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A particle moving in a homogeneous time-varying force

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The problem of a quantum-mechanical particle that moves in a spatially uniform time-varying field of force is solved in the Schrödinger picture. An explanation on the peculiar behavior for the spreading of an initial wave packet is given in terms of the dispersion relation obtained in the analytical solution.

A particle moving in a time-varying spatially uniform force $F(t)$ given by a potential energy $V(x,t)$ of the form

$$V(x,t) = -F(t)x \tag{1}$$

is quite an interesting example of a quantum-mechanical problem for two main reasons: It is completely solvable and the particle wave packet spreading in time is exactly equal to that of a free particle regardless of the applied force. The explanation of the phenomenon—which is the objective of this work—is very instructive and sheds light over the quantum-mechanical origin of the spreading of a wave packet. These details originate by the peculiar form of the potential energy (1) which is linear in x.

As an example, we may think of a charged particle moving in the time-varying electric field of a parallel plate capacitor, provided $v/c \ll 1$ to justify neglecting the interaction with the induced magnetic field.

The quantum-mechanical problem derived from (1) is more easily solved in the Schrödinger picture for the time-evolution operator $U(t)$.

Writing the Hamiltonian

$$H(p,x,t) = p^2/2m - F(t)x, \tag{2}$$

the Schrödinger equation for $U(t)$ to be solved is

$$i\hbar \frac{\partial U(t)}{\partial t} = HU(t), \quad U(0) = \mathbf{1}, \tag{3}$$

and, as we can see

$$[H(t), H(t')] = i\hbar (p/m) [F(t') - F(t)]$$

hence different from zero for a time-varying force, and therefore (3) cannot be solved by simple exponentiation of the time integral of H. However, following the suggestion given by G. Baym,[1] we write

$$U(t) = e^{i\alpha(t)x} \bar{U}(t), \tag{4}$$

where $\alpha(t)$ is a real c number and $\bar{U}(t)$ is a unitary operator. Imposing the initial conditions

$$\alpha(0) = 0, \quad \bar{U}(0) = \mathbf{1},$$

results in the original condition on $U(t)$.

Substitution of (4) into (3), and rearranging terms, results in a pair of differential equations equivalent to (3):

$$\frac{d\alpha(t)}{dt} = \frac{1}{\hbar} F(t), \tag{5a}$$

$$i\hbar \frac{d\bar{U}}{dt} = \mathscr{H}(p,t)\, \bar{U}(t), \tag{5b}$$

where $\mathscr{H}(p,t)$ is some "reduced Hamiltonian" given by

$$\mathscr{H}(p,t) = e^{-i\alpha(t)x} \frac{p^2}{2m} e^{i\alpha(t)x}$$

$$= \frac{(p + \hbar\alpha)^2}{2m}$$

$$= \frac{p^2}{2m} + \frac{\hbar\alpha(t)}{m} p + \frac{\hbar^2\alpha^2(t)}{2m},$$

where the translation operator $\exp(i\xi x/\hbar)$ in momentum space was used. Notice that $\mathscr{H}$ depends on p and t only, but not on x, and therefore $[\mathscr{H}(t), \mathscr{H}(t')] = 0$. The solutions to (5a) and (5b) are now easily found:

$$\alpha(t) = \frac{1}{\hbar} \int_0^t F(t')\, dt', \tag{6a}$$

$$\bar{U}(t) = \exp\left(\frac{-i}{\hbar} \int_0^t \mathscr{H}(p,t')\, dt'\right)$$

$$= \exp\left[\frac{-i}{\hbar}\left(H_0 t + \frac{\hbar\beta(t)}{m} p + \frac{\hbar^2\gamma(t)}{2m}\right)\right], \tag{6b}$$

where we have introduced the notation

$$\beta(t) = \int_0^t \alpha(t')\, dt', \tag{7a}$$

$$\gamma(t) = \int_0^t [\alpha(t')]^2\, dt', \tag{7b}$$

and $H_0 = p^2/2m$ is the kinetic energy operator.

With solutions (6a) and (6b) we can build up the evolution operator $U(t)$ as proposed in Eq. (4):

$$U(x,p,t) = e^{i\alpha(t)x} \times \exp\left[\frac{-i}{\hbar}\left(H_0 t + \frac{\hbar\beta(t)}{m} p + \frac{\hbar^2\gamma(t)}{2m}\right)\right] \tag{8}$$

and the problem is thus solved in a very general way for any arbitrarily time-varying force $F(t)$.

Now we are able to calculate any quantity of relevant interest. For example, the classical trajectory of the particle will be reproduced by $\langle x\rangle(t) = \langle\psi(t)|x|\psi(t)\rangle$, where

$$|\psi(t)\rangle = U(t)|\psi_0\rangle$$

and $|\psi_0\rangle$ is the initial state vector. Hence, the matrix element that has to be calculated is

$$\bar{x}(t) \equiv \langle x\rangle(t) = \langle\psi_0| U^\dagger(t) x U(t)|\psi_0\rangle,$$

and, by inspection of (8) we see that the x-dependent as well as the c-number-dependent phase [the term with $\gamma(t)$] commute with x and produce the unit operator, leaving expression

$$\bar{x}(t) = \langle\psi_0|\exp\left(i\frac{\beta(t)}{m}p\right)\exp\left(\frac{i}{\hbar}H_0t\right) \times x\exp\left(\frac{-i}{\hbar}H_0t\right)\exp\left(\frac{-i\beta(t)}{m}p\right)|\psi_0\rangle \quad (9)$$

to be calculated. By using the free evolution operator

$$U_0(p,t) = \exp\left(\frac{-i}{\hbar}H_0t\right) \quad (10)$$

to evolute x, it is obtained:

$$\exp\left(\frac{i}{\hbar}H_0t\right)x\exp\left(\frac{-i}{\hbar}H_0t\right) = x + \frac{p}{m}t, \quad (11)$$

and the x-space translation operator

$$e^{i\beta(t)p/m}\,x\,e^{-i\beta(t)p/m} = x + \frac{\hbar\beta(t)}{m}. \quad (12)$$

We finally obtain from (9):

$$\bar{x}(t) = \langle x\rangle_0 + \frac{\langle p\rangle_0}{m}t + \frac{\hbar\beta(t)}{m}, \quad (13)$$

which is exactly of the same form as the solution of the classical problem for the particle trajectory $x_{\text{class}}(t)$, in which $\hbar\alpha(t)$ represents the impulsion given by force $F(t)$ to the particle and $\hbar\beta(t)/m$ is the nonlinear t dependence in $x_{\text{class}}(t)$ produced by that nonuniform acceleration.

The quantity we are interested in is the dispersion $[\Delta x(t)]^2$ of an initial wave packet $\psi_0(x,0) = \langle x|\psi_0(0)\rangle$. We also want to verify that it is exactly equal to the dispersion of the wave packet of a freely moving particle.

The dispersion is defined by

$$[\Delta x(t)]^2 = \langle\psi(t)|[x-\bar{x}(t)]^2|\psi(t)\rangle = \langle\psi_0|U^\dagger(t)[x-\bar{x}(t)]^2U(t)|\psi_0\rangle, \quad (14)$$

while the dispersion for a free particle is given by

$$[\Delta x_0(t)]^2 = \langle\psi_0|U_0^\dagger(t)\,[x-\bar{x}_0(t)]^2U_0(t)|U_0\rangle, \quad (15)$$

where U_0 was introduced in Eq. (10), and $\bar{x}_0(t)$ is the expectation value of the coordinate of the free particle:

$$\bar{x}_0(t) = \langle\psi_0|U_0^\dagger(t)xU_0(t)|\psi_0\rangle = \langle x\rangle_0 + \frac{\langle p\rangle_0}{m}t. \quad (16a)$$

In order to transform expression (14) let us recall that, comparing Eqs. (13) and (16a), we may write

$$\bar{x}(t) = \bar{x}_0(t) + \frac{\hbar\beta(t)}{m}, \quad (16b)$$

and that $U(t)$, given by Eq. (8), has the form

$$U(x,p,t) = \exp\left[i\left(\alpha(t)x - \frac{\hbar\gamma(t)}{2m}\right)\right] \times U_0(p,t)\exp\left(-i\frac{\beta(t)}{m}p\right). \quad (16c)$$

Therefore Eq. (14) can be transformed in the following way:

$$[\Delta x(t)]^2 = \langle\psi_0|\{U^\dagger[x-\bar{x}(t)]U\}^2|\psi_0\rangle = \langle\psi_0|\left[U_0^+\left(x+\frac{\hbar\beta(t)}{m}-\bar{x}(t)\right)U_0\right]^2|\psi_0\rangle$$

[use of Eq. (12) was made] and using (16b) we finally obtain

$$[\Delta x(t)]^2 = \langle\psi_0|U_0^+[x-\bar{x}_0(t)]^2U_0|\psi_0\rangle,$$

which, compared with Eq. (15) says that

$$[\Delta x(t)]^2 = [\Delta x_0(t)]^2. \quad (17)$$

As proposed: *The dispersion of a particle wave packet moving in a spatially uniform but arbitrarily time-varying force is exactly the same as that of a freely moving particle, regardless of the applied force.*

The explicit calculation of this dispersion is not relevant for our purposes; however, a straightforward evaluation of Eq. (15) using Eq. (11) gives the well-known result

$$[\Delta x(t)^2] = (\Delta x_0)^2 + (\Delta p_0)^2\frac{t^2}{m^2} + (\langle xp+px\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0)\frac{t}{m}. \quad (18)$$

(Here zero subindices means initial value quantities.) The last term of (18), linear in time t, may be eliminated by a convenient choice of the initial state vector $|\psi_0\rangle$.[2]

To explain the surprising result (17) let us recall that the spreading of the wave packet of a free quantum-mechanical particle is a consequence of the fact that the frequency ω of every Fourier component of the packet is given by the De Broglie–Einstein relation

$$\hbar\omega = E(k), \quad (19)$$

where $E(k) = \hbar^2k^2/2m$ is the kinetic energy, and creates a nonlinear dispersion relation

$$\omega(k) = \hbar k^2/2m$$

that results in a phase velocity $V_{\text{ph}}^{(0)}$:

$$V_{ph}^{(0)} = \frac{\omega(k)}{k} = \frac{\hbar k}{2m}, \quad (20)$$

different in fact for each harmonic component of the wave packet. Hence the change in form of the wave packet while it moves with the constant group velocity. This is a very well understood and accepted idea: *Empty space is a dispersive medium for a quantum-mechanical particle.* This is in contrast with the free propagation of a pulse of electromagnetic radiation, that will never change form unless it enters into a dispersive medium. However, in that case it will no longer be a "free-propagation" phenomenon.

Let us search for the dispersion relation produced by our solution (8). To do so write the wave function $\langle x|\psi(t)\rangle$ explicitly. Starting from the initial state vector $|\psi_0\rangle$, and calculating

$$\langle x|\psi(t)\rangle = \langle x|U(t)|\psi_0\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\langle x|U(t)|p\rangle\psi_0(p), \quad (21)$$

we get

$$\psi(x,t) = \exp\left[i\left(\alpha(t)x - \frac{\hbar\gamma(t)}{2m}\right)\right] \times \int_{\infty}^{\infty}dk\,\psi_0(k)e^{i\Phi(x,k,t)}, \quad (22)$$

where we have introduced $\Phi(x,k,t)$—the phase of each Fourier component of wavenumber k, given by

$$\Phi(x,k,t) = kx - \frac{\hbar k^2}{2m}t - \frac{\hbar\beta(t)}{m}k. \quad (23)$$

The dispersion is controlled by the probability density $|\psi(x,t)|^2$, and we see that the factor in front of the integral in Eq. (22) disapears. Then, the phase velocity for each Fourier component of constant k is given by the velocity of the Φ = constant surfaces, that is

$$V_{ph} = \frac{dx}{dt} = -\left(\frac{\partial \Phi}{\partial t}\right)_k \bigg/ \left(\frac{\partial \Phi}{\partial x}\right)_k,$$

$$V_{ph} = V_{ph}^{(0)} + \frac{\hbar \alpha(t)}{m}, \tag{24}$$

which is exactly the same as Eq. (20) except for the second term on the right-hand side of Eq. (24) that, anyway, is a constant with respect to wavenumber k, and henceforth will not produce any further dispersion among the Fourier components than that produced already by $V_{ph}^{(0)}$. Hence the dispersion calculated with $|\psi(x,t)|^2$ will be exactly the same as that calculated with $|\langle x|U_0(t)|\psi_0\rangle|^2$; this is the space-time evolution picture of the phenomenon represented formally by Eq. (17).

Notice that our conclusions are independent of the peculiar form of the force and of the particular initial state vector. It represents quite a general result of the behavior of a quantum-mechanical particle moving in the type of force field here considered.

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[1]Gordom Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1977), p. 147.

[2]Write the coefficient of $(2t/m)$ as $\mathrm{Re}\langle xp\rangle_0 - \langle x\rangle_0\langle p\rangle_0$. Observe that an initial state vector $|\psi_0\rangle$ satisfying the condition $\langle x|\psi_0\rangle = \langle\psi_0|x\rangle$ or the condition $\langle p|\psi_0\rangle = \langle\psi_0|p\rangle$ will eliminate that coefficient.

SOLUTION TO THE PROBLEM ON PAGE 526

The problem with a transfer orbit to Counter Earth is that the planet shares the same orbit as the Earth. The solution entails launching the spacecraft into an elliptical orbit about the sun which has a period T of $1\frac{1}{2}$ years. When the spacecraft completes one revolution in this orbit, Counter Earth will be in the position the Earth was at launch. The semimajor axis of the transfer orbit can be found from Kepler's third law,

$$(T/T_E)^2 = (a/a_E)^3,$$

where $T_E = 1$ yr. $a_E = 1.00$ AU (Astronomical Unit). The result is $a = 1.31$ AU. The perihelon and aphelion distances therefore are $r_a = 1.00$ AU and $r_p = 1.62$ AU, respectively.

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