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Hence, the Einstein heat capacity is

$$C_V{}^E/Nk = X_E{}^2 \exp(-X_E)/[1-\exp(-X_E)]^2, \quad (6)$$

where $X_E = \hbar\omega_E/kT$. We determined the Einstein temperature X_E by requiring the first two terms in the high-temperature expansion of Eq. (6) to be identical to the exact result. This yields $X_E = X_M/2^{1/2}$. It can be shown[4] that with this choice for X_E, $C_V{}^E$ is a rigorous lower bound to the exact heat capacity at all temperatures.

In the Debye approximation the frequency distribution is set equal to the low-frequency limit of the exact result:

$$g_D(\omega) = 2N/\pi\omega_M. \quad (7)$$

The Debye cutoff frequency, ω_D, is determined from the requirement that the total number of modes is equal to N; i.e.,

$$\int_0^{\omega_D} g_D(\omega)\, d\omega = N. \quad (8)$$

This yields $\omega_D = \omega_M\pi/2$. Substituting Eq. (7) into Eq. (1) and setting $X = \hbar\omega/kT$, we find

$$\frac{C_V{}^D}{Nk} = \frac{2}{\pi X_M}\int_0^{X_M\pi/2} \frac{X^2\exp(-X)}{[1-\exp(-X)]^2}\,dX. \quad (9)$$

In Fig. 1 we present the exact, Einstein, and Debye frequency distribution [from Eqs. (2), (5), and (7), respectively] and heat capacities [from Eqs. (3), (6), and (9), respectively]. The exact and Debye heat capacities were obtained using Gaussian quadrature numerical integration. In the exact case the substitution $X = X_M \sin t$ was made to remove the singularity and facilitate the convergence of the numerical integration. The Debye heat capacity becomes exact at low temperatures ($X_M{}^{-1} \to 0$), while the Einstein heat capacity becomes exact at high temperatures ($X_M{}^{-1} \to \infty$), as we expect from the choices for the Debye frequency distribution and the Einstein temperature. It is of interest to note that, in contrast to the three-dimensional case, the Debye model yields a rather poor approximation of the exact heat capacity in one dimension. This is a simple consequence of the fact that in one dimension $g(\omega)$ itself is singular, whereas in three dimensions the singularity appears in the second derivative of the normal mode frequency distribution function.

We would like to thank Attila Szabo, who suggested this problem, for his advice and help.

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[2]T. L. Hill, *Statistical Thermodynamics* (Addison-Wesley, Reading, MA, 1960), p. 106.

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Trigonometric solution of the polar function of the Schroedinger equation for hydrogen

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In the usual elementary solution of the time-independent Schroedinger equation as applied to the hydrogen atom, it is customary to express the equation in polar coordinates using the variables, ϕ, θ, and r, to separate the functions $F(\phi)$, $T(\theta)$, and $R(r)$ relating to each of these variables, and then to obtain solutions for these respective functions. In this process, the separated polar equation for $T(\theta)$ generally appears as

$$\frac{1}{T\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right] - \frac{M^2}{\sin^2\theta} + L = 0, \quad (1)$$

where M^2 is a separation constant applied to $F(\phi)$ and L is the separation constant for $T(\theta)$ and $R(r)$.

At this point, it is conventional to transform the above differential equation from one involving trigonometric functions into one involving an algebraic function by assuming that $\cos\theta = x$ and solving for x.[1]

However, it is the purpose of this brief note to suggest that it may actually be simpler to solve the equation without making a trigonometric–alebraic transformation; the same results are obtained, and the solution is itself at least as instructive.

For such a solution, Eq. (1) is simplified to give

$$\frac{\partial^2 T}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial T}{\partial\theta} + \left[L - \frac{M^2}{\sin^2\theta}\right]T = 0. \quad (2)$$

A solution of this equation may be assumed to be given by $T(\theta) = \sin^M\theta K(\theta)$, where M is the absolute value of M, or $M = |(M^2)^{1/2}|$, and $K(\theta)$ is an unidentified function of θ.

Now, solving for the various derivatives of $T(\theta) = \sin^M\theta K(\theta)$, one obtains

$$\frac{\partial T}{\partial\theta} = \sin^M\theta\frac{\partial K}{\partial\theta} + M\sin^{M-1}\theta\cos\theta\, K, \quad (3)$$

$$\frac{\partial^2 T}{\partial\theta^2} = \sin^M\theta\frac{\partial^2 K}{\partial\theta^2} + 2M\sin^{M-1}\theta\cos\theta\frac{\partial K}{\partial\theta} + [M(M-1)\sin^{M-2}\theta\cos^2\theta - M\sin^M\theta]K = 0. \quad (4)$$

Substituting these derivatives back into the original equation for $T(\theta)$ results in the following differential equation for $K(\theta)$:

$$\frac{\partial^2 K}{\partial\theta^2} + (2M+1)\frac{\cos\theta}{\sin\theta}\frac{\partial K}{\partial\theta} + (L - M - M^2)K = 0. \quad (5)$$

Now, let it be assumed that $K(\theta)$ is a power series in $\cos\theta$ given by

$$K(\theta) = \sum_{k=0}^{k=n} a_k \cos^k\theta;$$

therefore,

$$K(\theta) = a_0 + a_1 \cos\theta + a_2 \cos^2\theta + \cdots + a_k \cos^k\theta + a_{k+1} \cos^{k+1}\theta + a_{k+2} \cos^{k+2}\theta + \cdots + a_n \cos^n\theta. \quad (6)$$

The respective derivatives of this power series for the $\cos^k\theta$ to $\cos^{k+2}\theta$ terms are found to be the following:

$$\partial K/\partial\theta = -\sin\theta[\cdots + ka_k \cos^{k-1}\theta + (k+1)a_{k+1} \cos^k\theta + (k+2)a_{k+2} \cos^{k+1}\theta + \cdots], \quad (7)$$

$$\partial^2 K/\partial\theta^2 = -\cos\theta[\cdots + ka_k \cos^{k-1}\theta + (k+1)a_{k+1} \cos^k\theta + (k+2)a_{k+2} \cos^{k+1}\theta + \cdots] + \sin^2\theta[\cdots + k(k-1)a_k \cos^{k-2}\theta + (k+1)ka_{k+1} \cos^{k-1}\theta + (k+2)(k+1)a_{k+2} \cos^k\theta + \cdots]. \quad (8)$$

Now, if the multiplier, $\sin^2\theta$, appearing as part of the second derivative of $K(\theta)$ given above is replaced by $1 - \cos^2\theta$, the resulting expression for the second derivative will involve only powers of $\cos\theta$ and be given by

$$\partial^2 K/\partial\theta^2 = [\cdots + k(k-1)a_k \cos^{k-2}\theta + (k+1)ka_{k+1} \cos^{k-1}\theta + (k+2)(k+1)a_{k+2} \cos^k\theta + \cdots] - \cos\theta[\cdots + ka_k \cos^{k-1}\theta + (k+1)a_{k+1} \cos^k\theta + (k+2)a_{k+2} \cos^{k+1}\theta + \cdots] - \cos^2\theta[\cdots + k(k-1)a_k \cos^{k-2}\theta + (k+1)ka_{k+1} \cos^{k-1}\theta + (k+2)(k+1)a_{k+2} \cos^k\theta + \cdots]. \quad (9)$$

From substituting the above values of $K(\theta)$, $\partial K/\partial\theta$, and $\partial^2 K/\partial\theta^2$ into the differential equation (5) involving $K(\theta)$, the coefficient of the $\cos^k\theta$ term is found to be

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - ka_k - (2M+1)ka_k + (L - M - M^2)a_k. \quad (10)$$

Now, as usual, a solution for the differential equation in $K(\theta)$ is obtained by finding the conditions necessary for the respective coefficients of each power of $\cos\theta$ to become zero. Accordingly, if the coefficient of $\cos^k\theta$, an arbitrarily chosen power of $\cos\theta$, is set equal to 0, then a recursion relationship involving various of the coefficients of the power series originally assumed [Eq. (6)] is found to be

$$(k+2)(k+1)a_{k+2} = [k(k-1) + k + (2M+1)k - (L - M - M^2)]a_k. \quad (11)$$

A solution for the power series itself will be obtained if it may be made to terminate at some power of $\cos\theta$, such as $\cos^{k+2}\theta$. Obviously, this will be the case if $a_{k+2} = 0$, which will occur if

$$k(k-1) + k + (2M+1)k - L + M + M^2 = 0.$$

It may be simply shown that this relationship is satisfied if $L = (k+M)(k+M+1)$. Setting $k + M = l$, this becomes $L = l(l+1)$, which is the accepted solution of the polar equation normally obtained after making the conventional trigonometric–algebraic transformation. Since any further comments on the mathematics or physics involved in this solution would be the same as those used with the analogous conventional algebraic approach, they are not repeated here.

[1]This trigonometric–algebraic transformation is used in all of the elementary quantum mechanics and introductory modern physics texts designed for the sophomore–junior level, where a solution of $T(\theta)$ is detailed. This is the case for about half of the some 20 current modern physics texts reviewed; the others merely state the results.

An interesting property of the Kepler ellipse

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Teachers of elementary physics who do not believe in the total elimination of mathematics from their courses appear to be looking continually for simplified analytical deductions for presentation to their students. It can hardly be denied that this is a laudable quest—provided, of course, that no tricks are played, consciously or otherwise.[1]

Over the last few years, interest in problems concerned with orbital motion has increased markedly, resulting in the publication of numerous analyses that purportedly make use of only elementary algebra or geometry.[2] The present note is a small contribution that, to my knowledge, has not yet appeared in the literature but which seems to generate considerable student interest. Moreover, it is directly related to recent work concerning the effect of the changing eccentricity of the Earth's orbit on the recurrence of the ice ages.[3]

Using only a simple property of the ellipse and some elementary algebra, it will be shown that *the orbital speed of a satellite when it is at its maximum distance from the major axis is equal to the geometric mean of the maximum and minimum speeds in the orbit.*

It is well known that an ellipse may be defined as the locus of a point, the sum of whose distances from two fixed points is constant. This corresponds directly to the familiar