 0, \quad (6b)$$

where $\rho = 2Zx/a_0[(n+S)^2 + Z^2\alpha^2]^{1/2}$ with $a_0 = \hbar^2/me^2$ the Bohr radius for the hydrogen atom. Here $a_1$ and $a_2$ are constants which are determined by the normalization of the wave function and the boundary conditions at the origin. For states of even parity $a_1 = a_2$, while for the state of odd parity $a_1 = -a_2$.

In the limit $Z\alpha \ll 1$, which in the three-dimensional case corresponds to the nonrelativistic limit, the energies and wave function for the ground and excited states of the 1D hydrogen atom reduce to

$$E = mc^2(Z\alpha - \tfrac{1}{2}Z^3\alpha^3) \quad n = 0, \quad (7a)$$

$$E = mc^2\left[1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{3}{4n} - 2\right)\right] \quad n = 1,2,3,4\cdots, \quad (7b)$$

$$\psi_0(x) = \left(\frac{1}{\alpha a_0}\right)^{1/2} \exp(-|x|/\alpha a_0) \quad n = 0, \quad (8a)$$

$$\psi_n(x) = \left(\frac{2Z^3}{a_0^3 n^3 (n+1)^2}\right)^{1/2} x \exp\left(\frac{-Z|x|}{na_0}\right) \times L_n^1\left(\frac{2Z|x|}{na_0}\right) \quad n = 1,2,3,4,\cdots, \quad (8b)$$

where we have kept corrections to the energy to order $Z^4\alpha^4$, which are the lowest-order correlations to the nonrelativistic energies of the 1D hydrogen atom. Here $L_n^m(x)$ are the associated Laguerre polynomials,[27] and $\lambda_c = (\hbar/mc)$ is the Compton wavelength of the electron. We see that in this limit, the ground-state energy of the 1D hydrogen atom is to lowest order in $Z\alpha$ much smaller than the rest mass energy of the electron but that it is positive and finite. The excited-state energies and wave function in this limit are the same as those for the excited states ($n \geq 1$) of the nonrelativistic 1D hydrogen atom up to terms $Z^2\alpha^2$. Therefore, when $Z\alpha \ll 1$, only the ground-state energy is significantly modified from the nonrelativistic results when relativistic correlations are taken into account. If we define the binding energy of the atom as the difference between the relativistic energy and the rest mass energy, $E = E_b + mc^2$, the binding energy of the relativistic 1D hydrogen atom in this limit is of order $-mc^2$. The fact that the total relativistic energy is positive in the ground state means that the 1D hydrogen atom is stable against the spontaneous creation of electron-positron pairs. This would not have been the case if the binding energy in the relativistic case had been greater than twice the rest mass energy of the electron. We also note that in the ground state of the 1D hydrogen atom the electron is only localized to a distance of a Compton wavelength to the origin of the Coulomb potential. In the nonrelativistic case, the electron was completely localized around the origin in the ground state with a delta function probability density. It is this difference in the amount of localization of the electron around the origin in the ground state which explains the difference between the finite binding energy predicted by our relativistic treatment and the infinite binding energy predicted by the nonrelativistic treatment. Finally, for the excited states, the relativistic energies for the 1D hydrogen atom are identical to those for the $s$ states ($l = 0$) for the 3D hydrogen atom using the Klein–Gordon equation.[28] The main difference between the 1D and 3D hydrogen atoms is the occurrence of the strongly bound ground state for the 1D case.

An additional point of interest involves the probabilities for transitions from the strongly bound ground state to the excited states of the 1D hydrogen atom. Because the excited state wave function vanish near the origin of the Coulomb potential while the ground-state wave function, even in the relativistic case, is strongly localized near the origin, we expect that the transitions probabilities from the ground to the excited states will be extremely small. As an illustration, we will present a calculation here of the matrix element for the transition between the ground state and an excited state of odd parity using the electric dipole approximation. (In this approximation, the matrix elements for transitions to even parity excited states, if they exist, would vanish identically.) The electric dipole matrix element in this case is

$$(n|x|1) = \left(\frac{2Z^3}{\alpha a_0^4 n^3 (n+1)^2}\right)^{1/2} \int_{-\infty}^{\infty} dx\, x^2 \exp\left[-\frac{|x|}{a_0}\left(\frac{1}{\alpha} + \frac{Z}{n}\right)\right] L_n^1\left(\frac{2Z|x|}{na_0}\right)$$
$$= \frac{2^{1/2}}{4}\left(\frac{a_0}{\alpha^{1/2}Z^{3/2}}\right) n^{3/2} q^{-3-n}(q-1)^{n-1}(2q-n-2), \quad (9)$$

where we have used the integral

$$\int_0^\infty dt\, t^r e^{-qt} L_n^r(t) = \frac{\Gamma(r+n+1)(q-1)^n}{n! q^{r+n+1}},$$

$$q = \frac{n}{2Z}\left(\frac{1}{\alpha} + \frac{Z}{n}\right) \tag{10}$$

from Gradshteyn and Ryzhik.[29] To obtain Eq. (9), we partially differentiate Eq. (10) with respect to $q$. Since the Compton wavelength of the electron, $\lambda_c = \alpha a_0$, is much smaller than the Bohr radius, the matrix element given by Eq. (9) is very small for the 1D hydrogen atom where $Z = 1$.

## III. DISCUSSION

In this paper, we have solved the one-dimensional relativistic Schrödinger equation for a charged particle without spin confined to move in at attractive Coulomb potential. For the excited states of the system, we find that in view of the small value of the fine-structure constant $\alpha = (1/137)$, that for the hydrogen where $Z = 1$, the energy eigenvalues and eigenfunctions do not differ significantly from those obtained in the nonrelativistic treatment. In addition, when the relativistic corrections are important, they are the same for the excited states as they are for the $s$ states of the 3D hydrogen atom. However, for the ground state, the results of the relativistic treatment differ significantly from those obtained using the nonrelativistic treatment. The nonrelativistic treatment yields an infinite binding energy in the ground state of the 1D hydrogen atom with a delta-function probability density. The relativistic treatment, on the other hand, predicts a finite binding energy of order of the rest mass energy of the electron and a probability density which falls off exponentially with distance from the origin with a characteristic decay length of order of the Compton wavelength of the electron. It is due to the fact that the relativistic treatment provides a characteristic length, i.e., the Compton wavelength for the localization of the electron in the ground state, that the binding energy of the 1D hydrogen atom is finite. Such characteristic lengths also occur in other problems in which the electron in a hydrogenic atom is confined in a quasi-one-dimensional system, either by the application of strong magnetic fields[4–6] or the confinement in a quasi-one-dimensional quantum wire structure.[7]

The results obtained here apply strictly to an electron without spin since we have solved the Klein–Gordon equation, which is valid for a particle without spin, instead of the Dirac equation which does take the spin of the electron into account. However, since our main motivation was to investigate the binding energy of the 1D hydrogen atom in its ground state, the use of the Klein–Gordon equation is justified in predicting the limiting energy of the 1D hydrogen atom in the ground state. Moreover, since the nonrelativistic treatments of the 1D hydrogen atom did not take electron spin into account, we can directly compare our results in the limit $Z\alpha \ll 1$ and see that except for the ground state, the energies and wave functions reduce to those obtained in the nonrelativistic treatment in this limit. A relativistic treatment of the 1D hydrogen atom has been given using the Dirac equation,[22] but a comparison with our results is made difficult because a delta-function potential was used instead of an attractive Coulomb potential for the interaction between the positive and negative charges in the hydrogen atom. We would expect that the use of the Dirac equation for the 1D hydrogen atom would lead to differences from our results only in terms of order $Z^4\alpha^4$ because of the contribution of terms involving the electron spin to the fine-structure corrections which occur relativistically. Finally, we expect that the transition probabilities between the ground and excited states of the 1D hydrogen atom will be extremely small because of the localization of the electron in the ground state in a region in which the excited-state wave function are negligible. However, in the relativistic treatment, these transition probabilities although small would be finite, unlike the nonrelativistic treatment where these transition probabilities would vanish identically because of the delta-function nature of the ground-state probability density.


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