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$$T \underset{d\to\infty}{\sim} \exp[-2\pi d(V_0 - q - k)]. \quad (3.22)$$

For the special case $q = k, E = V_0/2$, it is easy to show that

$$T \underset{d\to\infty}{\sim} \exp\left\{\left[-2\pi m^2 \Big/ \left(\frac{dV}{dx}\right)_0\right]\frac{1}{1+\beta}\right\}, \quad (3.23)$$

and that the right-hand side of (3.23) is greater than the right-hand side of (3.22), thus (3.23) provides an upper bound on the asymptotic form of the transmission coefficient. It is worth noting that (3.23) is identical to the result found by Sauter[4] with the Dirac theory. Thus we see that in the weak field limit the transmission to the states with negative kinetic energy is very small and the reflection is almost total.

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Quark confining potential in relativistic equations

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A linear potential, when used in the Schrödinger equation, confines a quark. In this paper, we discuss what happens when this potential is used in the relativistic Klein–Gordon and Dirac equations.

INTRODUCTION

Quarks were introduced in particle physics in 1964 by Gell-Mann,[1] and independently by Zweig,[2] essentially as physical realizations of the *fundamental* three-dimensional representation of the group[3] SU(3). Since all other representations of the group, which are identified with the observed hadrons, can be obtained as direct products from the fundamental representation,[4] all hadrons could then be described most naturally as bound states of the constituent quarks.[5] However, free quarks have not been observed except for the recent claims by La Rue *et al.*[6] of having seen fractional charges in their superconducting levitation experiment with niobium pellets. Consequently, theorists have constructed models in which quarks are permanently confined.[7] In potential models, based on the Schrödinger equation, the confinement is provided by a potential that rises indefinitely as a function of the radial distance. A most commonly used potential is a *linearly* rising potential.[8]

The linearly rising potential when used in the Schrödinger equation gives real energy eigenvalues corresponding to pure bound states,[9] thus keeping the quark on which it acts permanently confined. In this paper we discuss what happens when this potential is used in a relativistic equation such as the Klein–Gordon (KG) or Dirac. The physics revealed is illuminating.

LINEAR POTENTIAL IN THE KLEIN–GORDON EQUATION

Since there is no unique prescription to construct a two-body relativistic equation, we consider here only a one-body problem. Let us first use the linear potential $V = \alpha r (\alpha > 0)$ as the *fourth component* of a Lorentz four-vector[10] in the Klein–Gordon equation,[11] in analogy with the case of a Coulomb potential usually given in textbooks[12] on quantum mechanics:

$$(-\nabla^2 + m^2)\psi(\mathbf{r}) = (E - \alpha r)^2\psi(\mathbf{r}). \quad (1)$$

Equation (1) can be easily rewritten in the Schrödinger form

$$[-(1/2m)\nabla^2 + V_{\text{eff}}(r)]\psi(\mathbf{r}) = \bar{E}\psi(\mathbf{r}), \quad (2)$$

where

$$\bar{E} = (E^2 - m^2)/2m \quad (3)$$

and

$$V_{\text{eff}}(r) = (1/2m)(2E\alpha r - \alpha^2 r^2). \quad (4)$$

A plot of $V_{\text{eff}}(r)$ vs r is given in Fig. 1. Note that $V_{\text{eff}}(r)$ does

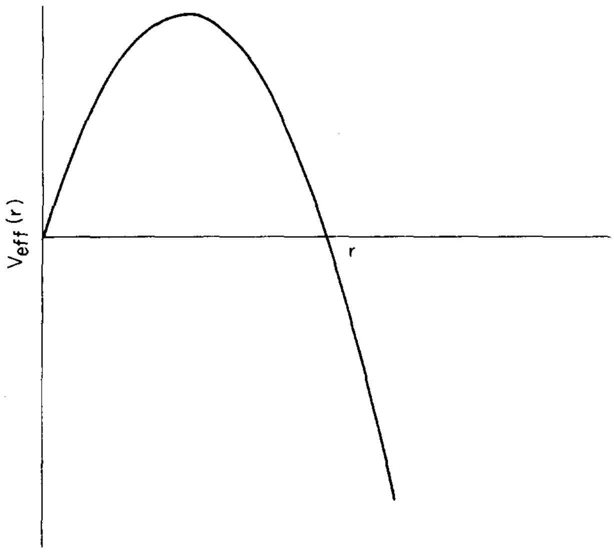

Fig. 1. Plot of $V_{eff}(r)$ in Eq. (4) versus r.

not keep on rising indefinitely; it turns over because of the second term ($-\alpha^2r^2$), which dominates at large r. Consequently, it does not confine a quark permanently; the quark leaks out. The effective potential $V_{eff}(r)$ is zero at $r=0$ and $r=2E/\alpha$, has a maximum value equal to $E^2/2m$ at $r=E/\alpha$, and goes to $-\infty$ as $r\to\infty$. The transmission coefficient T for the quark can be easily calculated using the WKB method.[13] For this purpose, we obtain from Eq. (2) the one-dimensional radial equation

$$-\frac{d^2U(r)}{dr^2}+2m\left(\frac{1}{2m}\frac{l(l+1)}{r^2}+V_{eff}(r)-\bar{E}\right)U(r)=0 \tag{5}$$

after setting

$$\psi(\mathbf{r})=R(r)Y_l^m(\theta,\phi)$$

and

$$U(r)=rR(r).$$

Then T is given by[13]

$$T=\frac{4}{(2\theta+1/2\theta)^2} \tag{6}$$

with

$$\theta=\exp\left(\int_{r_1}^{r_2}k(r)dr\right). \tag{7}$$

Here

$$k(r)=\left[2m\left(\frac{1}{2m}\frac{l(l+1)}{r^2}+V_{eff}(r)-\bar{E}\right)\right]^{1/2} \tag{8}$$

and r_1 and r_2 are the roots of the equation

$$k(r)=0.$$

The integral in Eq. (7) can be done analytically for $l=0$ with the result[14]

$$\int_{r_1}^{r_2}k(r)dr=\frac{m^2\pi}{2\alpha}. \tag{9}$$

From Eq. (9) it is clear that the transmission coefficient is independent of the total energy E; it depends only on the mass m of the quark and the strength parameter α of the linear potential.[15] *This is a very interesting result.* It says that the quark has the *same* probability of leaking through the barrier potential $V_{eff}(r)$ irrespective of whether it is very energetic or just crawling.[16] Thus a quark is no longer confined by a linearly rising potential when it is used as a vector in the KG equation. This is equivalent to saying that the energy eigenvalues are no longer real but are complex and so do not correspond to pure bound states.[17]

Confinement, however, is restored if the linear potential is used as a Lorentz *scalar*. A scalar potential transforms like the mass under Lorentz transformation. It is then added to the mass term in the KG equation:

$$[-\nabla^2+(m+\alpha r)^2]\psi(\mathbf{r})=E^2\psi(\mathbf{r}). \tag{10}$$

Equation (10) can be rewritten in the Schrödinger form

$$\left(-\frac{1}{2m}\nabla^2+\frac{1}{2m}(2m\alpha r+\alpha^2r^2)\right)\psi(\mathbf{r})=\bar{E}\psi(\mathbf{r}) \tag{11}$$

with $\bar{E}$ given by Eq. (3). Note that in Eq. (11) the effective potential

$$V_{eff}(r)=(1/2m)(2m\alpha r+\alpha^2r^2) \tag{12}$$

and that in it the term α^2r^2 appears with the plus sign. As a result $V_{eff}(r)$ in this case will keep on rising indefinitely, and keep a quark permanently confined. The energy eigenvalues will again be real. For $l=0$, these can be easily determined by noting that by a change of variable $\rho=m+\alpha r$, Eq. (11) can be rewritten as

$$-\frac{1}{2m}\frac{d^2U}{d\rho^2}+\tfrac{1}{2}K\rho^2U=E'U \tag{11'}$$

with

$$K=1/m\alpha^2$$

and

$$E'=(1/\alpha^2)(\bar{E}+m/2).$$

Equation (11') represents a one-dimensional harmonic oscillator and thus gives

$$E'=(n+\tfrac{1}{2})\surd(K/m),$$

or the energy eigenvalues

$$E_n=\surd[(2n+1)\alpha]. \tag{13}$$

Note that these are independent[16] of m.

LINEAR POTENTIAL IN THE DIRAC EQUATION

Quark is considered to be a spin-$\frac{1}{2}$ particle. Therefore the relevant equation to be used is that of Dirac, which we now discuss. The Dirac equation with both a vector potential V and a scalar potential S is given by

$$[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta(m+S)+V]\psi=E\psi. \tag{14}$$

With the usual method given in Schiff,[12] Eq. (14) can be reduced to two coupled equations in g and f, the large and small components of the Dirac wave function ψ:

$$(m+S-E+V)g-\frac{df}{dr}-\frac{1-\kappa}{r}f=0, \tag{15a}$$

$$(m+S+E-V)f-\frac{dg}{dr}-\frac{\kappa+1}{r}g=0. \tag{15b}$$

In Eqs. (15), $\kappa=-(l+1)$ when the total angular momentum $j=l+\frac{1}{2}$, and $\kappa=l$ when $j=l-\frac{1}{2}(l\neq0)$. It is straightforward though somewhat tedious to show that the transformation[18]

$$\psi_1=rg/\surd(m+S+E-V), \tag{16a}$$

$$\psi_2 = rf/\sqrt{(m+S-E+V)} \tag{16b}$$

transforms Eqs. (15) into the one-dimensional Schrödinger form

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2} + V^1_{\rm eff}\right)\psi_1 = \bar{E}\psi_1, \tag{17a}$$

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2} + V^2_{\rm eff}\right)\psi_2 = \bar{E}\psi_2, \tag{17b}$$

where $\bar{E}$ is given by Eq. (3),

$$V^1_{\rm eff} = \frac{1}{2m}\bigg(\frac{\kappa(\kappa+1)}{r^2} + 2EV - V^2 + 2mS + S^2 + \frac{1}{2}\frac{V'' - S'' + 2(S'-V')(\kappa/r)}{m+S+E-V} + \frac{3}{4}\frac{(S'-V')^2}{(m+S+E-V)^2}\bigg) \tag{18}$$

and

$$V^2_{\rm eff}(\kappa,E,V) = V^1_{\rm eff}(-\kappa,-E,-V). \tag{19}$$

In Eq. (18) primes mean differentiations with respect to r. Note that ψ_1 is related to the large component g of the Dirac wave function, and ψ_2 to the small component f.

Now if $S=0$ and $V=\alpha r$, from Eq. (18),

$$V^1_{\rm eff}(r) = \frac{1}{2m}\bigg(\frac{\kappa(\kappa+1)}{r^2} + 2E\alpha r - \alpha^2 r^2 - \frac{\alpha\kappa/r}{m+E-\alpha r} + \frac{3}{4}\frac{\alpha^2}{m+E-\alpha r^2}\bigg). \tag{20}$$

Thus for $\kappa = -1$, the plot of $V^1_{\rm eff}(r)$ vs r will be similar to that shown in Fig. 1. The quark will leak out and the energy eigenvalues will not be pure bound states. On the other hand, if $V=0$, and $S=\alpha r$, i.e., pure scalar potential, from Eq. (18),

$$V^1_{\rm eff}(r) = \frac{1}{2m}\bigg(\frac{\kappa(\kappa+1)}{r^2} + 2m\alpha r + \alpha^2 r^2 + \frac{\alpha\kappa/r}{m+E+\alpha r} + \frac{3}{4}\frac{\alpha^2}{(m+E+\alpha r)^2}\bigg), \tag{21}$$

which, for $\kappa = -1$, is similar to Eq. (12), and thus leads to permanent confinement of the quark with real energy eigenvalues. In short, a vector potential leads to leakage of the quark in both the Klein–Gordon and the Dirac equations whereas a scalar potential permanently confines it.

CLOSING REMARKS

The entire discussion above could have been made in terms of a general confining potential αr^β ($\alpha > 0, \beta > 0$), but then the physics would not have been as transparent as it is with the linear potential. The linear potential when used in the Schrödinger equation describes pure bound systems and thus has been used in the study of the spectroscopy of the charmonium, with the belief that quarks are confined. The linear potential when used in a relativistic equation, on the other hand, is capable of describing physically both pure bound states as well as quasistationary states (resonances); the former when it is used as a Lorentz scalar, and the latter when it is used as a vector. With the philosophy of quark confinement, the lighter systems such as the π meson, ρ meson, etc., are therefore to be described relativistically with pure scalar potentials. If quarks do exist[6] as physical particles, however, then one would have to abandon the philosophy of complete quark confinement and allow a pure vector or a mixture of vector and scalar potentials in relativistic models.

It is clear that the relativistic behavior of a nonrelativistic confining potential depends very much on its Lorentz character. This is not shown in existing textbooks on quantum mechanics. The above material, therefore, should play its supplementary role for students as well as instructors.

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[9]For $l=0$, the solution is the well-known Airy function that is usually written in terms of Bessel function of order 1/3; see, for example, *Handbook of Mathematical Functions*, edited by M. Abranowitz and I. A. Stegun (Dover, New York, 1965).

[10]From now on we shall call V a vector potential to be understood as $V = (0, \alpha r)$.

[11]We use natural units in which $c = \hbar = 1$.

[12]See, for example, L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968), Chap. 13.

[13]See, for example, E. Merzbacher, *Quantum Mechanics*, 2nd ed. (Wiley, New York, 1970), Chap. 7.

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[15]To give some feel for the numbers, $T = 3\times10^{-55}$ for $m = 2.0$ GeV and $\alpha = 0.1$ GeV^2; whereas for $m = 0.1$ GeV = 100 MeV and $\alpha = 0.1$ GeV^2, the transmission coefficient = 0.52!

[16]The reader may compare this result with the discussion [I. Bloch and H. Crater, Am. J. Phys. **49**, 67 (1981)] concerning criteria for using nonrelativistic quantum mechanics for Lorentz-invariant potentials, which has recently appeared in the pages of the Journal.

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