One-dimensional hydrogen atom in an infinite square well

I. Richard Lapidus

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energy. For $r \rightarrow \infty$ the total energy of the atom is dominated by the energy of the muon with $Z_2 = 2$ and $E/E_0 \rightarrow 4r$. For $r = 1, Z_1 = Z_2 = 1.75$ and $E/E_0 = 6.125$.[4] For $r \rightarrow 0$ the energy is that of a single electron with $Z_1 = 2$ and $E/E_0 \rightarrow 4$.

In muonic helium $r = 206.8$ and the electron is almost completely shielded by the muon in contrast to the situation in normal helium where $r = 1$ and the effective shielding is $0.25e$.

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One-dimensional hydrogen atom in an infinite square well

I. Richard Lapidus
Department of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030

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A typical problem in quantum mechanics courses involves the solution of a differential equation with certain boundary conditions such as "the wave function vanishes at infinity" or "the wave function vanishes at $x = \pm L$." Often it is difficult for students in introductory courses to appreciate how such boundary conditions determine the energy levels of the system. The example discussed here may be helpful in clarifying this point.

In this note an extremely simple system is discussed. The "hydrogen atom" in one dimension consists of an electron moving in an attractive δ-function potential. This atomic system has one negative energy level.[1–3]

If the atom is symmetrically enclosed in a one-dimensional infinite square well of width $2L$, the energy of the system is a function of L because of the boundary condition that the wave function must vanish at the walls. Despite the infinite attractive potential, if the well is sufficiently narrow the energy of the system must be positive. If the atom is asymmetrically enclosed in the well, the energy is also a function of the distance d from the center of the well.

Consider an "electron" with mass m that moves in a one-dimensional potential well

$$V(x) = -e^2\delta(x), \tag{1}$$

where $\delta(x)$ is the Dirac delta function. The Schrödinger equation

$$-(\hbar^2/2m)\psi'' + V(x)\psi = E\psi \tag{2}$$

has one negative energy solution

$$\psi(x) = (1/a_0)^{1/2} \exp(-|x|/a_0), \tag{3}$$

with

$$E = -E_0 = -e^2/2a_0 = -me^4/2\hbar^2, \tag{4}$$

where $a_0 = \hbar^2/me^2$.

The solution (3) is obtained by using the boundary condition $\psi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. If the atom is enclosed in an infinite square well with walls at $x = \pm L$ the energy of the atom is significantly altered as $L \rightarrow a_0$.

The problem of the one-dimensional hydrogen atom in an infinite square well may be solved exactly. If $L > a_0$, the wave function is

$$\psi_\pm(x) = \mp \left(\frac{1 - k^2a_0^2}{L(k^2a_0^2 - 1) + a_0}\right)^{1/2} \sinh k(x \mp L), \tag{5}$$

where $\pm$ signifies $x \gtrless 0$ and $E = -\hbar^2k^2/2m$.

Making use of the condition

$$\psi'_+(0) - \psi'_-(0) = -(2/a_0)\psi(0) \tag{6}$$

obtained by integrating Eq. (2), and the boundary condition $\psi(\pm L) = 0$, one obtains an implicit equation for the energy

$$ka_0 = \tanh(kL). \tag{7}$$

For $L \rightarrow \infty$, $\tanh(kL) \rightarrow 1$; $k = 1/a_0$ and one again obtains Eq. (4). For finite values of $L > a_0$ the energy is greater than $-E_0$. When $L = a_0$, $E = 0$. For $L < a_0$ the negative energy level becomes positive and one must replace Eq. (7) by

$$ka_0 = \tan(kL). \tag{8}$$

The corresponding wave function in this case is

$$\psi_\pm(x) = \mp \left(\frac{k^2a_0 + 1}{L(k^2a_0^2 + 1) - a_0}\right)^{1/2} \sin k(x \mp L) \tag{9}$$

and $E = \hbar^2k^2/2m$.

A plot of the energy of the one-dimensional hydrogen atom in an infinite square well as a function of L is shown in Fig. 1. The energy has been shifted by $2E_0$ in order to present the result on a logarithmic scale. For large values of L,

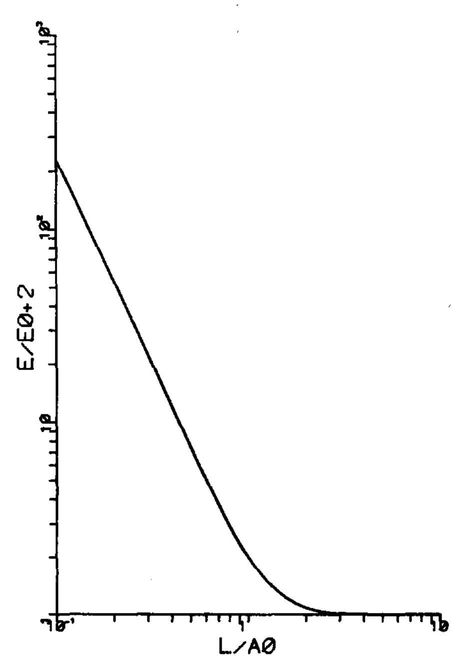

Fig. 1. Plot of the energy of the one-dimensional δ-function hydrogen atom as a function of the width of an infinite square well symmetrically enclosing the atom. For $E > 0$ only the first energy level is shown. The energy has been shifted by $2E_0$ in order to use a logarithmic scale.

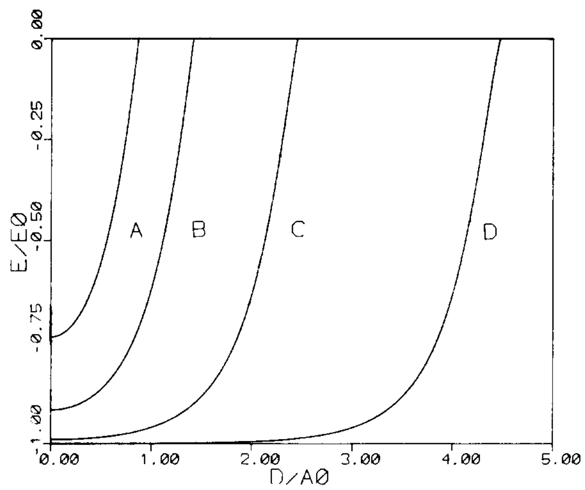

Fig. 2. Plots of the negative energy of the one-dimensional hydrogen atom as a function of the displacement d from the center of an infinite square well for several values of L. (a) $L/a_0 = 1.5$; (b) $L/a_0 = 2.0$; (c) $L/a_0 = 3.0$; (d) $L/a_0 = 5.0$.

E/E_0 approaches -1. When $L/a_0 = 1$, $E/E_0 = 0$. For $L/a_0 < 1$, $E/E_0 > 0$. As $L \to 0$ the energy approaches that of the lowest level of a square well independent of the δ-function potential. In addition, there are an infinite number of positive energy levels that are the eigenstates of the square well in the absence of the δ-function potential.

In the previous discussion the hydrogen atom was symmetrically placed in the center of the well. If the atom is displaced from this position the energy is also a function of the distance d from the center of the well. In this case the wave function has the form

$$\psi(x) = A \sinh k(L + x) \quad x \geqslant d ,$$
$$\psi(x) = B \sinh k(L - x) \quad x \leqslant d , \tag{10}$$

where $A \sinh k(L + d) = B \sinh k(L - d)$. The constants A and B in Eq. (10), which are determined by normalization of $\psi(x)$, are functions of the parameters k, L, and d.

For negative energies the energy of the system is determined from the implicit equation

$$ka_0 = \frac{\tanh(kL)[1 - \tanh^2(kd)/\tanh^2(kL)]}{1 - \tanh^2(kd)}, \tag{11}$$

where $E = -h^2k^2/2m$.

The energy of the one-dimensional hydrogen atom placed asymmetrically in an infinite square well of width $2L$ is plotted in Fig. 2 as a function of the displacement d of the atom from the center of the well for several values of L.

When $L \to \infty$, $E = -E_0$ independent of the value of d. As the well decreases in width the range of values of d for which a negative energy is possible also decreases. The value of d for which $E = 0$ is given by $a = L(1 - a_0/L)^{1/2}$. For $L \leqslant a_0$ there cannot be a negative energy state as found above. When $d \to 0$, Eq. (11) reduces to Eq. (7).

This example illustrates the effect on the wave function of boundary conditions at the walls in determining the energy levels of a quantum-mechanical system. If the width of the well is sufficiently small there cannot be negative energy states even for an attractive δ-function potential. This result is consistent with qualitative results obtained by appealing to the uncertainty principle. Because of the simple form of the δ-function potential it is possible to obtain exact expressions for the energy of this system. Thus it may provide a useful exercise for students in a quantum mechanics course.

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Formant filtering of the vocal source function in singing

Thomas D. Rossing
Department of Physics, Northern Illinois University, DeKalb, Illinois 60115

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I. INTRODUCTION

Although the singing voice is both the oldest and the most widely used musical instrument, its acoustical behavior is often slighted in musical acoustics courses as well as in the discussion of sound in general physics courses. It is the purpose of this brief paper to suggest a simple demonstration experiment illustrating the separate functions of the vocal cords and the vocal tract in a way that is appropriate for either type of course.

II. FORMANTS AND PITCH

In both speech and singing, there is a division of labor between the vocal cords and the vocal tract. The vocal cords determine the pitch of the sound, whereas the vocal tract determines the vowel sounds by creating formants or resonances, and also articulates the consonants.[1–3] The pitch and the formant frequencies in speech are essentially independent of each other, although trained singers (especially sopranos) sometimes tune their vowel formants to match one or more harmonics of the sung pitch.[4,5]

The vibrating vocal cords modify the flow of air through the larynx and act as the main source of sound in speech and singing. They can vibrate in several different ways, which singers sometimes refer to as voices or registers. Singers commonly employ two different registers that we will call chest (or normal) voice and head voice (or falsetto). Only the chest voice is normally used in speaking.

The waveform of the air flow (volume velocity) through the larynx in the chest voice is roughly described as consist-

