One-dimensional hydrogen atom

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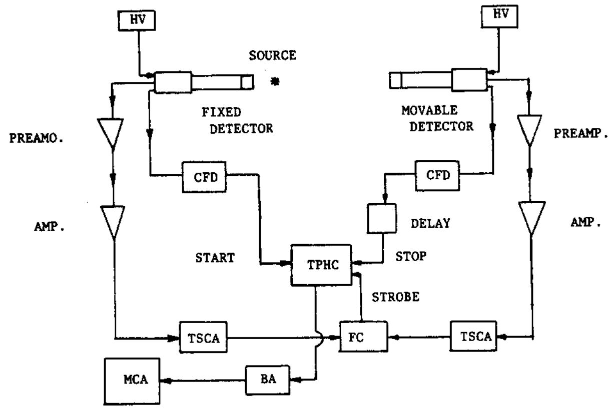

Fig. 1. Schematic diagram of experimental layout. HV: high-voltage power supply; CFD: constant fraction discriminator; TPHC: time to pulse height converter; FC: fast coincidence; TSCA: timing single channel analyzer; BA: biased amplifier; MCA: multichannel analyzer.

coupled to RCA 8850 photomultiplier tubes. Plastic was selected despite its relatively low photoelectric absorption coefficient because of its superior output pulse shape for timing applications. The tubes are plugged into special timing bases (Ortec model 270). The positive signals from the ninth dynodes of the tubes were used for the energy channels. After suitable amplification and shaping, these signals were routed through two timing single channel analyzers with windows set to the 0.51-MeV pulses. The analyzers thus limit the acceptable pulses to those that correspond to the absorption of an annihilation photon in the plastic scintillator. The analyzer outputs were taken to a fast coincidence circuit, and the signals from that circuit were used to strobe the output of a time to pulse height converter. The converter generates an output pulse of amplitude proportional to the elapsed time between the arrival of signals at its start and stop inputs, respectively. The output is not produced, however, unless the converter receives a signal from the coincidence circuit. The energy channels were used in this way to limit the dynamic range of the timing pulses.

The negative pulses from the photomultiplier anodes were routed directly to two constant fraction discriminators set to trigger at 30% of pulse amplitude. This type of discriminator improves timing performance by triggering on the leading edge of the input pulse at a preselected constant fraction of the maximum pulse amplitude rather than at a fixed amplitude level as is the case with conventional discriminators. The discriminator outputs were used as the start and stop signals to the time to pulse height converter. Nanosecond delays were inserted in the stop branch in order to ensure that the timing intervals were close to the midrange of the converter, thus improving the linearity of response of the converter. The delays were also used to calibrate the multichannel analyzer. A biased amplifier was used to amplify the output of the converter over the timing range of interest. This was found to be necessary because the timing intervals covered in the experiment extended up to a maximum of about 15 nsec, whereas the 0–10 V output of the converter covered an interval range of 50 nsec.

Measurements were taken by keeping one of the detectors fixed at 20 cm from the source and moving the other detector to successive positions at 50-cm intervals starting at 150 cm from the source and extending up to 350 cm from the source. A time spectrum was accumulated at each position, and the peak locations were determined by fitting Gaussian functions to the data using nonlinear least-squares techniques.

A straight line was fit to the resulting numbers, using the distance from the source as the dependent variable and the time interval as the independent variable. The slope of this line gave the velocity. This was found to be $(3.009 \pm 0.013) \times 10^{10}$ cm/sec, or a precision of about 0.43%. This result is within 0.37% of the currently accepted value of c.

The time resolution obtained in the above experiment was about 650 psec full width at half maximum. Better resolution could have been obtained by using a narrower dynamic range of acceptable pulses as set by the single channel analyzers, but this was not done because of the need to keep the count rates reasonably high. A bigger source would permit selection of a narrower dynamic range, thus improving resolution, and would also permit the accumulation of a greater number of counts in a reasonable time, thereby improving the precision, and possibly the accuracy of the results. The method probably does not possess sufficient inherent precision for the determination of the speed of light with an accuracy better than a fraction of 1%; but the experiment is of interest in pointing out the timing capabilities available using current state of the art methods.

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Some time ago Loudon[1] and Haines and Roberts[2] considered the bound-state energies of the one-dimensional "hydrogen atom" with potential energy given by $V(x) = -e^2/|x|$. Therefore the wave equation is written

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \frac{e^2}{|x|}\Psi = E\Psi. \quad (1)$$

The purpose of this note is to discuss the even states $\Psi(x) = \Psi(-x)$, which were proposed by the above-mentioned authors. Introducing the dimensionless variables[1,2] α and z,

$$E = -\frac{\hbar^2}{2ma_0^2\alpha^2}, \quad z = \frac{2x}{\alpha a_0}, \quad a_0 = \frac{\hbar^2}{me^2}, \quad (2)$$

Eq. (1) becomes

$$\frac{d^2\Psi}{dz^2} - \frac{1}{4}\Psi + \frac{\alpha}{|z|}\Psi = 0. \quad (3)$$

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Integrating it from $-\epsilon$ to $+\epsilon$, we obtain

$$\Psi'(\epsilon) - \Psi'(-\epsilon) - \frac{1}{4}\int_{-\epsilon}^{+\epsilon} \Psi(z)\,dz + \alpha \int_{-\epsilon}^{+\epsilon} \frac{\Psi(z)}{|z|}\,dz = 0. \quad (4)$$

For $z \geqslant 0$, the normalizable wave function is[2]

$$\Psi_+(z) = B_+ W_\alpha(z), \quad (5)$$

B_+ being an arbitrary constant and $W_\alpha(z)$ given by[2]

$$W_\alpha(z) = \frac{e^{-z/2}}{\Gamma(-\alpha)}\left(-\frac{1}{\alpha} + \sum_{r=1}^{\infty} \frac{(1-\alpha)_r}{r!(r+1)!} A_r z^{r+1} + [\ln z + \psi(1-\alpha) - \psi(1) - \psi(2)] \times \sum_{r=0}^{\infty} \frac{(1-\alpha)_r}{r!(r+1)!} z^{r+1}\right), \quad (6)$$

with A_r, $(1-\alpha)_r$, and $\psi(s)$ defined by Eqs. (11), (12), and (13) of Ref. 2.

For $z \leqslant 0$ the corresponding normalizable solution of Eq. (3) is

$$\Psi_-(z) = B_- W_\alpha(-z), \quad (7)$$

where B_- is an arbitrary constant.

For the case $\alpha = N$ ($N = 1,2\ldots$), from Eq. (4) it follows $\Psi'_+(0) = \Psi'_-(0)$ and therefore $B_+ = -B_-$, meaning that for this case $\Psi(z)$ is odd, ruling out the even states considered by Loudon.[1]

Haines and Roberts[2] proposed for Eq. (3) the following even states [their Eq. (9)]:

$$\Psi(z) = B W_\alpha(|z|) \quad (8)$$

with $\alpha \neq N$ ($N = 1,2\ldots$). In this way they claim to have obtained a continuous spectrum of even states, not degenerate with the odd ones [see the discussion which follows from Eqs. (17)–(22), as well as the abstract, of Ref. 2].

For small z, expression (6) for $W_\alpha(|z|)$ writes

$$W_\alpha(|z|) = \frac{1}{\Gamma(-\alpha)}\left\{\left(1 - \frac{|z|}{2} + \frac{|z|^2}{8} + \cdots\right) \times \left(-\frac{1}{\alpha} + \frac{3\alpha - 1}{4}|z|^2 + \cdots\right) + [\ln|z| + \psi(1-\alpha) - \psi(1) - \psi(2)] \times \left(|z| + \frac{1-\alpha}{2}|z|^2 + \cdots\right)\right\}. \quad (8')$$

Expression (8) is not a solution of Eq. (3). In order to see this, we introduce expression (8′) into Eq. (3) and study it for small z. We will obtain some extra terms proportional to $\delta(z)\ln|z|$ which do not cancel out.

Another way to see this is the following one: if expression (8) is a solution of Eq. (3) even for $z = 0$, then it would be also a solution of expression (4). Now we can verify that this is not true.

For that purpose let us introduce expression (8′) into Eq. (4). Evidently it must be satisfied for any value of ϵ. The part of expression (8′) equal to

$$\frac{1}{\Gamma(-\alpha)}\left[|z|\left(\frac{1}{2\alpha} + \ln|z| + \psi(1-\alpha) - \psi(1) - \psi(2)\right)\right]$$

gives

$$\Psi'(\epsilon) - \Psi'(-\epsilon) = [1/\Gamma(-\alpha)][2\ln\epsilon + 2 + (1/\alpha) + 2\psi(1-\alpha) - 2\psi(1) - 2\psi(2)], \quad (9)$$

which is of order $\ln\epsilon$ and ϵ^0. The remaining terms of Eq. (8′) giving contributions of order $\epsilon\ln\epsilon$, ϵ, and so on for expression (9).

The first integral of Eq. (4) is at least of order $\epsilon\ln\epsilon$, while the second integral, if we keep terms of order up to ϵ^0, receives only contribution from the constant term $-1/\alpha\Gamma(-\alpha)$ from expression (8′), giving

$$\alpha\int_{-\epsilon}^{\epsilon} \frac{\Psi(z)}{|z|}\,dt = \frac{-2}{\Gamma(-\alpha)}(\ln\epsilon - \ln 0). \quad (9')$$

These two last expressions when introduced in the left-hand side of Eq. (4), give a zero contribution at order $\ln\epsilon$, while at order ϵ^0, give

$$[1/\Gamma(-\alpha)][(1/\alpha) + 2\psi(1-\alpha) - 2\psi(1) - 2\psi(2) + 2 + 2\ln 0] \quad (10)$$

for $\alpha \neq N$. Because of the presence of the infinite term $\ln 0$, there are no values of $\alpha \neq N$ and real which satisfy Eq. (10) and therefore the even states with finite energy are excluded.

In order to confirm this result we shall make use of the virial theorem which states[3]

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle = -\langle V\rangle$$

and, therefore,

$$\langle E\rangle = \tfrac{1}{2}\langle V\rangle = -\frac{\alpha}{2}\int_{-\infty}^{+\infty} \frac{\Psi^*(z)\Psi(z)}{|z|}\,dz. \quad (11)$$

This last equation can be written as

$$\langle E\rangle = -\frac{\alpha}{2}\int_{-\infty}^{-M} \frac{\Psi^*(z)\Psi(z)}{|z|}\,dz - \frac{\alpha}{2}\int_{M}^{\infty} \frac{\Psi^*(z)\Psi(z)}{|z|}\,dz - \frac{\alpha}{2}\int_{-M}^{M} \frac{\Psi^*(z)\Psi(z)}{|z|}\,dz, \quad (12)$$

where $-M$ and M are any two points. We can choose them as the points where $\Psi^*(z)\Psi(z)$ is maximum (see for instance Fig. 1 of Ref. 2). As $\Psi^*(z)\Psi(z)$ is limited in the whole interval $(-\infty, +\infty)$ and is different from zero at the origin, it follows that the last term of Eq. (12) diverges. Therefore for any even state proposed by the authors of Ref. 2, the energy is not bounded. Physically it means that, as $\Psi^*(z)\Psi(z)$ is different from zero at the origin, the electron will feel a very strong attraction, giving collapse of the system.

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