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Thus for an exoergic reaction, the desired triangle *mnq* is drawn, and then circles *a*, *b*, and *d* are drawn as before with centers at *p*, but with radius $k > 1$ times as large as for an elastic collision. Circle *e* is not so affected.

Figure 3 illustrates an exoergic 90°C case with $k = 1.1$. Note that angle v_{1i}, v_{2i} decreases as k increases, other things being unchanged.

For endoergic reactions, $k < 1$, the circles *a*, *b*, and *d* have smaller radii, than for elastic, by the factor k. Evidently, for $k \ll m_1/m_2 < 1$ (particles nearly sticking together) the circle *a*, outer bound, can be well inside circle *e*, everywhere. Then for a 90°C v_{1i} is nearly antiparallel to v_{2i}.

In the last step of going from Fig. 3 for an inelastic collision to the construction analogous to Fig. 2, as before triangle *mnr* must be rotated through $(180° - \phi)$, in the plane of the figure, to align $\mathbf{V}_{ci}$ and $\mathbf{V}_{cf}$ as parallel. Then triangle *mnq* must be redrawn k times as large in order that all parts of the final drawing have the same scale. These two operations give $\mathbf{V}_{ci} = \mathbf{V}_{cf}$ as required by conservation of momentum. Triangle *mnq* may then be rotated about $\mathbf{V}_c$ to any orientation.

This analysis has only used conservation of linear momentum and conservation of energy. If the two particles change their internal energy by spinning, or through other degrees of freedom, then the collision is inelastic (usually), and the construction still applies. Nothing is assumed about whether the force of interaction is central since conservations of momentum and energy are always required and they are the basis of the construction. However, the probability of reaching various possible final configurations will depend on the details of the force of interaction.

In summary, the wide variety of cases of 90°C (orthogonal final velocities) for any mass ratio, any degree of elasticity (except $k = 0$ where 90°C is meaningless) and the wide variety of ratios of initial speeds may be obtained from the simple construction shown here.

An exactly soluble quantum model of an atom in an arbitrarily strong uniform magnetic field

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The isotropic oscillator in a uniform magnetic field is an exactly soluble quantum problem which can serve as a useful model of an atom in a magnetic field of arbitrary strength. The exact eigenvalue spectrum is derived, compared with perturbation theory for low field strengths, and shown to yield a quasi-Landau spectrum for high field strengths. The frequency, polarization, and relative intensity of the spectral lines of the Zeeman effect are determined as a function of the field strength. The effect of the field on atomic size and on the atomic wave functions is discussed; the three-dimensional system is shown to collapse into a needlelike one-dimensional system for a sufficiently intense field.

I. INTRODUCTION

Magnetic fields have long been used in the study of atomic and molecular structure. Although a static uniform magnetic field may produce a dramatic modification in the number and polarization of atomic spectral lines through removal of level degeneracy, the energy changes involved are generally small for the range of field strengths ordinarily encountered in the laboratory. Indeed, it is not difficult to show that the frequencies $\omega_0, \omega_1, \omega_2$ associated, respectively, with the zero-field, paramagnetic, and diamagnetic terms of an atomic Hamiltonian satisfy $\omega_1/\omega_0 \sim \omega_2/\omega_1 \ll 1$ even for fields as great as 10^5–10^6 G. Thus perturbation theory has customarily provided a suitable analytic method for treating the effects of magnetic fields on atoms.

The investigation of atoms and molecules in intense magnetic fields, however, is currently an active area of research. Atoms in the vicinity of neutron stars, for example, may be subjected to magnetic fields of 10^{12} G or more.[1] Under these circumstances the magnetic terms of the Hamiltonian will dominate. The same effects can be simulated terrestrially by subjecting highly excited atoms or excitons in semiconductors to large, but accessible, laboratory fields. Fields up to about 4×10^4 G have permitted atomic physicists to examine the transition of the spectrum characteristic of the Coulomb potential to a quasi-Landau spectrum lying above the field-free ionization limit.[2] The theoretical treatment of these and similar systems can not be accomplished through perturbation theory.

Of all atomic systems the hydrogen atom for obvious reasons has always been a system of particular importance in the investigation of the interactions of bound electrons with external fields. A partial understanding of the behavior of a hydrogen atom in a strong magnetic field may be obtained by an extension of the Bohr model to include the Lorentz force. Such a semiclassical model is exactly soluble and leads to useful expressions for the quantized electron energies and orbital radii.[3,4] It cannot replace a purely quantum description, however. Unfortunately, despite the seeming simplicity of the one-electron atom in a magnetic field and despite the many theoretical studies which have been devoted to it, there is no comprehensive quantum

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theory of this system valid for arbitrary field strengths.[5,6]

The interaction of a one-electron atom with an arbitrarily strong magnetic field is an interesting problem not only for the researcher, but for the teacher as well. It would be highly instructive to supplement the standard perturbation theoretic treatment of the hydrogen atom Zeeman effect, paramagnetism, and diamagnetism with a discussion of an alternative three-dimensional spherically symmetric one-electron quantum system for which an exact analytic solution incorporating the magnetic interaction is available. For small field strengths one can then expand the system eigenvalues in terms of a suitable field-dependent parameter and compare with the results of perturbation theory. With the increase in field strength the transition of the energy spectrum from that characteristic of the initial spherical symmetry to one characteristic of the lower cylindrical symmetry can be seen. The frequency, polarization, and relative intensity of spectral lines in the Zeeman effect can be derived exactly for intense magnetic fields. Moreover, having the exact wave functions one can demonstrate the extreme contraction of the system in the plane perpendicular to the field as the field strength is greatly increased. This alternative quantum system would thereby display a number of the features that make the theoretical and experimental research on "real" atoms in intense magnetic fields so interesting.

The charged isotropic harmonic oscillator is an example of such a quantum system, i.e., an electron subjected simultaneously to a potential $(1/2)\,m\omega^2 r^2$ and to a uniform magnetic field can be treated exactly without recourse to either perturbation theory or semiclassical approximation.[7] The solubility of the Schrödinger equation in this case stems from the similarity of form of the two-dimensional oscillator potential and the diamagnetic interaction term.

There are, of course, differences between a harmonic oscillator and an atom. An electron in an oscillator potential does not have an ionization threshold as does an electron in a Coulomb potential. Moreover, as pointed out by Katriel and Adam the oscillator in a magnetic field is a system with infinitely degenerate bound states.[8] This infinite degeneracy can be accounted for in terms of a noncompact invariance group.[9] The invariance group of the nonrelativistic hydrogen atom is O(4), a compact group.[10,11] SO(4,2) is a noncompact dynamical group of the hydrogen atom, but it is also a *non*invariance group.[12] Nevertheless, there are strong enough correspondences between a one-electron atom and the isotropic oscillator for the latter to provide a simple and useful atomic model—a model that has been frequently and profitably used since the days of Lorentz's classical electron theory.

In the remainder of this article I present the details of the solution and explore the suggestions made above.

II. SOLUTION

The Hamiltonian of a charged particle oscillating isotropically at frequency ω_0 and subject to a magnetic field $\mathbf{B}$ characterized by a vector potential $\mathbf{A}$ is

$$H = \mathbf{p}^2/2m + m\omega_0^2\mathbf{r}^2/2 - e\mathbf{A}\cdot\mathbf{p}/mc + e^2\mathbf{A}\cdot\mathbf{A}/2mc^2. \tag{1}$$

This is derived by the usual minimal substitution $\mathbf{p} \to \mathbf{p} - e\mathbf{A}/c$ in the field-free Hamiltonian followed by algebraic expansion with imposition of the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ or, equivalently $[\mathbf{p},\mathbf{A}] = 0$. A suitable vector potential for a uniform magnetic field is $\mathbf{A} = -(1/2)\mathbf{r}\times\mathbf{B}$; this satisfies the choice of gauge. The Hamiltonian can then be expressed in the form

$$H = \mathbf{p}^2/2m + m\omega_0^2\mathbf{r}^2/2 - e\mathbf{B}\cdot\mathbf{L}/2mc + e^2B^2r_\perp^2/8mc^2, \tag{2}$$

where $\mathbf{L}$ is the orbital angular momentum operator and $r_\perp$ is the component of the coordinate operator $\mathbf{r}$ perpendicular to $\mathbf{B}$. Without loss of generality one can choose a coordinate system such that $\mathbf{B} = B\hat{z}$. The Hamiltonian can then be written as the sum of two commuting terms:

$$H = H_\parallel + H_\perp, \tag{3a}$$

where

$$H_\parallel = p_z^2/2m + m\omega_0^2 z^2/2, \tag{3b}$$

$$H_\perp = (p_x^2 + p_y^2)/2m + m\omega^2(x^2+y^2)/2 + \omega_L L_z. \tag{3c}$$

Here

$$\omega_L = -eB/2mc \tag{3d}$$

is the Larmor frequency (positive for the electron since $e = -|e|$), and

$$\omega = (\omega_0^2 + |\omega_L|^2)^{1/2}. \tag{3e}$$

Since $[H_\parallel, H_\perp] = 0$, the solution to the Schrödinger equation

$$H|\Phi\rangle = E|\Phi\rangle \tag{4a}$$

is the tensor product of the solutions of $H_\parallel$ and $H_\perp$ individually, i.e.,

$$|\Phi\rangle = |\varphi_\parallel\rangle \otimes |\varphi_\perp\rangle; \quad E = E_\parallel + E_\perp, \tag{4b}$$

where

$$H_\parallel|\varphi_\parallel\rangle = E_\parallel|\varphi_\parallel\rangle \tag{4c}$$

and

$$H_\perp|\varphi_\perp\rangle = E_\perp|\varphi_\perp\rangle. \tag{4d}$$

Moreover, since $[H_\perp, L_z] = 0$, the state vector $|\varphi_\perp\rangle$ can be constructed to be an eigenvector of L_z,

$$L_z|\varphi_\perp\rangle = m\hbar|\varphi_\perp\rangle. \tag{4e}$$

Thus the initial three-dimensional problem has been reduced to the problems of (a) a one-dimensional oscillator along the field axis and (b) a two-dimensional oscillator in the plane perpendicular to the field with a component of angular momentum on the field axis.

The solution to the one-dimensional oscillator is well known; I merely record it below in notation suitable for future use. Defining the customary annihilation and creation operators

$$a_z = (\beta_0 z + ip_z/\beta_0)/\sqrt{2}, \tag{5a}$$

$$a_z^\dagger = (\beta_0 z - ip_z/\beta_0)/\sqrt{2}, \tag{5b}$$

with

$$\beta_0 = (m\omega_0/\hbar)^{1/2} \tag{5c}$$

and

$$[a_z, a_z^\dagger] = 1. \tag{5d}$$

One can express the Hamiltonian in terms of the diagonal number operator

$$N_z = a_z^\dagger a_z \tag{5e}$$

as follows:

$$H_\parallel = \hbar\omega_0(N_z + 1/2), \tag{5f}$$

where

$$N_z|\varphi_\parallel\rangle = n_0|\varphi_\parallel\rangle. \tag{5g}$$

The eigenvalues and eigenvectors are then

$$E_\parallel(n_0) = \hbar\omega_0(n_0 + \tfrac{1}{2}), \tag{5h}$$

$$|\varphi_\parallel{}^{(n_0)}\rangle = (a_z^\dagger)^{n_0}|0\rangle/\sqrt{n_0!}. \tag{5i}$$

Recall that

$$a_z^\dagger|n_0\rangle = (n_0+1)^{1/2}|n_0+1\rangle; \quad a_z|n_0\rangle = n_0^{1/2}|n_0-1\rangle. \tag{5j}$$

The two-dimensional isotropic oscillator with angular momentum can be solved in an analogous way. From Eqs. (5a) and (5b) one constructs *mutatis mutandis* the operators $a_x, a_y, a_x^\dagger, a_y^\dagger$ for excitations in the x–y plane; β_0 is replaced, however, by

$$\beta = (m\omega/\hbar)^{1/2}. \tag{6}$$

The tensor product $|\varphi_\perp{}^{(n_x)}\rangle \otimes |\varphi_\perp{}^{(n_y)}\rangle$, where each factor is constructed as in Eq. (5i), is not in general an eigenvector of L_z. To construct such eigenvectors one takes advantage of the cylindrical symmetry about **B** to introduce the commuting operators for right and left circularly polarized quanta

$$a_r = (a_x - ia_y)/\sqrt{2}, \tag{7a}$$

$$a_l = (a_x + ia_y)/\sqrt{2}. \tag{7b}$$

The nonvanishing commutation relations are similar to Eq. (5d),

$$[a_r, a_r^\dagger] = [a_l, a^\dagger] = 1. \tag{7c}$$

The number operators analogous to Eq. (5e) are

$$N_r = a_r^\dagger a_r, \tag{7d}$$

where

$$N_r|\varphi_\perp{}^{(n_r n_l)}\rangle = n_r|\varphi_\perp{}^{(n_r n_l)}\rangle \tag{7e}$$

and

$$N_l = a^\dagger a_l, \tag{7f}$$

where

$$N_l|\varphi_\perp{}^{(n_r n_l)}\rangle = n_l|\varphi_\perp{}^{(n_r n_l)}\rangle. \tag{7g}$$

The eigenvalues n_r and n_l give the number of quanta of angular momentum $\hbar$ parallel or antiparallel to **B**, respectively.

The Hamiltonian $H_\perp$ is now expressible in the diagonal representation

$$H_\perp = \hbar\omega(N_r + N_l + 1) + \hbar\omega_L(N_r - N_l), \tag{8a}$$

from which follows readily the eigenvalues and eigenvectors

$$E_\perp(n_r n_l) = \hbar\omega(n_r + n_l + 1) + \hbar\omega_L(n_r - n_l), \tag{8b}$$

$$|\varphi_\perp{}^{(n_r n_l)}\rangle = \frac{(a_r^\dagger)^{n_r}(a^\dagger)^{n_l}}{\sqrt{n_r!n_l!}}|00\rangle. \tag{8c}$$

Thus the entire solution to Eq. (4a) is given by

$$E(n_r n_l n_0) = (n_r + n_l + 1)\hbar\omega + (n_0 + \tfrac{1}{2})\hbar\omega_0 + (n_r - n_l)\hbar\omega_L, \tag{9a}$$

$$|n_r n_l n_0\rangle = \frac{(a_r^\dagger)^{n_r}(a^\dagger)^{n_l}(a_z^\dagger)^{n_0}}{\sqrt{n_r!n_l!n_0!}}|000\rangle. \tag{9b}$$

The wave functions are obtained by projection of Eq. (9b) onto a coordinate basic bra,

$$\Phi^{(n_r n_l n_0)}(\mathbf{r}) = \langle \mathbf{r}|n_r n_l n_0\rangle. \tag{9c}$$

The ground-state wave function is determined by solution of the differential equations

$$\langle \mathbf{r}|a_i|000\rangle = a_i(\mathbf{r}, -ih\nabla)\Phi^{(000)}(\mathbf{r}) = 0 \quad (i = r,l,z). \tag{9d}$$

The excited-state wave functions can then be generated recursively through sequential application of the creation operators.

III. DISCUSSION

Let us consider first the energy spectrum for several interesting cases.

In the absence of a magnetic field $\omega_L = 0$ and $\omega = \omega_0$. The energy eigenvalues, Eq. (9a), reduce to

$$E(n_r n_l n_0) = (n_r + n_l + n_0 + \tfrac{3}{2})\hbar\omega_0 = (n + \tfrac{3}{2})\hbar\omega_0, \tag{10a}$$

the spectrum of the isotropic oscillator. The energy depends only on $n = n_r + n_l + n_0$ and each level of given n has a degeneracy $g_n = (n+1)(n+2)/2$. Levels of even (odd) n comprise states of even (odd) l with $0 \le l \le n$.

In the opposite limiting case of an unbound particle, one has $\omega_0 = 0$ and $\omega = |\omega_L|$. The energy of motion perpendicular to the field becomes

$$E_\perp(n_r n_l) = |\omega_L|[(n_r + n_l + 1) + (\mathrm{sgn}\omega_L)(n_r - n_l)] \tag{10b}$$

$$= \begin{cases} 2\omega_L(n_r + \frac{1}{2}) & \text{electron} \\ 2|\omega_L|(n_l + \frac{1}{2}) & \text{positron} \end{cases}$$

and is therefore composed of right or left circularly polarized quanta depending on whether the particle is charged negatively or positively, respectively. From Eq. (3b) we see, however, that the energy of motion parallel to the field becomes

$$E_\parallel = (\tfrac{1}{2})mv_z^2, \tag{10c}$$

where the velocity along the field v_z can be any real number (for which $|v_z|/c \ll 1$ so that the Schrödinger equation is valid). The total energy of the so-called Landau levels is therefore (for an electron)

$$E(n_r n_l; v_z) = (\tfrac{1}{2})mv_z^2 + 2\omega_L(n_r + \tfrac{1}{2}). \tag{10d}$$

Since $E(n_r n_l; v_z)$ is independent of n_l, the Landau levels are infinitely degenerate and span a continuum of energies. (It should be noted that the canonical momentum **p** is not generally proportional to the velocity **v** of a particle interacting with an electromagnetic field; nor is $\mathbf{p}^2/2m$ to be interpreted as the particles's kinetic energy. From the correct relation $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$ it follows that the old associ-

ations are retained for only those components p_i for which $A_i = 0$.)

In the general case where neither ω_0 nor ω_L is exactly zero, the eigenvalues, expressed in terms of the dimensionless parameters

$$\epsilon(n_r n_l n_0) \equiv E(n_r n_l n_0)/\hbar\omega_0, \tag{11a}$$

$$\alpha \equiv \omega_L/\omega_0 \tag{11b}$$

are

$$\epsilon(n_r n_l n_0) = (n_r + n_l + 1)(1 + \alpha^2)^{1/2} + (n_0 + \tfrac{1}{2}) + (n_r - n_l)\alpha. \tag{11c}$$

For a low magnetic field strength, $\alpha \ll 1$, the Taylor expansion of $(1 + \alpha^2)^{1/2}$ to terms in α^2 leads to

$$\epsilon'(n_r n_l n_0) = (n_r + n_l + n_0 + \tfrac{3}{2}) + (n_r - n_l)\alpha + (\tfrac{1}{2})(n_r + n_l + 1)\alpha^2. \tag{12a}$$

The three terms may be identified, respectively, with the expectation values of (i) the isotropic harmonic oscillator Hamiltonian

$$H_0/\hbar\omega_0 = (\mathbf{p}^2/2m + m\omega^2\mathbf{r}^2/2)/\hbar\omega_0 = N_r + N_l + N_z + \tfrac{3}{2}; \tag{12b}$$

(ii) the paramagnetic interaction

$$V_1/\hbar\omega_0 = \omega_L L_z/\omega_0 = \alpha(N_r - N_l); \tag{12c}$$

and (iii) the diamagnetic interaction

$$V_2/\hbar\omega_0 = (\tfrac{1}{2})m\omega_L^2(x^2 + y^2)/\hbar\omega_0 = (\tfrac{1}{2})\alpha^2[(N_r + N_l + 1) + (a_r^\dagger a_l^\dagger + a_r a_l)]. \tag{12d}$$

Note that V_1 does not contribute in second order to the energy. Hence Eq. (12a), accurate to α^2, is equivalent to a first-order perturbation calculation in each interaction Hamiltonian.

Over what range of field strengths is perturbation theory reasonably applicable? Equation (12a) is valid so long as $(1 + \alpha^2)^{1/2} \sim 1 + (\tfrac{1}{2})\alpha^2$. This relation is satisfied to within an error <0.1 for α up to$\sim$1, i.e., for $\omega_L \sim \omega_0$. Let us assume that the oscillator restoring force acts over a distance of roughly one Bohr radius, i.e., $F_{\text{osc}} = ka_0$, and is equal to the electrostatic attractive force within a Bohr atom at that distance, i.e., $F_{\text{el}} = e^2/a_0^2$. It then follows that the characteristic oscillator frequency is

$$\omega_0 = \sqrt{k/m} = (e^2/ma_0^3)^{1/2} \sim 4 \times 10^{16}\ \text{sec}^{-1}$$

and therefore an upper limit to B will be

$$B_{\max} \sim 2mc\omega_0/|e| \sim 5 \times 10^9\ \text{G}.$$

For large magnetic field strengths, $\alpha \gg 1$, Eq. (12b) reduces to (for electrons)

$$\epsilon''(n_r n_l n_0) = 2(n_r + \tfrac{1}{2})\alpha + (n_0 + \tfrac{1}{2}). \tag{13}$$

In Fig. 1 is shown the variation in energy with B for all oscillator states in the levels $n = 0,1,2$, $(n = n_r + n_l + n_0)$.

Having obtained the energy eigenvalues and state vectors, one can determine the essential characteristics of the Zeeman effect for this system, in particular the frequency, polarization, and relative intensity of the spectral lines observed in directions perpendicular and parallel to the magnetic field. We consider within the electric dipole approximation the radiation produced by transitions from an excited state $|n_r n_l n_0\rangle$ to the ground state $|000\rangle$.

Suppose at time $t = 0$ the system is characterized by the state vector

$$|\psi_{n_r n_l n_0}(0)\rangle = \cos\gamma|000\rangle + \sin\gamma|n_r n_l n_0\rangle, \tag{14a}$$

where γ is an arbitrary real angle. For an ensemble of atoms γ can assume all values over the range 0 to 2π; final expressions for observable quantities (e.g., spectral line intensities) should therefore be averaged over γ. At time t later the state vector has evolved under the time translation operator $e^{-iHt/\hbar}$ and can be expressed as

$$|\psi_{n_r n_l n_0}(t)\rangle = e^{-i\omega_0\epsilon(000)t}[\cos\gamma|000\rangle + e^{-i\Omega^{(n_r n_l n_0)}t}\sin\gamma|n_r n_l n_0\rangle], \tag{14b}$$

where

$$\Omega^{(n_r n_l n_0)} \equiv [\epsilon(n_r n_l n_0) - \epsilon(000)] = \omega_0[(n_r + n_l)(1 + \alpha^2)^{1/2} + (n_r - n_l)\alpha + n_0]. \tag{14c}$$

The expectation value of the electric dipole operator $\mathbf{D} = e\mathbf{x}$ in the state (14b) is given by

$$\langle\mathbf{D}\rangle(t) = \langle\psi_{n_r n_l n_0}|\mathbf{D}|\psi_{n_r n_l n_0}\rangle. \tag{15a}$$

Expressing the components of $\mathbf{x}$ in terms of the creation and annihilation operators allows one to evaluate facilely the matrix elements appearing in (15a); $\langle\mathbf{D}\rangle(t)$ then reduces to

$$\langle\mathbf{D}\rangle(t) = \frac{e}{2\beta_0}(1 + \alpha^2)^{-1/4}\sin 2\gamma[(\hat{x}\cos\Omega^{(100)}t + \hat{y}\sin\Omega^{(100)}t)\delta_{n_r 1}\delta_{n_l 0}\delta_{n_0 0} + (\hat{x}\cos\Omega^{(010)}t - y\sin\Omega^{(010)}t)\delta_{n_r 0}\delta_{n_l 1}\delta_{n_0 0}]$$

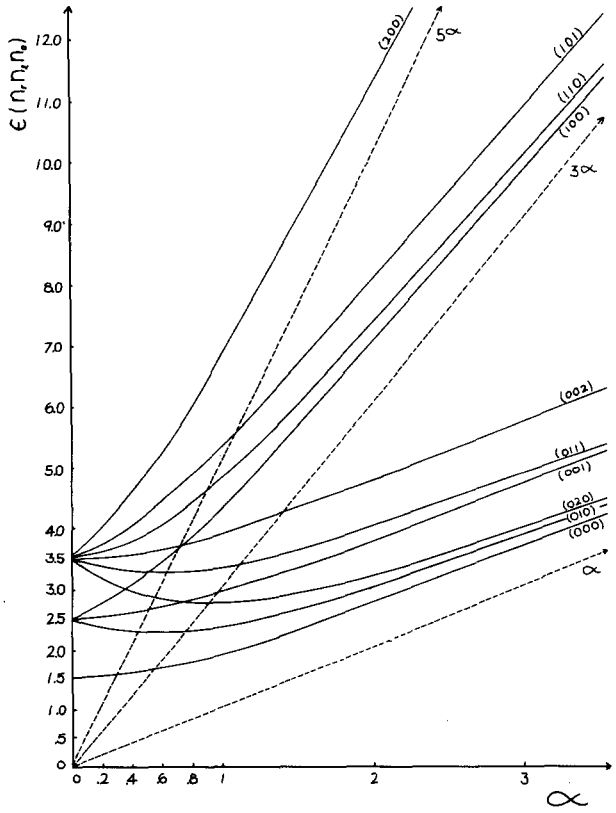

Fig. 1. Variation of energy eigenvalues with magnetic field strength for all oscillator states in the levels $n = 0, 1, 2$, where $n = n_r + n_l + n_0$.

$$+\frac{e}{\sqrt{2}\beta_0}\sin 2\gamma[\hat{z}\cos\Omega^{(001)}t]\delta_{n_r0}\delta_{n_l0}\delta_{n_01}, \quad (15b)$$

where

$$\beta_0 = \sqrt{m\omega_0/\hbar}.$$

One sees that only those states for which $n = n_r + n_l + n_0 = 1$ can contribute to the spectrum. Moreover, the expectation value of the dipole moment either rotates (counterclockwise or clockwise, the observer facing the emitter) about **B** or oscillates along **B** depending on whether the excitation is r, l, or z, respectively.

In the semiclassical treatment of spontaneous emission the radiation frequency corresponds to the frequency of oscillation or rotation of the electric dipole. Thus the spectrum can consist of at most the frequencies

$$\Omega^{(100)} = \omega_0[(1+\alpha^2)^{1/2} + \alpha],$$
$$\Omega^{(010)} = \omega_0[(1+\alpha^2)^{1/2} - \alpha], \quad (15c)$$
$$\Omega^{(001)} = \omega_0.$$

For observations transverse to the magnetic field, e.g., along the x axis, only the dipole components $\langle D_y\rangle(t)$ and $\langle D_z\rangle(t)$ contribute. A dipole oscillating along the y axis radiates light polarized perpendicular to **B** (σ polarization) at frequencies $\Omega^{(100)}$ and $\Omega^{(010)}$; a dipole oscillating along **B** radiates light parallel to **B** (π polarization) at the unperturbed frequency ω_0. For low field strengths the frequencies of the σ-polarized light reduce to $\omega_0 \pm \omega_L$. The intensity of each spectral line is proportional to the square of the time-independent coefficient of the dipole component which produces it and to the fourth power of the radiation frequency. At low field strengths the σ to π intensity ratio is easily seen from Eq. (15b) to be $\frac{1}{2}$:1. Thus the transverse spectrum is the normal Zeeman triplet: two σ-polarized lines shifted $\pm\omega_L$ from the π-polarized central frequency ω_0 and with relative intensities

$$I(\omega_0+\omega_L):I(\omega_0):I(\omega_0-\omega_L) = \tfrac{1}{2}:1:\tfrac{1}{2}.$$

The high-field spectrum, however, is quite different. For $\alpha \to \infty$ the frequency $\Omega^{(100)} \to 2\omega_L \equiv \omega_c$ (the cyclotron frequency); the intensity of this line increases relative to that of the central frequency as $8\alpha^3$. The frequency $\Omega^{(010)} \to \omega_0/\alpha$ and its intensity relative to that of the central component falls to zero as $\alpha^{-5}/32$. The symmetric triplet therefore collapses into an asymmetric doublet as B increases greatly.

An analysis similar to the above can be performed for observations along the magnetic field, i.e., along the z axis. From Eq. (15b) it is clear that the observer receives radiation only from the two components of $\langle\mathbf{D}\rangle(t)$ rotating counterclockwise and clockwise in the x–y plane. These two spectral lines at the weak-field frequencies $\omega_0 \pm \omega_L$ are said to be polarized σ^+ and σ^-, respectively. The weak-field relative intensity ratio is $I(\omega_0+\omega_L):I(\omega_0-\omega_L) = 1{:}1$. This again corresponds to the normal Zeeman effect. For large fields, however, the frequency of the σ^- component falls to zero as $(2\alpha)^{-1}$ and its intensity relative to the σ^+ component diminishes as $(2\alpha)^{-8}$. Thus the Zeeman doublet collapses into a single σ^+ line at the cyclotron frequency ω_c. A summary of the Zeeman effect is given in Table I.

We consider now the effect of the magnetic field on the size of the oscillator atom. One possible measure of this attribute is the expectation value of the squared components of the coordinate operator **r**. In view of the cylindrical symmetry about **B**, we will evaluate z^2 and $\rho^2 = x^2 + y^2$. Expressing these operators in terms of the appropriate annihilation and creation operators and using Eq. (5j) (and analogous relations for the r and l quanta) one readily obtains

$$\langle n_r n_l n_0|z^2|n_r n_l n_0\rangle = (n_0+\tfrac{1}{2})\hbar/m\omega_0, \quad (16a)$$
$$\langle n_r n_l n_0|\rho^2|n_r n_l n_0\rangle = (n_r+n_l+1)\hbar/m\omega. \quad (16b)$$

Thus

$$\frac{\langle\rho^2\rangle^{1/2}_{n_r n_l n_0}}{\langle z^2\rangle^{1/2}_{n_r n_l n_0}} = (1+\alpha^2)^{-1/4}\left(\frac{n_r+n_l+1}{n_0+\tfrac{1}{2}}\right). \quad (16c)$$

The ground state of the isotropic oscillator is spherically symmetric. Placed in a magnetic field, the electron distribution of the atom is distorted so that

$$\xi \equiv \frac{\langle\rho^2\rangle^{1/2}_{000}}{\langle z^2\rangle^{1/2}_{000}} = \sqrt{2}\,(1+\alpha^2)^{-1/4}. \quad (16d)$$

For a low field $\alpha \to 0$ and $\xi \to \sqrt{2}$, as expected since for spherical symmetry $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = (\frac{1}{3})\langle\mathbf{r}^2\rangle$. For high fields $\alpha \to \infty$ and $\xi \to \sqrt{2/\alpha} \ll 1$. The atomic diameter in the x–y plane is severely contracted whereas its diameter along z is unaffected. The atom becomes needle shaped or, in effect, a one-dimensional system.

One can see more clearly how this comes about by examining the ground-state wave function directly. The oscillator wave functions can be derived in the standard manner from Eqs. (9c) and (9d). Alternatively, the ground-state wave function can be obtained simply and immediately by noticing that for $n_r = n_l$ the state has zero angular momentum. It is therefore the ground-state eigenfunction of $H_\parallel + H_\perp$ for $L_z = 0$. [See Eqs. (3b) and (3c).]
Thus

$$\Phi^{(000)}(\mathbf{r}) = \phi_0(x;\omega)\phi_0(y;\omega)\phi_0(z;\omega_0), \quad (17a)$$

where $\phi_0(u;\nu)$ is the one-dimensional oscillator ground-state wave function of coordinate u and frequency ν. Substitution of the well-known expression for ϕ_0 leads to

Table I. Zeeman effect in the states $n = n_r + n_l + n_0 = 1$.

Observation	Frequency	$\alpha \ll 1$	$\alpha \gg 1$	Polarization	Relative intensity	$\alpha \ll 1$	$\alpha \gg 1$
Transverse ($\hat{x}$)							
	$\Omega^{(100)} = \omega_0(1+\alpha^2)^{1/2} + \omega_L$	$\omega_0 + \omega_L$	$\omega_c = 2\omega_L$	σ	$\frac{1}{2}(1+\alpha^2)^{1/2}[\Omega^{(100)}]^4$	$\frac{1}{2}$	$8\alpha^3$
	$\Omega^{(010)} = \omega_0(1+\alpha^2)^{1/2} - \omega_L$	$\omega_0 - \omega_L$	$\omega_0/2\alpha$	σ	$\frac{1}{2}(1+\alpha^2)^{1/2}[\Omega^{(010)}]^4$	$\frac{1}{2}$	$\alpha^{-5}/32$
	$\Omega^{(001)} = \omega_0$	ω_0	ω_0	π	ω_0^4	1	1
Longitudinal ($\hat{z}$)							
	$\Omega^{(100)} = \omega_0(1+\alpha^2)^{1/2} + \omega_L$	$\omega_0 + \omega_L$	$\omega_c = 2\omega_L$	σ^+	1	1	1
	$\Omega^{(010)} = \omega_0(1+\alpha^2)^{1/2} - \omega_L$	$\omega_0 - \omega_L$	$\omega_0/2\alpha$	σ^-	$[\Omega^{(010)}/\Omega^{(100)}]^4$	1	$(2\alpha)^{-8}$

$$\Phi^{(000)}(\mathbf{r}) = (1+\alpha^2)^{1/4}(m\omega_0/\pi\hbar)^{3/4} \times e^{-m\omega_0 z^2/2\hbar} e^{-m\omega_0(1+\alpha^2)^{1/2}\rho^2/2\hbar}. \quad (17b)$$

For field strengths such that $\alpha < 1$, Eq. (17b) reduces to

$$\Phi^{(000)}(\mathbf{r}) = [1 + (\tfrac{1}{4})\alpha^2] e^{-m\omega_0\rho^2\alpha^2/4\hbar}\phi_{000}(r), \quad (17c)$$

where $\phi_{000}(r)$ is the ground-state wave function of the isotropic oscillator. The wave function is contracted in the x–y plane for large $\rho = (x^2 + y^2)^{1/2}$. When the field strength is so intense that $\alpha \gg 1$ Eq. (17b) becomes

$$\Phi^{(000)}(\mathbf{r}) = \alpha^{1/2}(m\omega_0/\pi\hbar) e^{-m\omega_0 z^2/2\hbar} e^{-m\omega_0\alpha\rho^2/2\hbar}. \quad (17d)$$

The wave function is severely attenuated in the x–y plane except for $\rho \sim 0$; thus the electron distribution is "needle-like" along the field axis.

As a final point of interest we look at the macroscopic response of an ensemble of ground-state oscillator atoms to a magnetic field. The isotropic oscillator in its ground state has no permanent magnetic moment; placed in a magnetic field it will exhibit a diamagnetic response. A measure of this response is the magnetic susceptibility

$$\chi = -N_0\, \partial^2 E/\partial B^2, \quad (18a)$$

where N_0 is the number of atoms/cm^3. Using Eqs. (11a)–(11c) one can express the ground-state susceptibility as

$$\chi_{000} = -N_0 \frac{\mu_B^2}{\hbar\omega_0} \partial^2\epsilon(000)/\partial\alpha^2 = -N_0\mu_B/\hbar\omega_0, \quad (18b)$$

where μ_B is the Bohr magneton.

The origin of the diamagnetism is a field-induced magnetic moment in each oscillator. This implies that there is an induced current in the oscillator. We can interpret this electric current as the electric charge multiplying the conserved probability current $\mathbf{J}$. For the field-free isotropic oscillator the probability current $\mathbf{J} = \mathrm{Re}[\phi^*_{n_r n|n0}(-i\hbar\nabla)\phi_{n_r n|n0}]/m$ vanishes. However, in the presence of a magnetic field

$$\mathbf{J} = \mathrm{Re}[\Phi^{(n_r n|n0)*}[-i\hbar\nabla - (e/c)\mathbf{A}]\Phi^{(n_r n|n0)}]/m = -e|\Phi^{(n_r n|n0)}|^2\mathbf{A}/mc. \quad (19a)$$

If the field is uniform, the current induced in the ground-state oscillator is

$$\mathbf{J}(\mathbf{r}) = |\Phi^{(000)}|^2\omega_L \times \mathbf{r}, \quad (19b)$$

indicating a block rotation of the electron density (like a probability "fluid") counterclockwise about the field direction at the Larmor frequency.

IV. CONCLUSION

The problem of an electron bound by a Coulomb potential and subjected to an external magnetic field is one of current research interest but of sufficient mathematical complexity that only numerical or approximate solutions can be given. The isotropic harmonic oscillator in a uniform field is an exactly soluble quantum problem which can serve as a useful model of an atom in a magnetic field of arbitrary strength. We have derived the exact energy eigenvalues and used them to test the standard perturbation theory of the magnetic interaction at low field strengths where (for judicious choice of gauge) the field-free, paramagnetic, and diamagnetic contributions are clearly displayed. The transition to the quasi-Landau spectrum for high fields was also shown. A study of the Zeeman effect showed that for low fields the expected Zeeman triplet for transverse observation and doublet for longitudinal observation emerge, but for very high fields the dominant emission line is the σ^+ component at the cyclotron frequency. Finally we demonstrated the contraction of the wave functions in the plane perpendicular to the field and showed that in the limit of very strong fields the three-dimensional system becomes effectively a one-dimensional system directed along the field axis.

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Men love to wonder
and that is
the seed of our science.

Emerson