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Hydrogen atom and relativistic pi-mesic atom in N-space dimensions

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We derive in simple analytic closed form the eigenfunctions and eigenenergies for the hydrogen atom in N dimensions. A section is devoted to the specialization to one dimension. Comments are made on the relation to the harmonic oscillator, the ground-state energy per degree of freedom, the raising and lowering operators, and the radial momentum operators. By particular changes of variables, the relativistic pi-mesic atom is solved in the same functional form.

I. INTRODUCTION

The two problems which are commonly discussed in three-dimensional quantum mechanics are the hydrogen atom and the harmonic oscillator. These problems can be exactly solved for all angular momentum L, not just for $L = 0$. The relationship between these problems is very deep since, as was first observed by Schrödinger,[1] the solutions to the quantum hydrogen atom and harmonic oscillator problems are related,[1-5] although not totally equivalent.[6] Schrödinger's interest in this connection was part of the incentive for deriving his factorization method to obtain the solutions of second-order differential equations.[1,7,8] This method was beautifully expounded in the work of Infeld and Hull.[9]

The harmonic oscillator has been thoroughly discussed in N-space dimensions.[2,10-13] The N-dimensional hydrogen atom, on the other hand, is not so well known. The peculiar one-dimensional atom has been controversially discussed,[14-17] and literature exists as well on the two-dimensional atom.[18,19] One can find[20] a discussion of the N-dimensional case in terms of unnormalized confluent hypergeometric functions.

In this paper we will first give a complete normalized solution of the general N-dimensional hydrogen atom, which reduces to the standard[21-25] three-dimensional atom when the parameter N is set equal to 3. Next, in Sec. III we will use our N-dimensional results to discuss what one-dimensional potential is the appropriate specialization. This being done, we use known quantum-mechanical principles to predict the exact closed-form normalized bound-state solutions. This prediction is verified explicitly. In Sec. IV we make observations on the effects of the number of dimensions on the generalized eigenenergies and ground-state energies, on the raising and lowering operators of the radial eigenfunctions, and on the radial momentum operators. Finally we show that in N-space dimensions particular changes of variables map the relativistic pi-mesic atom into the nonrelativistic hydrogen atom, so that the solutions follow immediately.

Before continuing, however, we wish to make the physically important point that, although the hydrogen atom in N-space dimensions is standardly defined by the potential

$$-e\phi = V(r) = -e^2/r, \tag{1.1}$$

$$r^2 = \sum_{i=1}^{N} x_i^2, \tag{1.2}$$

Equation (1.1) is *not* the static potential which would correspond to the solution of Maxwell's equations in N-space dimensions. Equation (1.1) is only the solution of Poisson's equation

$$\nabla^2\phi = -4\pi\rho \tag{1.3}$$

in three-space dimensions. In general, the solution is

$$V(r) = \begin{cases} \text{const}\, r^{2-N}, & N \neq 2, \\ \text{const}\ln(r/r_0), & N = 2. \end{cases} \tag{1.4}$$

For $N = 2$ the reader should recognize Eq. (1.4) as the electrostatic solution for an infinite line charge in three-space dimensions. For $N = 1$, the solution (1.4) is an infinitely rising confining potential. This potential is one of the reasons for modern field-theoretic interest in the electrodynamics of one-space and one-time dimensions. It provides insight into the problem of quark confinement.

II. N-DIMENSIONAL HYDROGEN ATOM

The Schrödinger equation for the N-dimensional hydrogen atom is ($Ze^2 \to e^2$)

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}\right)\psi = E\psi, \tag{2.1}$$

$$r^2 = \sum_{i=1}^{N} x_i^2. \tag{2.2}$$

The first step is to change to spherical coordinates in N dimensions. As in three dimensions, this allows a factorization of the problem into a wave function which is the product of a radial wave function $R_{nl}(r)$, which will turn out to be labeled by the two quantum numbers n and l, and a spherical harmonic Y labeled by the quantum numbers $Y(l_1, l_2, \ldots, l_{N-2}, l_{N-1} \equiv l)$, $l = |m|$ for $N = 2$. These spherical harmonics are well known and studied. They are given in terms of the coordinates θ_i, $1 \leq i \leq N-1$, defined by

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$$
\begin{aligned}
x_1 &= r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1},\\
x_2 &= r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1},\\
x_3 &= r\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{N-1},\\
&\vdots\\
x_j &= r\cos\theta_{j-1} \sin\theta_j \cdots \sin\theta_{N-1},\\
&\vdots\\
x_{N-1} &= r\cos\theta_{N-2} \sin\theta_{N-1},\\
x_N &= r\cos\theta_{N-1},
\end{aligned}
\tag{2.3}
$$

for $N = 2,3,\ldots$, and $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ for $N = 2$. We refer the reader to Louck[2,26,27] for the appropriate discussion. The main point for us is that with these variables,

$$
\nabla^2 R_{nl} Y = Y\left(\frac{d^2}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr} - \frac{l(l+N-2)}{r^2}\right) R_{nl}(r). \tag{2.4}
$$

Making with foresight the changes of variables

$$
E = -\mathcal{E}_0/k^2(n), \quad \mathcal{E}_0 = me^4/2\hbar^2, \tag{2.5}
$$

$$
\rho = r/r_0 k, \quad r_0 = \hbar^2/2me^2, \tag{2.6}
$$

Eq. (2.1) thus becomes

$$
\left(\frac{d^2}{d\rho^2} + \frac{(N-1)}{\rho}\frac{d}{\rho} - \frac{l(l+N-2)}{\rho^2} + \frac{k}{\rho} - \frac{1}{4}\right) R_{nl} = 0. \tag{2.7}
$$

For $N = 3$ our equation reduces to the radial equation for the standard hydrogen atom. Motivated by this, we go through a procedure which is similar to that for the standard solution of the hydrogen atom.

Consider the $\rho \to \infty$ limit of (2.7). In this case one has

$$
\left(\frac{d^2}{d\rho^2} - \frac{1}{4}\right) R_{nl} = 0. \tag{2.8}
$$

The solution of (2.8) which is finite at $\rho \to \infty$ is

$$
\lim_{\rho\to\infty} R_{nl} = \mathcal{N}(n,l) \exp(-\rho/2), \tag{2.9}
$$

where $\mathcal{N}(n,l)$ is the normalization constant.

Equation (2.9) leads us to propose the trial solution

$$
R_{nl} = \mathcal{N}(n,l) \exp(-\rho/2)\, g(\rho). \tag{2.10}
$$

Putting this back into (2.7) yields an equation for g of

$$
g'' + \left(\frac{N-1}{\rho} - 1\right) g' + \left(\frac{k - [(N-1)/2]}{\rho} - \frac{l(l+N-2)}{\rho^2}\right) g = 0, \tag{2.11}
$$

where the prime refers to the derivative with respect to ρ. Because exponential behavior has already been taken out, one hopes that the solution for g is a polynomial. Indeed, if one considers the special value for k of

$$
k - [(N-1)/2] = l, \tag{2.12}
$$

then a solution to (2.11) is

$$
g(k = (N-1)/2 + l) = \rho^l. \tag{2.13}
$$

One is thus led to take the further trial solution

$$
g = \rho^l h, \tag{2.14}
$$

which, when put into (2.11), gives the equation for h,

$$
\rho h'' + [(2l + N - 1) - \rho] h' + [k - (N-1)/2 - l]\, h = 0. \tag{2.15}
$$

If the quantity $[k - (N-1)/2 - l]$ which multiplies h in (2.15) is equal to a non-negative integer, then a finite polynomial solution is allowed. (This, when combined with the rest of R_{nl}, yields a normalizable solution.) In particular, this solution to (2.15) is the generalized Laguerre polynomial $L^{(2l+N-2)}_{k+1/2-N/2-l}(\rho)$.

Please observe that our Laguerre polynomials are defined as

$$
L_n^{(\alpha)}(t) \equiv \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-t)^j}{j!} \tag{2.16}
$$

$$
\equiv \frac{(-1)^\alpha L^\alpha_{n+\alpha}(t)}{(n+\alpha)!} \tag{2.17}
$$

[α a non-negative integer in (2.17)]. The $L_n^{(\alpha)}$ are mathematically more useful than the $L^\alpha_{n+\alpha}$ often used in Coulomb wave functions, and are the ones discussed in comprehensive books on the functions of mathematical physics[28,29] as well as in tables of integrals.[30]

Changing to the commonly used principle quantum number $n = 1,2,3,\ldots$, the condition $[k(n) - (1/2)N + 1/2 - l] = 0,1,2,\ldots$, used with the limiting case $l_{\max} = n - 1$ coming from Eqs. (2.12) and (2.13), gives the following allowed values for n and l:

$$
n = k - (1/2)N + 3/2 = 1,2,3,\ldots, \tag{2.18a}
$$

$$
l = 0,1,2,\ldots, n-1. \tag{2.18b}
$$

Combining everything, one finally has

$$
R_{nl}(\rho) = \mathcal{N}(n,l) \exp(-\rho/2) \rho^l L^{(2l+N-2)}_{n-l-1}(\rho), \tag{2.19}
$$

$$
\rho = \frac{r}{r_0[n + (N-3)/2]}, \tag{2.20}
$$

$$
E_n = \frac{-\mathcal{E}_0}{[n + (N-3)/2]^2}. \tag{2.21}
$$

Since the radial volume element in N-dimensional space is $r^{N-1}\,dr$, one can obtain

$$
\begin{aligned}
\mathcal{N}(n,l) &= \left[\int_0^\infty dr\, r^{N-1} e^{-\rho} \rho^{2l} \left(L^{(2l+N-2)}_{n-l-1}(\rho)\right)^2\right]^{-1/2}\\
&= \left[r_0\left(n + \frac{N-3}{2}\right)\right]^{-N/2} \left(J^{(1)}_{n-l-1,2l+N-2}\right)^{-1/2},
\end{aligned}
\tag{2.22}
$$

where the generalized Coulomb-like integral $J^{(\beta)}_{n,\alpha}$ for non-integer α and β has been derived elsewhere[31] as

$$
\begin{aligned}
J^{(\beta)}_{n,\alpha} &= \int_0^\infty e^{-t} t^{\alpha+\beta} [L_n^{(\alpha)}(t)]^2\, dt\\
&= \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n-k-\beta)}{\Gamma(-k-\beta)}\\
&\quad \times \frac{\Gamma(\alpha+k+1+\beta)}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(k+1)\,\Gamma(n-k+1)},\\
&\qquad \mathrm{Re}(\alpha+\beta+1) > 0.
\end{aligned}
\tag{2.23}
$$

$J^{(1)}_{n,\alpha}$ has only two nonzero contributions (for $k = n-1$ and n) in the sum (2.23) because of the gamma functions of negative integers. The result is

$$J^{(1)}_{n,\alpha} = \frac{(\alpha+n+1)}{\Gamma(n+1)}(2n+\alpha+1), \tag{2.24}$$

meaning the normalization constant $\mathcal{N}(n,l)$ is

$$\mathcal{N}(n,l) = \frac{1}{\{r_0[n+(N-3)/2]\}^{N/2}} \times \left(\frac{\Gamma(n-l)}{2[n+(N-3)/2]\,\Gamma(n+l+N-2)}\right)^{1/2}. \tag{2.25}$$

III. ONE-DIMENSIONAL HYDROGEN ATOM

The one-dimensional hydrogen atom has been involved in an interesting controversy.[14-17] From our point of view we will be able to discuss an appropriate one-dimensional specialization in closed form, making a connection to the previous literature.

From the eigenenergy equation (2.21), one would naively guess that a specialization to one dimension would have the same eigenspectra as the three-dimensional atom, but with the three-dimensional ground state disappearing since it would be infinitely bound. [That is, in Eq. (2.21), $(n-1) = 1,2,\ldots$]. Indeed, taking the potential

$$V(x) = \begin{cases} -e^2/x, & x > 0, \\ +\infty, & x \leq 0, \end{cases} \tag{3.1}$$

this turns out to be the case. Further, by known quantum-mechanical principles[32,33] one can give the functional forms of the wave functions even before they are derived, as we now do.

If one starts with a symmetric one-dimensional potential, $V(x) = V(-x)$, the eigensolutions $\psi_m(x)$ are even for $m = 0,2,\ldots$, and odd for $m = 1,3,5,\ldots$. The differential equation satisfied by these one-dimensional ψ_m is the same differential equation satisfied by the three-dimensional wave functions $\chi_{m,l=0}(r)$, where

$$\chi_{m,l=0}(r) = rR_{m,l=0}(r). \tag{3.2}$$

However, since on *physical* grounds $R_{m,l=0}$ must have a continuous first derivative at the origin, $r = 0$, only the odd solutions $\chi_{m,l=0}(r)$ are allowed. Thus, the *odd* solutions of the one-dimensional symmetric problem correspond to *all* the allowed radial solutions of the three-dimensional problem for $l = 0$.

Given the above, if we now destroy the symmetry of the one-dimensional potential by putting in an infinite potential barrier at the origin, the effect is to rule out the even ψ_m and allow only the odd ψ_m. Then, *all* the allowed one-dimensional solutions correspond to *all* the allowed radial solutions of the three-dimensional problem for $l = 0$.

The above paragraph exactly describes the situation for the one-dimensional potential of Eq. (3.1). Thus, by taking the three-dimensional $l = 0$ hydrogen-atom wave functions of (2.19) and then using (3.2) with $n = (m+1)/2 = 1,2,3,\ldots$, one obtains the solutions to the one-dimensional hydrogen atom as,

$$\psi_n(x) = \mathcal{N}_n e^{-\rho/2}\rho L^{(1)}_{n-1}(\rho), \quad n = 1,2,3,\ldots, \tag{3.3}$$

$$\rho \equiv x/(nr_0), \tag{3.4}$$

$$E_n = -\mathcal{E}_0/n^2. \tag{3.5}$$

The lost[15,17] "infinitely bound state" is the $m = 0$ even ground state of the symmetric one-dimensional problem.

Now that we have predicted the solution (3.3), let us verify it. As in Sec. II we attempt the trial solution

$$\psi_n(x) = \mathcal{N}_n e^{-\rho/2}g(\rho) = \mathcal{N}_n e^{-\rho/2}\rho h(\rho), \tag{3.6}$$

$$\rho \equiv x/[k(n)x_0]. \tag{3.7}$$

Putting this into the one-dimensional Schrödinger equation, the differential equations for g and h are

$$0 = \rho g'' - \rho g' + kg, \tag{3.8}$$

$$0 = \rho h'' + (2-\rho)h' + (k-1)h. \tag{3.9}$$

Equation (3.8) is Kummar's equation and (3.9) is once more the Laguerre equation. One can obtain a finite polynomial solution to (3.9) if $k-1$ is a nonnegative integer $0,1,2,\ldots$. Then the solution is $L^{(1)}_{k-1}(\rho) \equiv L^{(1)}_{n-1}(\rho)$, $n = 1,2,3,\ldots$. Note that since the factor multiplying h in Eq. (3.9) is $k-1$ instead of k, $k = n$ instead of differing by unity from n. Technically this is where the "infinitely bound ground state" is avoided.

Summarizing, Eqs. (3.3)-(3.5) are the correct wave functions and eigenspectra for the potential (3.1), with the normalization constants being

$$\mathcal{N}_n = (r_0 n J^{(1)}_{n-1,1})^{-1/2} = (2n^3 r_0)^{-1/2}. \tag{3.10}$$

The "physical" reason we prefer the potential (3.1) to

$$V(x) = -e^2/|x| \tag{3.11}$$

is that although (3.11) yields the same eigenspectra as (3.5), it is doubly degenerate. In higher dimensions the eigenenergy degeneracies are due to the angular coordinates instead of the radial one. Therefore, since there is no radial coordinate in one dimension, we feel a nondegenerate eigenspectra is more "physical." However, the above must certainly be taken with a giant grain of salt, since our "physical" system is one dimensional. Thus, the choice between (3.1) and (3.11) is somewhat a matter of opinion.

In any event, it is interesting to compare the eigenfunctions for[14] (3.11) to our Eq. (3.3) for $V(x)$ of (3.1). For $x > 0$ they are the same. For $x \leq 0$ the ρ becomes $|\rho|$ in the exponential and Laguerre polynomials. However the factor ρ remains ρ for the odd wave functions but becomes $|\rho|$ for the even wave functions.

IV. PROPERTIES OF THE SOLUTIONS

There are a number of properties which are interesting to look at. The first is the monotonic increase of the nth eigenenergy with increasing dimension N. From (2.21),

$$\Delta E_n \equiv E_n(N) - E_n(N-1) = \mathcal{E}_0\left(\frac{[n+(1/2)N-7/4]}{[n+(1/2)N-2]^2\,[n+(1/2)N-3/2]^2}\right). \tag{4.1}$$

In particular, this means that the added ground-state energy per degree of freedom is

$$\Delta E_1 = 4\mathcal{E}_0\left(\frac{2N-3}{(N-2)^2(N-1)^2}\right). \tag{4.2}$$

A next property to consider is the raising and lowering

operators to adjacent eigenstates. Such calculations have been discussed in great detail by Infeld and Hull[9] in their classic "factorization-method" paper, even though at times their factorizations turn out not to be the physically useful ones.[33,34] For the Coulomb potential, Infeld and Hull focused on what remains to this day a fundamental problem in the n raising and lowering operators. Even though one can use the raising and lowering operators for the generalized Laguerre polynomials to obtain what appear to be the raising and lowering operators for the wave functions $R_{nl}(\rho)$, what is devastating, but often unnoticed, is that ρ depends on n as given in (2.20). Thus, applying a raising operator to $R_{nl}(\rho)$ will yield

$$A_n^+ R_{n,l}(\rho_n) = \hat{C}_{nl}\hat{R}_{n+1,l}(\rho_n) \neq C_{nl}R_{n+1,l}(\rho_{n+1}), \tag{4.3}$$

$$\rho_n \equiv \frac{r}{r_0 n}. \tag{4.4}$$

Thus, one has to apply a "shift operator" to change ρ_n to ρ_{n+1}. Effectively, the "shift operator" really means changing ρ_n to ρ_{n+1} by hand. This is a deep problem, whose origin has been discussed by Aebersold and Biedenharn.[6]

Contrariwise, one can still discuss the l raising and lowering operators since ρ does not depend on l. However, partially because the eigenvalues of L^2 are not evenly spaced but $l(l+N-2)$, these operators are l (and n) dependent. This is contrary to the most usually discussed raising and lowering operators, those for the one-dimensional harmonic oscillator and L_z in three dimensions. There the eigenvalues are evenly spaced, and $a^+(a)$ for the harmonic oscillator and $L_\pm$ do *not* depend upon n or m, respectively.

Using the relationships

$$(n+1)(n+\alpha)L_{n+1}^{(\alpha-2)}(y) = [\alpha(\alpha-1) - y(\alpha+n)] \times L_n^{(\alpha)}(y) + (\alpha-1)y\left(\frac{d}{dy}L_n^{(\alpha)}(y)\right), \tag{4.5}$$

$$yL_{n-1}^{(\alpha+2)}(y) = -nL_n^{(\alpha)}(y) - (\alpha+1)\left(\frac{d}{dy}L_n^{(\alpha)}(y)\right), \tag{4.6}$$

which can be obtained from the standard recurrence relations for the generalized Laguerre polynomials,[28] the reader can directly verify that with

$$A_l^- \equiv \left[\frac{l+N-2}{\rho} + \frac{d}{d\rho} - \frac{1}{2}\left(\frac{2n+N-3}{2l+N-3}\right)\right], \tag{4.7}$$

$$A_l^+ \equiv \left[\frac{l}{\rho} - \frac{d}{d\rho} - \frac{1}{2}\left(\frac{2n+N-3}{2l+N-1}\right)\right], \tag{4.8}$$

one has

$$A_l^- R_{n,l} = \frac{[(n-l)(n+l+N-3)]^{1/2}}{(2l+N-3)} R_{n,l-1}, \tag{4.9}$$

$$A_l^+ R_{n,l} = \frac{[(n-l-1)(n+l+N-2)]^{1/2}}{(2l+N-1)} R_{n,l+1}. \tag{4.10}$$

Further, and ultimately due to the unequal level spacing, the $A_l^\pm$ are not Hermitian conjugates of each other. $(A_l^\pm)^\dagger$ can be calculated easily by using the technique discussed by Dirac[35] to obtain

$$\left(\frac{d}{dr}\right)^\dagger = \left(\sum_{i=1}^N \frac{x_i}{r}\frac{d}{dx_i}\right)^\dagger = -\sum_{i=1}^N \frac{d}{dx_i}\frac{x_i}{r} = -\frac{(N-1)}{r} - \frac{d}{dr}, \tag{4.11}$$

so that

$$(A_l^-)^\dagger = \left[\frac{l-1}{\rho} - \frac{d}{d\rho} - \frac{1}{2}\left(\frac{2n+N-3}{2l+N-3}\right)\right], \tag{4.12}$$

$$(A_l^+)^\dagger = \left[\frac{l+N-1}{\rho} + \frac{d}{d\rho} - \frac{1}{2}\left(\frac{2n+N-3}{2l+N-1}\right)\right]. \tag{4.13}$$

Equation (4.11) means the radial momentum operator is

$$p_r = \frac{\hbar}{i}\left(\frac{d}{dr} + \frac{N-1}{2r}\right), \tag{4.14}$$

consistent with other discussions of radial operators.[36,37]

For $N = 3$, the above raising and lowering operators have already been used to discuss the "natural quantum operators" which yield the radial coherent states of the three-dimensional hydrogen atom.[34,38] In the future a more detailed discussion will appear on the coherent states of the hydrogen atom. For now we simply observe that $[A_l^- + (A_l^+)^\dagger] \pm [A_l^+ + (A_l^-)^\dagger]$ are proportional to the "natural quantum operators"

$$X = \left(\frac{1}{r} - \frac{me^2}{\hbar^2[l+(N-3)/2]\,[l+(N-1)/2]}\right), \tag{4.15}$$

$$P = p_r. \tag{4.16}$$

We call these the "natural quantum operators" because if we associate $\hbar^2(l+N/2-3/2)(l+N/2-1/2)$ and p_r with the classical angular momentum squared and the classical radial momentum, respectively, then the corresponding classical variables[38] are those objects which vary as $\sin[\theta(t)]$ and $\cos[\theta(t)]$, $\theta(t)$ the angular velocity in the exact solution of the Kepler problem.[39]

V. RELATIVISTIC PI-MESIC ATOM

The treatment of the relativistic pi-mesic atom using the Klein-Gordon (KG) equation is usually geared to finding the energy levels,[40–42] and not the normalized wave functions. Historically it was the fact that the KG energy levels did not agree with the Sommerfeld formula for the hydrogen atom that necessitated the search which culminated in the Dirac equation.

Ironically, although the Klein-Gordon equation was the first relativistic quantum-mechanical wave equation, it was only recently[43,44] that the predictions for the energy levels of the pi atom have been experimentally verified. This was because to prevent the electromagnetic splittings from being overwhelmed by strong interaction effects due to the nucleus, one needs to observe transitions from large n and l states which are relatively far from the nucleus. It is experimentally difficult to obtain mesic atoms in large $n-l$ states.

However, what we are concerned with here is that, with well known[40] changes of variables, the relativistic pi-mesic atom problem can be mapped into the hydrogen atom problem in three- or arbitrary N-space dimensions. Thus, functionally the wave equation solutions are identical. This last point is *not* well known.

The relativistic pi-mesic atom in N-space dimensions, treated with the Klein-Gordon equation, is described as $(Ze^2 \to e^2)$

$$(E + e^2/r)^2\psi = (-c^2\hbar^2\nabla^2 + m^2c^4)\psi. \quad (5.1)$$

By defining

$$\rho = \frac{r}{a_0}, \quad a_0^2 = \frac{\hbar^2c^2}{4(m^2c^4 - E^2)}, \quad (5.2)$$

$$\lambda = \frac{2Ee^2}{c^2\hbar^2 a_0}, \quad \gamma^2 = \frac{e^4}{\hbar^2c^2}, \quad (5.3)$$

Eq. (5.1) becomes

$$0 = \left(\frac{d^2}{d\rho^2} + \frac{N-1}{\rho}\frac{d}{d\rho} - \frac{[l(l+N-2) - \gamma^2]}{\rho^2} - \frac{1}{4} + \frac{\lambda}{\rho}\right)R. \quad (5.4)$$

Comparing this to Eq. (2.7) one sees that it is the same except for $k = (n - N/2 + 3/2)$ being replaced by λ and the numerator of the ρ^{-2} term having $-\gamma^2$ added to it. The same considerations which led to the trial function defined by (2.10) and (2.14) now lead to the trial function

$$R = \mathcal{N}(\lambda,s)e^{-\rho/2}\rho^s h(\rho), \quad (5.5)$$

where s will be similar to l, but modified from it by the $-\gamma^2$ term. Putting (5.5) into (5.4) one will obtain the equation

$$0 = \rho h'' + [(2s + N - 1) - \rho]h' + [\lambda - (1/2)(N-1) - s]h, \quad (5.6)$$

with the normalizable solution

$$h = L^{(2s+N-2)}_{\lambda-1/2(N-1)-s}(\rho), \quad (5.7)$$

if

$$s = -\frac{1}{2}(N-2) + \left[\left(l + \frac{N-2}{2}\right)^2 - \gamma^2\right]^{1/2} \quad (5.8)$$

and

$$\lambda - (1/2)(N-1) - s = n' = n - l - 1. = 0, 1, 2, \ldots. \quad (5.9)$$

But the above is a solution with the same functional form as (2.19). Therefore the normalization constants are functionally the same, yielding the final solutions

$$R(\rho) = N \exp\left(-\frac{1}{2}\rho\right)\rho^s L^{(2s+N-2)}_{n-l-1}(\rho), \quad (5.10)$$

$$\mathcal{N} = \frac{1}{(a_0)^{N/2}}\left(\frac{\Gamma(n-l)}{(2\lambda)\Gamma[s + \lambda + (N-1)/2]}\right)^{1/2}. \quad (5.11)$$

where in (5.10) and (5.11) we have inserted the condition (5.9).

The energy levels come from Eqs. (5.3), (5.8), and (5.9), giving

$$E = mc^2\Big/\left[1 + \left(\frac{\gamma^2}{([n + (N-3)/2] - [l + (1/2)N - 1] + \{[l + (1/2)N - 1]^2 - \gamma^2\}^{1/2})^2}\right)^{1/2}\right] \quad (5.12)$$

The nonrelativistic limit is

$$E \cong mc^2 - \frac{\mathcal{E}_0}{[n + (N-3)/2]^2} - \frac{\mathcal{E}_0}{[n + (N-3)/2]^2} \times \frac{\gamma^2}{[n + (N-3)/2]}\left(\frac{1}{l + (1/2)N - 1} - \frac{3}{4[n + (N-3)/2]}\right). \quad (5.13)$$

Finally, we note that because the Dirac equation inherently involves a mixture of two solutions, it is not possible to write out the eigensolutions of the Dirac equation in terms of single generalized Laguerre polynomials. Rather, they are mixtures of these polynomials.[45]

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