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Cancellation of internal forces

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Most readers will be familiar with the parable of the horse and the cart, which focuses attention on an important feature of Newton's third law. The horse pulls forward and the cart pulls equally hard backward, and yet the horse and the cart can accelerate. The elucidation of the "paradox" emphasizes that the two action-reaction forces are exerted on different bodies and that therefore there is no question of their cancelling one another.

Thoughtful students who have grasped the meaning of the horse-and-cart parable cannot fail to be puzzled by the treatment given in some textbooks of the momentum of a system of particles.

Defining the momentum $\mathbf{P}$ of the system as the sum of the momenta $\mathbf{p}_j$ of the constituent particles and differentiating with respect to time, one obtains

$$\frac{d\mathbf{P}}{dt} = \sum_j \frac{d\mathbf{p}_j}{dt} = \sum_j \mathbf{f}_j, \tag{1}$$

where $\mathbf{f}_j$ is the resultant of all forces, internal and external, acting on the jth particle. The statement is then made that, by Newton's third law, the internal forces cancel in pairs, so that Eq. (1) becomes

$$\frac{d\mathbf{P}}{dt} = \sum_j \mathbf{f}_{j,\mathrm{ex}} = \mathbf{F}_{\mathrm{ex}}, \tag{2}$$

where $\mathbf{f}_{j,\mathrm{ex}}$ is the resultant of the external forces acting on the jth particle and $\mathbf{F}_{\mathrm{ex}}$ is the resultant of all the external forces acting on the system.

If it is wrong to suppose that the force exerted by the horse on the cart cancels the force exerted by the cart on the horse, then it is equally wrong to suppose that the force exerted by the ith particle on the jth cancels the force exerted by the jth particle on the ith.

There is no need to resort to an erroneous application of Newton's third law in order to proceed from Eq. (1) to (2). It is easy to show that if two particles exert equal and opposite forces on each other, then the resulting motions of the particles are such that the motion of their center of mass is unchanged. The one obtains Eq. (2), not by incorrectly invoking a cancellation of the internal forces, but simply by referring to the fact that the internal forces are irrelevant to the motion of the center of mass of the system.

Hydrogen atom in two dimensions

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The purpose of this note is to solve the Schrödinger equation for a hydrogen atom in two dimensions. We believe that this will help an average undergraduate physics major to fill the gap between solving one-dimensional Schrödinger equations and that of the hydrogen atom in three dimensions.

The time-independent Schrödinger equation for the two-dimensional hydrogen atom is

$$(-\hbar^2/2\mu)\,\nabla^2\Psi + V\Psi = E\Psi, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta}$$

in two-dimensional polar coordinates, μ is the reduced mass of the electron-proton system, $V = -ke^2/r$, E is the energy eigenvalue, and Ψ is the wave function to be found.

Using the method of separation of variables, we let $\Psi(r,\theta) = R(r)\Theta(\theta)$. Equation (1) becomes

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{\hbar} + \frac{2\mu r^2}{\hbar^2}(E\text{-}V) = \frac{-1}{\Theta}\frac{d^2\Theta}{d\theta}.$$

Since the left-hand side is a function only of r and the right-hand side is a function only of θ, both must be equal to a constant, say m^2. We have

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = m^2.$$

The solution of this equation is $\Theta = ce^{im\theta}$, where $m = 0, \pm1, \pm2, \ldots$ because Θ and $d\Theta/d\theta$ must be continuous.
We also have

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{2\mu}{\hbar^2}[V(r)\text{-}E] + \frac{m^2}{r^2}\right)R = 0. \tag{2}$$

Let $\alpha^2 = -2\mu E/\hbar^2$. Then since $R(r) \to e^{-\alpha r}$ as $r \to \infty$, we are led to try $R(r) = e^{-\alpha r}F(r)$. Substituting into Eq. (2),

$$\frac{d^2F(r)}{dr^2} + \left(-2\alpha + \frac{1}{r}\right)\frac{dF}{dr} + \left(-\alpha - \beta - \frac{m^2}{r}\right)\frac{F(r)}{r} = 0;$$

where $\beta = -2\mu ke^2/\hbar^2$. Letting $\rho = 2\alpha r$ and $S(\rho) = F(r)$, the above equation becomes

$$\frac{dS(\rho)}{d\rho} + \left(-1 + \frac{1}{\rho}\right)\frac{dS}{d\rho} + \left(-\frac{1}{2} - \frac{\beta}{2} - \frac{m^2}{\rho}\right)\frac{1}{\rho}S(\rho) = 0. \tag{3}$$

Let $S(\rho) = \rho^s L(\rho)$; $s =$ const; $L(0) \neq 0$ and finite. Substituting into Eq. (3):

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$$\rho \frac{d^2L}{d\rho^2} + (2s\rho - \rho^2 + \rho)\frac{dL}{d\rho} + \left(s(s-1) - s\rho + s - \frac{\rho}{2} - \frac{\beta}{2\alpha} - m^2\right) L = 0.$$

Looking at the behavior for $\rho \to 0$, we have $s(s-1) + 1 - m^2 = 0$, i.e., $s = \pm|m|$. But in order for $S(\rho)$ to converge as $\rho \to 0$, we choose $s = +|m|$ and hence

$$\rho \frac{d^2L}{d\rho^2} + (2|m| + 1 - \rho)\frac{dL}{d\rho} + \left(-|m| - \frac{1}{2} - \frac{\beta}{2\alpha}\right) L(\rho) = 0. \quad (4)$$

The above equation is of the same form as the associated Laguerre equation which is

$$\rho \frac{d^2L_K^j(\rho)}{d\rho^2} + (j + 1 - \rho)\frac{dL_K^j(\rho)}{d\rho} + (K - j)L_K^j(\rho) = 0. \quad (5)$$

This equation can be solved by power series and this series must terminate for reasons similar to those encountered in the harmonic oscillators. The solutions of Eq. (5) are the associated Laguerre polynomials of degree $K - j$ and order j:

$$L_K^j(\rho) = \frac{d^j}{d\rho^j} L_K(\rho),$$

where K and j are zero or positive integers, and where $L_K(\rho)$, the Laguerre polynomials, of degree K, are given by

$$L_K(\rho) = e^\rho \frac{d^K}{d\rho^K} \rho^K e^{-\rho}.$$

Comparison of Eq. (4) with (5) indicates that the solutions $L(\rho)$ for Eq. (3) are given by the associated Laguerre polynomial $L_K^j(\rho)$ if $j = 2|m|$, and $K = |m| - 1/2 - \beta/2\alpha$. Since K must be zero or a positive integer, then it follows that $(-1/2 - \beta/2\alpha)$ must also be zero or a positive integer which we will now define as n, the principal quantum number: $n = -1/2 - \beta/2\alpha$, hence $\alpha^2 = \beta^2/(2n+1)$. Substituting the value of α^2 and β into $\alpha^2 = -2\mu E/h$, one finds that $E_n = -E_0/(2n+1)^2$, where $n = 0, 1, 2, 3, 4 \ldots$ and $E_0 = -2\mu k^2 e^4/\hbar^2 = -54.4ev$.

It is interesting to consider the degeneracy of E_n. Since n is equal to $K - |m|$ and K and $|m|$ are zero or positive integers such that $2|m| = j \leq K$, each E_n is $(n+1)$ fold degenerate: $n = 0$; $K = 0, m = 0$; $n = 1$; $K = 1, m = 0, k = 2, m = 1$; $n = 2$; $K = 2, m = 0, K = 3, m = 1, K = 4, m = 2$; $n = 3$; $K = 3, m = 0, K = 4, m = 1, K = 5, m = 2, K = 6, m = 3$. The quantum number K is always $\geq n$. While in the case of three-dimensional (3D) hydrogen atoms the degeneracy of a state of given n is

$$\sum_{l=0}^{l=n-1} (2+1) = n^2,$$

where l, the counterpart of K, is $\leq n - 1$. The degeneracy of the two-dimensional (2D) hydrogen atom is the same as the 2D simple harmonic oscillator,[1] but the degeneracies for 3D hydrogen atom and 3D simple harmonic oscillator are different.[2]

[1] See, for example, Robert E. White, *Basic Quantum Mechanics* (McGraw-Hill, New York, 1966), p. 91.

[2] See, for example, Robert H. Dicke and James P. Wittke, *Introduction to Quantum Mechanics* (Addison-Wesley, Reading, MA, 1960), p. 167.

Reply to Professor Walstad's "The equivalence principle"

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In my paper I presented a few *Gedankenexperimente* (some admittedly rather idealized, but others quite realistic) which detect the tidal forces within an arbitrarily small region and thereby demonstrate violations of the conventional formulation of the Principle of Equivalence. It is Walstad's contention that these violations should be ignored because they are an abuse of the Principle of Equivalence and because they are a pedagogical inconvenience. In this I perceive some parallel to the attitude expressed by the famous Cremonini of the University of Padua who refused to look through the telescope and chose to ignore Galileo's discovery of "imperfections" of the Sun and the Moon. Of course, for most purposes an almost perfectly round and smooth Sun is a very useful approximation—but for the purposes of some solar astronomers the deviations are of much greater interest. Likewise, the almost perfect elimination of gravity by free fall is a very useful approximation—but this must not blind us to the importance of residual effects.

Furthermore, it is Walstad's contention that my counterexamples rest on an unrealistic and contrived experimental procedure. In Walstad's view, an experimenter seeking to test the Principle of Equivalence should first design and build the apparatus and then hand it over to Walstad who, naturally, will perform measurements only in those places where the tidal force is so weak that its detection is bound to fail. No sane experimenter would agree to such a farcical procedure. The sound and realistic procedure is quite the reverse: Walstad must first tell the experimenter at what place the tidal force is to be measured, and the experimenter will then try to design an apparatus that does the job. My claim, substantiated by detailed examples, is that given any prescribed tidal field, one can always design a sufficiently sensitive apparatus to detect this field, even within an arbitrarily small region, provided only that classical physics remains applicable.