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H. A. Gersch

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irreducible representation of the Poincaré group and does not have the possibility of Zitterbewegung in any unitarily transformed version of the theory.

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[22]It is not known whether all of the fundamental interactions in nature may be written as local vectors. Whether or not this turns out to be true, it still may be the case that useful phenomenological approximations to them may not be able to be written as local four-vectors.

[23]In the canonical representation, the Coulomb interaction is an algebraic function of the position $\hat{W}^{-1}\hat{\mathbf{X}}\hat{W}$ which is nonlocal in the eigenstates of $\hat{\mathbf{X}}$. Thus the Coulomb interaction is nonlocal in the eigenstates of $\hat{\mathbf{X}}$. This point of view is emphasized in Ref. 18.

An exactly soluble one-dimensional, two-particle problem

H. A. Gersch
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

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We examine a one-dimensional problem involving two interacting particles in an external field for which the Schrödinger equation can be solved analytically for the ground state. The Hartree approximation for this system also has a simple analytic solution of the form previously derived by Foldy. Comparison of exact and Hartree solutions leads to precise estimates of the effect of correlations on the binding energy and on the two-particle probability distribution.

I. INTRODUCTION

The paucity of exactly soluble realistic problems involving interacting particles in three dimensions has led to consideration of more tractable one-dimensional models for introductory pedagogical treatments of the many-body problem. Among these, use of Dirac delta function potentials has received a good deal of attention.[1–3] Recently, a one-dimensional problem for two interacting particles in the Hartree approximation was independently solved analytically by Foldy[4] and by Nogami *et al.*[5] The model was shown to have close similarities to the three-dimensional problem for the helium atom. That the one-dimensional model with δ function potentials exhibits features similar to the three-dimensional helium atom with Coulomb potentials is to be expected from the work of Herrick and Stillinger.[6] They treated the helium atom in variable space dimensions, and showed that the Coulomb potential reduces to a δ function potential as the space dimension is scaled to unity.

The results of Refs. 4 and 5 led us to the initial speculation as to whether one could go even further and solve their model exactly, which would provide a means of assessing the accuracy of the Hartree method for the model. Unfortunately, we soon realized that the exact solution for the bound state wave function for the helium-like, one-dimensional model is not simply expressible in terms of exponential functions, a conclusion which had already been reached by others.[3,7] However, at the expense of adding a term to the two-particle interaction which essentially restores a symmetry not present in the helium-like model, the bound state wave function is easily expressible in terms of simple exponential functions. Lost in this process is the distinct advantage of having a close connection to a real physical problem (the model no longer relates directly to helium-like atoms), but gained is a very simple bound state solution for a two-particle interacting system which includes the particle correlations and correlation energy not present in the Hartree approximation. So although the system treated here suffers somewhat from the rather unphysical character of the two-particle interaction introduced in order to obtain a simple exact bound state solution, it may still have use for pedagogical purposes since it affords a precise comparison between Hartree and exact solutions.

In Sec. II we briefly review the Hartree solution of Refs. 4 and 5 for the helium-like model. A minor modification of this treatment will later be shown (in Sec. IV) to give the Hartree solution for an exactly soluble model.

Section III describes the modified model and its exact bound state wave function and energy. Section IV compares these with the Hartree solution for this model.

II. THE He-LIKE MODEL IN THE HARTREE APPROXIMATION

The model problem treated in Refs. 4 and 5 deals with the Hamiltonian H':

$$H' = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - g\delta(x_1) - g\delta(x_2) + g'\delta(x_1 - x_2) \qquad (1)$$

representing two particles attracted to the origin with δ function potentials of strength g and repelling each other with strength g'. With $g = ze^2$, $g' = g/z = e^2$ this Hamiltonian describes a one-dimensional model of a He-like ion, in which Coulomb interactions are replaced by delta functions. These functions, although very different in form from Coulomb potentials nevertheless share an identical virial theorem with the Coulomb potentials.[4] In terms of dimensionless variables y_1, y_2 defined by

$$y_i = (mg/\hbar^2)\, x_i\,; \quad i = 1,2 \tag{2}$$

the Hamiltonian in dimensionless energy h',

$$h' = (\hbar^2/mg^2)\, H'\,, \tag{3}$$

is

$$h' = -\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right) - \delta(y_1) - \delta(y_2) + \frac{1}{z}\delta(y_1 - y_2)\,. \tag{4}$$

The bound state wave function in the Hartree approach is

$$\psi(y_1,y_2) = \phi(y_1)\,\phi(y_2)\,. \tag{5}$$

Minimizing the expectation value of h' for this ψ with respect to arbitrary variations in ϕ, subject to normalization $\int\phi^2(y)dy = 1$, gives the Hartree equation

$$-\frac{1}{2}\frac{d^2\phi}{dy^2} - \delta(y)\phi + \frac{1}{z}\phi^3 = \eta\phi\,, \tag{6}$$

where η represents the Lagrange multiplier introduced by the normalization constraint. Equation (6) was shown in Refs. 4 and 5 to have the solution

$$\phi(y) = 2\alpha z^{1/2} c e^{-\alpha|y|}(1 - c^2 e^{-2\alpha|y|})^{-1}\,, \tag{7}$$

where

$$c^2 = (4z - 1)^{-1}\,, \tag{8}$$

$$\alpha = 1 - (2z)^{-1}\,, \tag{9}$$

and to correspond to the Hartree bound state energy

$$\langle h'\rangle = -\left(1 - \frac{1}{z} + \frac{1}{12z^2}\right). \tag{10}$$

In Sec. IV we adapt these results to the modified Hamiltonian introduced in Sec. III.

III. MODIFIED HAMILTONIAN AND EXACT BOUND STATE SOLUTION

As remarked in Sec. I, the ground state solution of the Hamiltonian h' in Eq. (4) is not expressible in terms of simple exponentials, and it does not appear possible to give a neat closed form expression for ψ. A nice discussion of the difficulties involved is given in Ref. 7. A precursor of such difficulties is perhaps evident by noting a lack of spatial symmetry for the h' of Eq. (4). Namely, there is a potential trough along $y_1 = 0$ and an identical trough along $y_2 = 0$, exhibiting thus a symmetry with respect to a $\pi/2$ rotation in the y_1, y_2 plane. Then, in addition, there is a potential hill along $y_1 = y_2$. For h' to be invariant to a $\pi/2$ rotation, there should be an identical potential hill along $y_1 = -y_2$, which, of course is not present in Eq. (4). So one might expect things to become simple if the missing symmetry is restored to h' by adding to it the interaction $g'\delta(y_1 + y_2)$, giving a modified h,

$$h = -\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right) - \delta(y_1) - \delta(y_2) + (1/z)[\delta(y_1 - y_2) + \delta(y_1 + y_2)]\,. \tag{11}$$

The bound state wave function for this h is

$$\psi(y_1,y_2) = [(1 - 2\alpha)(1 - \alpha)]^{1/2} \times e^{-|y_1|}e^{-|y_2|}e^{\alpha[|y_1 - y_2| + |y_1 + y_2|]} \tag{12}$$

which can be arrived at by at least two ways. One of these consists of enumerating the four boundary conditions imposed on ψ by the singular potential energy terms, which require that a continuous ψ have jump discontinuities in its first derivative along the lines $y_1 = 0$, $y_2 = 0$, and $y_1 = \pm y_2$. Integrating $h\psi = \epsilon\psi$ successively over tiny regions of y_1 and y_2 enclosing each singular line one gets four equations which are satisfied by Eq. (12) if

$$\alpha = 1/2z \tag{13}$$

and the dimensionless energy ϵ is

$$\epsilon = -(1 - 1/z + 1/2z^2)\,. \tag{14}$$

Alternatively, one can substitute Eq. (12) for ψ into $h\psi = \epsilon\psi$ and show that ψ satisfies this equation if α and ϵ are as given above. This method makes use of the identities

$$\frac{d|y|}{dy} = \frac{y}{|y|} = 2\theta(y) - 1 \tag{15}$$

and

$$\frac{d^2|y|}{dy^2} = 2\delta(y)\,, \tag{16}$$

which give

$$\begin{aligned}\frac{\partial^2\psi}{\partial y_1^2} = {} & [1 + 2\alpha^2 - 2\delta(y_1) + 2\alpha(\delta(y_1 - y_2) + \delta(y_1 + y_2))]\,\psi \\ & - 2\alpha\frac{y_1}{|y_1|}\left(\frac{y_1 - y_2}{|y_1 - y_2|} + \frac{y_1 + y_2}{|y_1 + y_2|}\right)\psi \\ & + 2\alpha^2\frac{y_1 - y_2}{|y_1 - y_2|}\frac{y_1 + y_2}{|y_1 + y_2|}\psi\end{aligned} \tag{17}$$

and a similar equation for $\partial^2\psi/\partial y_2^2$, obtained from Eq. (17) by interchanging y_1 and y_2. Adding these two second derivatives, the last term on the right-hand side of Eq. (17) is seen to cancel against its counterpart in $\partial^2\psi/\partial y_2^2$, while the two middle terms add to give $-4\alpha\psi$, so that

$$\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\psi = [1 + 2\alpha^2 - 2\alpha - 2\delta(y_1) - 2\delta(y_2) + 2\alpha(\delta(y_1 - y_2) + \delta(y_1 + y_2))]\,\psi\,. \tag{18}$$

Therefore $h\psi = \epsilon\psi$, with h given by Eq. (11), is satisfied if $2\alpha = 1/z$ and $\epsilon = 1 - 2\alpha + 2\alpha^2$, which are the results quoted in Eqs. (13) and (14).

Using this method shows that a simple solution of the form $\psi = ce^{-|y_1|}e^{-|y_2|}e^{\alpha|y_1 - y_2|}$ will not work for the He-like h' of Eq. (4). This ψ, inserted in the Schrödinger equation $h'\psi = \epsilon\psi$ requires $\alpha = 1/2z$ and

$$\epsilon = \begin{cases} -(1 + \alpha^2) & \text{for } y_1 y_2 > 0 \text{ and } y_1 y_2 < 0 \\ -(1 - \alpha)^2 & \text{for } y_1 > 0, y_2 < 0 \text{ and } y_1 < 0, y_2 > 0\,. \end{cases} \tag{19}$$

Thus this simple exponential ψ is not an acceptable solution for the He-like Hamiltonian h' since it does not correspond to a single energy.

IV. COMPARISON BETWEEN HARTREE AND EXACT SOLUTION

The additional repulsive interaction term $(1/z)\delta(y_1+y_2)$ in the modified h of Eq. (11) adds to the Hartree expectation value $\langle h'\rangle$ the term $(1/z)\int\phi^2(y_1)\phi^2(-y_1)\,dy_1$. Since $\phi^2(y_1)=\phi^2(-y_1)$ from symmetry, this added term is identical to that appearing in the He-like $\langle h'\rangle$ coming from the $(1/z)\delta(y_1-y_2)$ interaction. Consequently, the Hartree solutions, Eqs. (7)–(10) are applicable to the modified Hamiltonian h by the replacement $1/z\to 2/z$ in them. The Hartree wave function $\psi_{\rm H}$ for the modified Hamiltonian is then

$$\psi_{\rm H}=\phi(y_1)\phi(y_2)=\frac{2(z-1)^2}{z(2z-1)}\,\frac{e^{-(1-1/z)|y_1|}e^{-(1-1/z)|y_2|}}{\{1-[1/(2z-1)]\,e^{-2(1-1/z)|y_1|}\}\{1-[1/(2z-1)]\,e^{-2(1-1/z)|y_2|}\}} \tag{20}$$

and the Hartree energy is

$$\epsilon_H=-(1-1/z+1/3z^2)\,. \tag{21}$$

These are to be compared with exact results, from Eqs. (12) and (14):

$$\psi=\frac{[(z-1)(2z-1)]^{1/2}}{2^{1/2}z}\times e^{-|y_1|}e^{-|y_2|}e^{(1/2z)(|y_1-y_2|+|y_1+y_2|)} \tag{22}$$

and

$$\epsilon=-(1-1/z+1/2z^2)\,. \tag{23}$$

The energy difference, usually called the correlation energy, is

$$\epsilon-\epsilon_{\rm H}=-1/6z^2\,. \tag{24}$$

From the virial theorem $\langle T\rangle=-\langle V\rangle/2$, and from $\epsilon=\langle T\rangle+\langle V\rangle$, there follows $\langle T\rangle=-\epsilon$, indicating that the kinetic energy for the exact bound state is higher than that for the Hartree solution by the factor $1/6z^2$. The attractive part of the potential energy is given by $\langle V\rangle_a^{\rm HF}=-2\int\delta(y)\phi^2(y)dy$ for the Hartree solution and by $\langle V\rangle_a=-2\int\psi^2(0,y)dy$ for the exact solution. A simple calculation, using Eqs. (20) and (22) shows them to be identical: $\langle V\rangle_a^{\rm HF}=\langle V\rangle_a=-2(1-1/z)$. Then, from $\epsilon=\langle T\rangle+\langle V\rangle$ and $\langle V\rangle=\langle V\rangle_a+\langle V\rangle_r$, the repulsive potential energies are found to be $\langle V\rangle_r^{\rm HF}=1/z-2/3z^2$, $\langle V\rangle_r=1/z-1/z^2$, so the repulsive interactions are smaller in the correlated exact state by the amount $1/3z^2$. Summarizing, the increase in kinetic energy required to produce the spatial correlations in the exact ψ, equal to $1/6z^2$, is just half of the decrease in potential energy produced by the correlations, $1/3z^2$, so that the net difference, $-1/6z^2$, is the correlation energy given in Eq. (24).

To compare wave functions, we plot contour maps of constant ψ and $\psi_{\rm H}$ choosing $z=2$. Figure 1 shows lines of constant $\psi_{\rm H}(y_1,y_2)/\psi_{\rm H}(0,0)$ for the Hartree solution given by Eq. (20), and Fig. 2 is a plot of the same ratio for the exact solution, Eq. (22). In the absence of repulsive interactions, $(1/z\to 0)$, $\psi=e^{-|y_1|}e^{-|y_2|}$, and the contour maps would have the general shape shown by the dashed squares in Fig. 1. For both Hartree and exact solutions, the contours of constant ψ are concave toward the origin, indicating a decreased probability whenever the two particles either coincide ($|y_1-y_2|=0$) or whenever their sum of distances from the origin is zero ($|y_1+y_2|=0$). For the Hartree solution, the concavity is due to the terms in the denominator of Eq. (20), and is made evident by introducing variables $s=2^{-1/2}(y_1+y_2)$, $u=2^{-1/2}(y_1-y_2)$, corresponding to a $\pi/4$ rotation in the y_1,y_2 plane. Then, for $z=2$ and for the $y_1>0$, $y_2>0$ quadrant, Eq. (20) becomes

$$\psi_{\rm H}=\frac{e^{-s/\sqrt{2}}}{3+e^{-\sqrt{2}s}/3-2e^{-s/\sqrt{2}}\cosh u/\sqrt{2}} \tag{25}$$

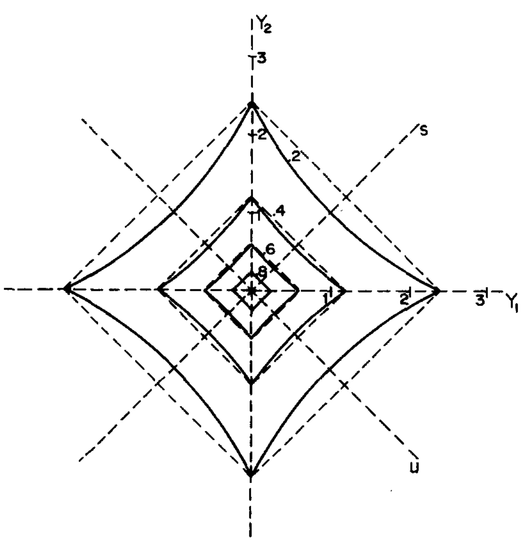

Fig. 1. Bound state wave function in the Hartree approximation. Solid curves are lines of constant $\psi_{\rm H}(y_1,y_2)/\psi_{\rm H}(0,0)=0.2$, 0.4, 0.6, and 0.8, obtained from Eq. (20) for the case $z=2$. The dashed squares indicate lines of constant ψ for the case of no interparticle interactions.

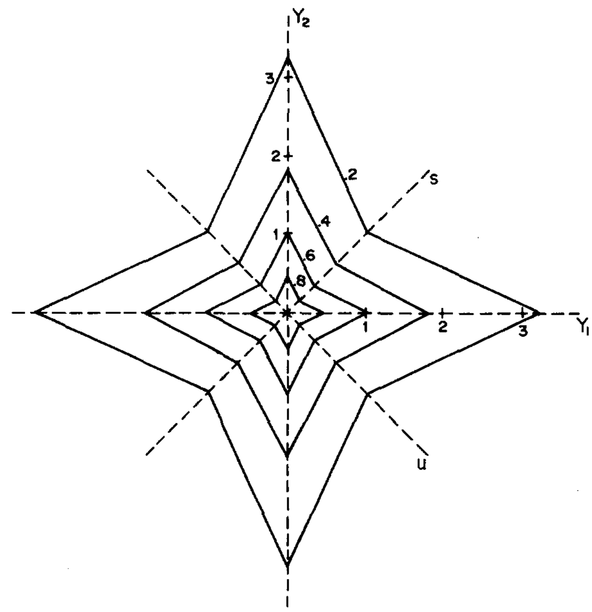

Fig. 2. Bound state wave function for the exact solution. Solid curves are lines of constant $\psi(y_1,y_2)/\psi(0,0)=0.2$, 0.4, 0.6, and 0.8, obtained from Eq. (22) for the case $z=2$.

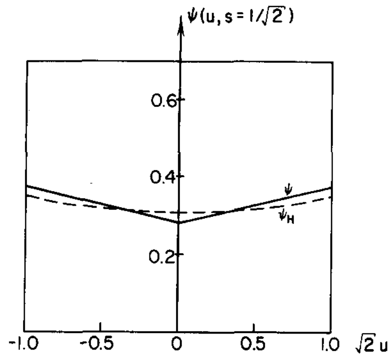

Fig. 3. Exact and Hartree bound states as given by Eqs. (16) and (25), plotted for fixed $s = 2^{-1/2}(y_1 + y_2) = 2^{-1/2}$ as a function of $u = 2^{-1/2}(y_1 - y_2)$.

indicating a minimum ψ_H for fixed s (along dashed line in Fig. 1) when $u = 0$ ($y_1 = y_2$). In terms of s, u variables the exact wave functions, for $z = 2$ and in the same y_1, y_2 quadrant is

$$\psi = (\sqrt{6}/4)e^{-3\sqrt{2}s/4}e^{\sqrt{2}u/4}, \tag{26}$$

again indicating a minimum ψ for fixed s when $u = 0$. Figure 3 shows the dependence of both wave functions on u for a fixed s, $s = 1/\sqrt{2}$. The minimum of the wave function at $u = 0$ being smaller for the exact ψ than for the Hartree ψ_H accounts for the smaller repulsive interaction in the exact state.

One may also compare the single particle density $P(y)$ for the two solutions. For the Hartree, $P(y) = \phi^2(y)$, while for the exact solution, $P(y) = \int \phi^2(y, y_1)\, dy_1$ which is easily calculated from Eq. (22). The result of such a calculation shows that the two densities are everywhere within a few percent of each other, for $z = 2$. As expected from Figs. 1 and 2, the exact probability density extends slightly further out than the Hartree approximation. As a result, the "atom" size $\langle Y \rangle \equiv \langle |y| \rangle / 2 = \int_0^\infty y P(y) dy$ is slightly larger for the exact solution: one finds $\langle Y \rangle = 0.417$ for the exact bound state and $\langle Y \rangle = 0.411$ for the Hartree approximation.

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Conversion of electromagnetic quantities from mksa to Gaussian units (and vice versa) using dimensional analysis

Bernard Leroy
Observatoire de Paris, Département d'astrophysique fondamentale, 92195 Meudon Principal Cedex, France

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It is shown how matrix methods in dimensional analysis can be used to convert the numerical values and units of electromagnetic quantities from the mksa system to the Gaussian system and vice versa. Examples are given which should convince the reader that the mentioned conversions can be performed with very little tedium.

I. INTRODUCTION

Recently, W. J. Remillard[1] showed how dimensional analysis in matrix form could be used to convert a given amount of a physical quantity from one system of units to another, and to change from one basic set of dimensions to another. Though this is nicely done, in our opinion too little emphasis has been put on the conversion between the mksa and Gaussian systems, a job of not so rare an occurrence for those who have the double practice of teaching and research. It is the purpose of the present paper to expand on Remillard's presentation, though without repeating it, and in particular to show how the matrix technique of dimensional analysis enables the "nasty" conversion factors which arise when comparing electromagnetic quantities in these two systems to be easily deduced.

II. THE mksa AND GAUSSIAN SYSTEMS OF UNITS

As is well known, in the mksa system any physical quantity is represented with the help of four independent dimensions: mass (M), length (L), time (T), and electric current (I). For example, a force has dimensions $[M\,L\,T^{-2}I^0]$, and a capacitance $[M^{-1}L^{-2}T^4I^2]$.

In the Gaussian system, only three independent dimensions are used to represent any physical quantity: mass (M), length (L), and time (T). For example, a force has dimensions $[M\,L\,T^{-2}]$ and a capacitance $[M^0L\,T^0]$. However, this is not only typical of the Gaussian system; it is also true of any CGS system. The peculiarity of the Gaussian system lies in that the permittivity ϵ_0, and permeability μ_0 of free space are taken as dimensionless and put equal to unity; whereas in the mksa system they are