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Quantum pendulum

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The problem of the mathematical pendulum is discussed in its classical, semiclassical, and quantum aspects. The energy spectrum and the eigenfunctions are presented under the usual requirement of single valuedness of the solutions.

I. INTRODUCTION

The mathematical pendulum, a massive particle constrained to move on a vertical circle under the action of gravity, is one of the classical problems of mechanics. Although with only one degree of freedom, its well-known solution is not trivial, as the equation of motion, being nonlinear, requires for its exact solution the machinery of elliptic functions. Physically, the system is rather peculiar: it exhibits three different regimes of motion, according to its total energy: rotational motion, vibration motion, and an asymptotic approach to a single angular deflection of 2π. The first two cases are periodic. The third appears as a limiting case and is nonperiodic. This distinctive behavior comes from the different topologies assumed by the available configuration space in each case.

The general quantum problem is rather involved, as it is a case of quantization in a multiply connected configuration space,[1] for which the only known approach is via path integrals. Strictly speaking, quantization in the circle would require multiply valued wave functions.[2] In the more realistic case, however, when the circle is imbedded in the Euclidean space R^3, the wave function on the circle is only a restriction to this multiply connected manifold of a wave function in R^3, and so will be single valued. However surprising it may be, we have not been able to find any self-contained treatment in the literature even for the special case of periodic (that is, single-valued) solutions. We shall here suppose it to have been missed, and try to fill the gap. As will be seen, the Schrödinger equation admits of exact solutions in terms of a certain class of Mathieu functions. The energy spectrum for a given pendulum can be calculated in terms of the "characteristic values," well known in the theory of the Mathieu equation.

Our presentation will be fairly elementary. In Sec. II, a brief account of the classical solutions[3,4] is given in terms of Jacobian elliptic functions and the phase-space orbits are discussed. Although very well known, we emphasize certain points which are important for the semiclassical quantization performed in Sec. III. The Schrödinger equation, its reduction to a Mathieu equation, a method to compute the eigenvalues and eigensolutions, as well as a discussion of some limiting cases, are the subject of Sec. IV. Section V is devoted to some final remarks.

II. CLASSICAL SOLUTIONS AND THE PHASE-SPACE ORBITS

The Hamiltonian for the plane pendulum of length l and mass m is

$$H = (1/2ml^2)P_\theta^2 - mgl\cos\theta, \tag{1}$$

where θ is the angle of deflection.

The Hamilton equations give immediately

$$P_\theta = ml^2\dot{\theta}; \qquad \dot{P}_\theta = -mgl\sin\theta, \tag{2}$$

from which it follows that

$$\ddot{\theta} + \Omega^2\sin\theta = 0 \tag{3}$$

if we define

$$\Omega^2 = g/l. \tag{4}$$

Equation (3) is an autonomous ordinary nonlinear differential equation. In order to solve it, we may use the fact that the total energy E is an integral of motion. From Eqs. (1) and (2),

$$E = (1/2)ml^2\dot{\theta}^2 - mgl\cos\theta. \tag{5}$$

It is convenient to introduce the quantity ω_0, defined as the value of the angular velocity at the point $\theta = 0$. As

$$E = (1/2)ml^2\omega_0^2 - mgl,$$

Eq. (5) yields

$$\dot{\theta}^2 = \omega_0^2 - 4\Omega^2\sin^2(\theta/2). \tag{6}$$

This first-order differential equation can be solved in terms of Jacobian elliptic functions.[5] Three cases, corresponding to three different regimes of motion, are to be distinguished, according to the relative values of ω_0^2 and 4 Ω^2:

(i) $\omega_0^2 > 4\Omega^2$, or $E > mgl$—rotational motion. The solutions are given by

$$\sin(\theta/2) = \mathrm{sn}(\Omega/k)(t - t_0), \tag{7}$$

where the modulus of the Jacobian elliptic function is

$$k = \left(\frac{2\,mgl}{E + mgl}\right)^{1/2}. \tag{8}$$

They correspond to motions with complete revolutions, with period given in terms of a complete Legendre elliptic integral of the first kind:

$$T = \frac{4}{\omega_0}\mathcal{K}\left(\frac{2\Omega}{\omega_0}\right). \tag{9}$$

An interesting limit is the pure rotor: $g \to 0$, $k \to 0$. As $\mathcal{K}(0) = \pi/2$, the period becomes energy independent, $T = 2\pi/\omega_0$.

(ii) $\omega_0^2 < 4\Omega^2$, or $E < mgl$—oscillatory motion.

The solutions are given by

$$\sin(\theta/2) = \bar{k}\,\mathrm{sn}\Omega\,(t - t_0), \tag{10}$$

with the modulus now being

$$\bar{k} = \left(\frac{E + mgl}{2mgl}\right)^{1/2} = \sin\left(\frac{\alpha}{2}\right), \tag{11}$$

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where α is the amplitude of the oscillation. The period is

$$T = \frac{4}{\Omega} \mathcal{K}\left(\frac{\omega_0}{2\Omega}\right). \quad (12)$$

The case of small oscillations, $\alpha \cong 2\bar{k} \ll 1$, leads again to isochronism.

(iii) $\omega_0^2 = 4\Omega^2$, or $E = mgl$—actually, this case is to be considered as a limiting case of (i) for $k = 1$, as it will be clear later on; the solution is

$$\theta = 4\tan^{-1}[\tanh(\Omega/2)(t - t_0)], \quad (13)$$

or

$$\sin(\theta/2) = \tanh[\Omega(t - t_0)]. \quad (13')$$

This corresponds to a deflection from $\theta = -\pi$ to $\theta = +\pi$ in an infinite time. If we choose to think of this case as periodic, Eq. (9) gives $T = \infty$. There is no really periodic solution in this case. Notice that Eq. (13) is always a solution of Eq. (3). In cases (i) and (ii), it is incompatible with the conditions on the energy. When k and $\bar{k}$ tend to one, both Eqs. (7) and (10) tend to Eq. (13′). This is not quite a surprise: Eqs. (7) and (10) can be shown to be identical by using the properties of the Jacobian functions.[5] The cases have been displayed separately in order to stress their different regimes and corresponding configuration spaces.

The orbits in phase space are easily obtained from Eqs. (2) and (5):

$$P_\theta = \pm[2ml^2(E + mgl\cos\theta)]^{1/2}. \quad (14)$$

Although this expression is the same, its interpretation is distinct in the three cases above:

(i) $$P_\theta = \pm ml^2\omega_0[1 - k^2\sin^2(\theta/2)]^{1/2}, \quad (15)$$

where the two signs correspond to different motions (counterclockwise, clockwise); P_θ is never zero, and the orbits in phase space are open of period 2π;

(ii) $$P_\theta = \pm 2ml^2\bar{k}\Omega\left(1 - \frac{1}{\bar{k}^2}\sin^2\frac{\theta}{2}\right)^{1/2}, \quad (16)$$

where both signs are to be used in order to give the complete closed orbit in phase space, corresponding to one given motion;

(iii) $$P_\theta = \pm ml^2\omega_0\cos(\theta/2).$$

Here, one motion is given by each sign; despite the apparent closed orbits in phase space, positive and negative P_θ correspond to distinct motions, limits of the two motions in case (i).

These differences will be important for the Bohr–Sommerfeld quantization to which we shall proceed in Sec. III. The orbits in phase space are usually depicted in textbooks.[4,6] They are, however, usually put all together and the above distinctions should be kept in mind.

III. ACTION VARIABLES: SEMICLASSICAL QUANTIZATION

The Bohr–Sommerfeld quantization rule is

$$J = \oint P_\theta d\theta = 2\pi N\hbar, \quad (17)$$

where N is an integer and the integration is to be done over a whole period of the motion. It is not difficult to calculate (17) for the three cases above.

Case $E > mgl$. One has

$$J = l[2m(E + mgl)]^{1/2} \times \int_{-\pi}^{+\pi} [1 - k^2\sin^2(\theta/2)]^{1/2} d\theta. \quad (18)$$

By the simple change of variables $\theta = 2\varphi$, this can be readily written in terms of the complete Legendre elliptic integral of the second kind:

$$J = 4l[2m(E + mgl)]^{1/2}\mathcal{E}(k). \quad (19)$$

Here a word of caution is worth giving: as stressed in Sec. II, a given motion is related to only one of the signs in Eq. (14). The quantized area in phase space is entirely contained either in the upper or in the lower half (θ, P_θ) plane.

In order to make contact with the quantum-mechanical treatment of Sec. IV, it is convenient to introduce a dimensionless (Mathieu) parameter q defined by

$$q = -2mgl/(\hbar^2/2ml^2) = -4(mgl/\hbar\Omega)^2. \quad (20)$$

Then, Eqs. (8), (17), and (19) can be put together in

$$(\pi/2)Nk = |q|^{1/2}\mathcal{E}(k); \quad (21a)$$

$$E = mgl(2/k^2 - 1). \quad (21b)$$

For fixed values of $|q|$ and N, Eq. (21a) can be numerically solved for k. The quantized energies are then readily obtained from Eq. (21b).

A limiting case of interest is the *plane rotor*, obtained for $g \to 0$ [that is, $k \to 0$, $q \to 0$ but $k/|q|^{1/2} \to \hbar/(2ml^2)^{1/2}$]. As $\mathcal{E}(k) \to \pi/2$ when $k \to 0$, the result is

$$E = (1/2I_0)N^2\hbar^2, \quad (22)$$

where we have denoted the moment of inertia ml^2 by I_0. This is, of course, the usual exact result for the rotor.

Case $E < mgl$. In this case the quantized area in the phase space is the whole one contained inside the curves of Eq. (16). Another difference is that the period is no more 2π. So,

$$J = 2\int_{-\alpha}^{+\alpha} dx[2I_0(E + mgl\cos\alpha)]^{1/2},$$

where α is the amplitude given by Eq. (11). Then,

$$J = 4[2I_0(E + mgl)]^{1/2} \times \int_0^{2\sin^{-1}\bar{k}} [1 - k^2\sin^2(\theta/2)]^{1/2} d\theta.$$

As in the previous case, this can be written in terms of a Legendre elliptic integral of the second kind, but now incomplete:

$$J = 16I_0\Omega(1/k)\mathcal{E}(k, \varphi = \sin^{-1}\bar{k}). \quad (23)$$

There is, however, an important difference: now, the modulus

$$k = (\bar{k})^{-1} > 1.$$

Making use of the relation[7] between incomplete elliptic functions of reciprocal moduli,

$$\mathcal{E}(k^{-1}, \varphi) = (1/k)[\mathcal{E}(k, \bar{\varphi}) - k'^2\mathcal{F}(k, \bar{\varphi})], \quad (24)$$

where $k'^2 = 1 - k^2$ is the complementary modulus and

$$\sin\varphi = k\sin\bar{\varphi}; \quad (25)$$

one finds $\varphi = \pi/2$ and Eq. (23) becomes simply

$$J = 16I_0\Omega[\mathscr{E}(\bar{k}) - \bar{k}'^2\mathscr{K}(\bar{k})]. \quad (26)$$

Inside the brackets, only complete elliptic integrals appear. Again in terms of the parameter q of Eq. (20),

$$N\pi/4|q|^{1/2} = \mathscr{E}(\bar{k}) - \bar{k}'^2\mathscr{K}(\bar{k}), \quad (27a)$$

$$E = mgl(2\bar{k}^2 - 1). \quad (27b)$$

Really, the integer N in the above equation should be replaced by $N + 1/2$, in the WKB approximation. In fact, for $E < mgl$ there are two turning points at $\theta = \pm\alpha$ and in such a case, as is well known, $N \to N + 1/2$, for $N = 0, 1, 2, \ldots$.

An interesting limit of the present case may be obtained from Eq. (27a) for *small oscillations*, that is, for small values of the amplitude α and [from Eq. (11)] of the modulus $\bar{k}$. By developing the right-hand side of Eq. (27a) up to quadratic terms in $\bar{k}$, one has

$$\mathscr{E}(\bar{k}) - \bar{k}'^2\mathscr{K}(\bar{k}) \approx (\pi/4)\bar{k}^2. \quad (28)$$

Equations (20) and (27) give then

$$E + mgl \approx N\hbar\Omega, \quad (29)$$

as it should be expected.

Case $E = mgl$. This would be, in principle, a limit of the first case when $k \to 1$. However, as $E = mgl$, the very meaning of quantization is lost. In a neighborhood of $k = 1$, as $\mathscr{E}(k) \to 1$ more slowly than k, one gets

$$E \approx -mgl + \frac{n^2\hbar^2}{2I_0}\frac{\pi^2}{4}, \quad (30)$$

a rotorlike spectrum with a "vacuum" deepening and a "renormalized" moment of inertia.

IV. SCHRÖDINGER EQUATION FOR THE PENDULUM

The time-independent Schrödinger equation corresponding to the Hamiltonian (1) is

$$-\frac{\hbar^2}{2ml^2}\frac{d^2}{d\theta^2}\psi(\theta) - mgl\cos\theta\psi(\theta) = E\psi(\theta). \quad (31)$$

The boundary condition to be imposed on the wave function is that it be single valued in the angular variable θ, that is, that $\psi(\theta)$ be periodic in θ of period 2π:

$$\psi(\theta + 2\pi) = \psi(\theta). \quad (32)$$

Equation (31) can be rewritten as a Mathieu equation, whose standard form is

$$\frac{d^2}{dv^2}\psi(v) + (p - 2q\cos 2v)\,\psi(v) = 0, \quad (33)$$

simply by defining

$$\theta = 2v, \quad (34)$$

$$P = \frac{4E}{\hbar^2/2ml^2}, \quad (35)$$

and recalling Eq. (20),

$$q = -\frac{2mgl}{\hbar^2/2ml^2}. \quad (20)$$

In view of Eq. (34), condition (32) reads

$$\psi\,[2(v + \pi)] = \psi(2v), \quad (36)$$

that is, as a function of v, the wave function, which we will denote by $\psi(v)$, has to be periodic of period π. Such solutions are *Mathieu functions of even order:* $ce_{2n}(v)$ and $se_{2n+2}(v)$, for $n = 0, 1, 2, \ldots$. The Mathieu equation has also odd-order solutions $ce_{2n+1}(v)$, $se_{2n+1}(v)$, which are periodic of period 2π, as well as nonperiodic solutions. These are all excluded by our periodicity condition above.

We notice further that Eq. (31) really corresponds to a Mathieu equation with a negative q. In order to meet the usual definitions for the Mathieu functions, we must perform a change of variable, replacing v by $(\pi/2 - v)$ in the equation. One can now use the following properties[7] of the Mathieu functions:

$$\begin{aligned} ce_0(v, -q) &= ce_0(\pi/2 - v, q); \\ ce_{2n}(v, -q) &= (-)^n\, ce_{2n}(\pi/2 - v, q); \\ se_{2n}(v, -q) &= (-)^{n+1}\, se_{2m}(\pi/2 - v, q). \end{aligned} \quad (37)$$

The factors $(-)^n$ and $(-)^{n+1}$ have been inserted in order to ensure the validity of the formulas for $q = 0$.

The Mathieu functions are conventionally normalized to π in the interval $(0, 2\pi)$ and so we finally obtain the normalized solutions of the Schrödinger equation (31), with boundary condition (32), in the form

$$\psi_0^{(e)}(\theta, -q) = (2\pi)^{-1/2}ce_0((\pi - \theta)/2, q); \quad (38a)$$

$$\psi_{2n}^{(e)}(\theta, -q) = \pi^{-1/2}(-)^n ce_{2n}((\pi - \theta)/2, q); \quad (38b)$$

$$\psi_{2n}^{(0)}(\theta, -q) = \pi^{-1/2}(-)^{n+1}\,se_{2n}((\pi - \theta)/2, q), \quad (38c)$$

with $n = 1, 2, 3, \ldots$.

The solutions $\psi_{2n}^{(e)}$ and $\psi_{2n}^{(0)}$ are even and odd real functions of θ, respectively. They are linearly independent and obey the orthogonality conditions

$$\int_0^{2\pi}\psi_{2n}^{(0)}(\theta)\psi_{2n}^{(e)}(\theta)\,d\theta = 0;$$

$$\int_0^{2\pi}\psi_{2m}^{(0)}(\theta)\psi_{2n}^{(0)}(\theta)\,d\theta = \int_0^{2\pi}\psi_{2m}^{(e)}(\theta)\psi_{2n}^{(e)}(\theta)\,d\theta = \delta_{mn}. \quad (39)$$

Their respective eigenvalues P_{2n} are denoted by $a_{2n}(-q)$ and $b_{2n}(-q)$. For them, the following relations hold:

$$\begin{aligned} a_{2n}(-q) &= a_{2n}(q), \\ b_{2n}(-q) &= b_{2n}(q), \\ a_{2n}(0) &= b_{2n}(0) = 4n^2. \end{aligned} \quad (40)$$

These eigenvalues ("characteristic values" in the mathematical literature) and the related eigenfunctions can be computed[8] by assuming trigonometrical expansions for the eigenfunctions. The coefficients of these trigonometric series obey certain three-term recursion relations and can be numerically calculated by a well-established procedure. The calculations are rather tedious. We shall here content ourselves with a sketchy how-to-do-it recipe, whose justification requires a large utilization of the properties of Mathieu functions that are found in the standard texts on the subject.[9,10]

The two kinds of Mathieu functions needed here are represented by the trigonometric series:

$$ce_{2n}(z, q) = \sum_{j=0}^{\infty} A_{2j}^{(2n)} \cos 2jz;$$

$$se_{2n}(z, q) = \sum_{j=0}^{\infty} B_{2j+2}^{(2n)} \sin[(2j + 2)z]. \qquad (41)$$

The wave functions (38) can then be put into the form

$$\psi_0^{(e)}(\theta, -q) = (2\pi)^{-1/2} \sum_{j=0}^{\infty} A_{2j}^{(0)} \cos(j\theta);$$

$$\psi_{2n}^{(e)}(\theta, -q) = \pi^{-1/2} \sum_{j=0}^{\infty} A_{2j}^{(2n)} \cos(j\theta); \qquad (42)$$

$$\psi_{2n}^{(0)}(\theta, -q) = \pi^{-1/2} \sum_{j=0}^{\infty} B_{2j+2}^{(2n)} \sin[(j+1)\theta].$$

The orthonormality conditions (39) give relations among the coefficients, which we shall write down in a form convenient for future discussion:

$$\frac{1}{A_0^2} = 2 + \left(\frac{A_2}{A_0}\right)^2 + \left(\frac{A_4}{A_0}\right)^2 + \left(\frac{A_6}{A_0}\right)^2 + \cdots, \qquad (43)$$

$$\frac{1}{B_2^2} = 1 + \left(\frac{B_4}{B_2}\right)^2 + \left(\frac{B_6}{B_2}\right)^2 + \cdots. \qquad (44)$$

The value of the parameter q is a characteristic of the pendulum under consideration and is supposed to be given from the beginning. Now to our recipe:

(a) *Eigenvalues.* Introduce

$$V_n = (P - n^2)/q, \qquad (45)$$

and define a function $g(P)$ through a continued fraction,

$$g(P) = V_2 - \frac{1}{V_4 -} \frac{1}{V_6 -} \frac{1}{V_8 -} \cdots. \qquad (46)$$

The zeros of this function are the eigenvalues corresponding to the odd functions, that is,

$$g(b_{2n}) = 0. \qquad (47)$$

The eigenvalues corresponding to the even eigenfunctions are the solutions of the equation

$$g(P) - 2q/P = 0, \qquad (48)$$

that is,

$$g(a_{2n}) = 2q/a_{2n}. \qquad (49)$$

Here, approximate results such as those given by the semiclassical approach are of great help, as the numerical methods to find the zeros in Eqs. (47) and (48) frequently require an initial guess.

(b) *Eigenfunctions.* Once an eigenvalue is given, the remaining task is the computation of the coefficients in Eqs. (42). This is done by using the following recursion relations:

(i) *even case:*

$$\frac{A_2}{A_0} = \frac{P}{q};$$

$$\frac{A_4}{A_0} = \frac{P-4}{q}\frac{A_2}{A_0} - 2; \qquad (50)$$

$$\frac{A_{2j+2}}{A_0} = \frac{P - 4j^2}{q}\frac{A_{2j}}{A_0} - \frac{A_{2j-2}}{A_0} \quad \text{for} \quad j \geq 2.$$

Once the ratios A_{2n}/A_0 are obtained, A_0^2 can be calculated from Eq. (43). Conventionally A_0 is positive in the series (41) and so in principle all the coefficients are known.

(ii) *odd case:*

$$\frac{B_4}{B_2} = \frac{P-4}{q};$$

$$\frac{B_{2j+2}}{B_2} = \frac{P - 4j^2}{q}\frac{B_{2j}}{B_2} - \frac{B_{2j-2}}{B_2}. \qquad (51)$$

B_2 is conventionally positive and can be obtained from Eq. (44).

So, in principle, the above rules allow one to obtain all the eigenvalues and respective eigenfunctions for the pendulum, with as good a precision as wanted.

The convergence of the series (41) is extremely fast.[9] This fact, first realized by Ince, made possible the first tabulation of the Mathieu functions, in a much better way than Mathieu's original procedure based on a power series expansion in q.

An extra case, for which this procedure can only be used through successive approximations, appears when $g = 0$, i.e., $q = 0$. Then Eq. (45) loses all meaning. The simplest issue is to go directly to Eq. (31) or (33), which become those describing a *free plane rotor.* The normalized wave functions (42) reduce simply to

$$\psi_0^{(e)}(\theta) = (2\pi)^{-1/2}$$

$$\psi_{2n}^{(e)}(\theta) = \pi^{-1/2}\cos n\theta, \qquad (52)$$

$$\psi_{2n}^{(0)}(\theta) = \pi^{-1/2}\sin n\theta;$$

and the related eigenvalues [from Eq. (35) and the last relation (40)] are

$$E_0 = 0,$$

$$E_{2n} = \frac{1}{4}\frac{\hbar^2}{2ml^2}P_{2n} = \frac{\hbar^2}{2ml^2}n^2, \qquad (53)$$

which are just Eq. (22). Except for the fundamental state, one has the well-known double degeneracy of the rotor, which is, of course, removed when $q \neq 0$.

Another interesting limit of Eq. (31) is obtained for $|\theta| \ll 1$. In this case, one easily obtains, in terms of the variable $x = l\sin\theta \approx l\theta$, the equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \left(-(E + mgl) + \frac{m\Omega^2}{2}x^2\right)\psi = 0, \qquad (54)$$

the Schrödinger equation for a linear harmonic oscillator with an angular frequency $\Omega = (g/l)^{1/2}$. The energy eigenvalues are

$$E_n = -mgl + (N + 1/2)\hbar\Omega, \qquad (55)$$

or, in terms of the Mathieu parameters p and q,

$$P_N = -2|q| + 4(N + 1/2)|q|^{1/2} \qquad (q < 0). \qquad (56)$$

This formula coincides with the approximate asymptotic formulas for the eigenvalues of the Mathieu equation[9] obtained by taking $q \to -\infty$ and keeping qv^2 fixed, that is, small values of v. One can see that in this case, the eigenvalues P_N for even N correspond to $a_{2n}(|q|)$ and those for odd N to $b_{2n+2}(|q|)$:

$$a_{2n} \sim -2|q| + 4(2n + 1/2)\,|q|^{1/2},$$

$$b_{2n+2} \sim -2|q| + 4(2n + 3/2)|q|^{1/2}. \qquad (57)$$

The accuracy of these approximations, for fixed N, increases for increasing $|q|$. As an example, the exact numerical values for $P_0(-9)$ and $P_0(-1600)$ are, respectively,

(-12.26) and (-3120.25), while the above formulas give (-12) and (-3120.00). On the other side, for fixed q the accuracy decreases with increasing N: for example, for $P_2(-1600) = -2803.28$ is given a value (-2800.00). One sees that the semiclassical results are very good even for the lower states. By using tables for the Legendre elliptic functions, straightforward calculations show that Eqs. (27) are an still improved approximation as compared to the above ones.

V. FINAL COMMENTS

We have shown how to obtain the energy levels and respective wave functions for the mathematical pendulum. As it should be expected, it somehow interpolates between the rotor and the harmonic oscillator, to which it tends in extreme opposite limits. The energy spectrum retains an interesting feature of the classical case, that the lower states are vibrational-like and the higher ones are rotational-like. The ground state is of vibrational origin for any value of the parameter q. The excited states form pairs of opposite parity states whose separations exhibit a very peculiar behavior as a function of q. This behavior is best visualized in the graphs of the "characteristic values" versus q, available in the quoted textbooks on the Mathieu equation. Of course, another point worth stressing is the fact that no simple closed formula exists for the spectrum. Moreover, the Hilbert space is fairly restricted and no degeneracy allowed. This seems rather surprising for such a smooth nonsingular potential as the one considered here. Although the deep reasons for all these intricacies are difficult to know exactly, the good results given by the semiclassical spectrum suggest that the quantum system is never too far from the classical one, at least not far enough not to inherit the effects of its nonlinearity.

There is a point we would like to stress here: we have simply solved the problem of a mathematical pendulum in the Euclidean space R^3. In this case, the problem is quite different from the one-dimensional sinusoidal model used in solid-state physics, which has an Eq. as (31) with $\theta = 2\pi x/L$. The difference lies in the fact that the physical variable then is x, extending from $-\infty$ to $+\infty$, and not an angle. The phase space is the whole plane whereas for the pendulum it is really an infinite cylinder of circumference 2π. Were we to stick to the motion of a particle under the action of gravity and constrained to move on a circle, and both problems would probably coincide. This is suggested by the case of the pure rotor[2] and by a general result obtained from path-integration methods,[1] following which the quantization in a multiply connected configuration space has to be done as if the system were really envolving in its universal covering space, which for the circle is the straight line.

Note added in proof. After this paper has been submitted, A. O. Barut kindly informed us about E. U. Condon's paper [Phys. Rev. **31**, 891 (1928)], as well as that by T. Pradham and A. V. Khare [Am. J. Phys. **41**, 59 (1973)], on the quantum pendulum. In spite of the overlapping with the present paper, we believe that the differences, both in emphasis and character, are still relevant.

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[1]See, for instance, M. G. G. Laidlaw and C. Morette De Witt, Phys. Rev. D **3**, 1375 (1971).

[2]The special case of the pure rotator is treated by L. Schulman, Phys. Rev. **176**, 1558 (1968).

[3]H. C. Corben and P. Stehle, *Classical Mechanics* (Wiley, New York, 1960).

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1959).

[5]S. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1968).

[6]V. I. Arnold, *Ordinary Differential Equations* (MIT, Cambridge, MA, 1973).

[7]I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965).

[8]E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University, London, 1935).

[9]R. Campbell, *Théorie Générale de l'Equation de Mathieu* (Masson, Paris, 1955).

[10]N. W. McLachlan, *Theory and Applications of Mathieu Functions* (Clarendon, Oxford, 1951).