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The Quantum Bouncer

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Examples in one and two dimensions for motion in a uniform gravitational field are considered quantum mechanically. The examples of bouncing in one dimension and sliding down an incline are proposed for use as conceptual aids in an introductory course.

I. INTRODUCTION

In the first course of Modern Physics, we seek to give the student as clear as possible an understanding of the role of $\Psi^*\Psi$. In the early stages the free particle example may not be a good one for this purpose as $\Psi^*\Psi$ is not normalized in the usual fashion. The common remedy is to use the particle-in-the-box example to illustrate the probability distributions. The harmonic oscillator and hydrogen atom are, of course, excellent conceptual models but require the usual mathematical detours which sometimes become cumbersome at this level. Thus, one may concentrate on the particle-in-the-box to make sure all the salient points of the quantum theory are pointed out before proceeding to the next stages.

Students are apt to ask about quantum theory applied to the standard elementary mechanics problems even though the instructor has taken pains to map out the areas in which quantum theory applies. Instead of giving standard answers, two related problems can be presented which are quite easily visualized. First, consider a point mass which would fall due to gravity and bounce off a flat surface with no loss of kinetic energy (coefficient of restitution $R=1$). (This was called the quantum bouncer since we would use the Schrödinger equation to describe the dynamics.) The inclined plane can then be discussed by utilizing an incline with a reflecting surface perpendicular to and at the bottom of the incline. The new two-dimensional problem is still mathematically tractable and presents an interesting situation for a discussion of the correspondence principle.

I would propose that these problems follow the particle-in-the-box problems in the beginning course because they are good conceptual models for the student and the amusement derived provides a good spiritual lift. For these reasons the problems are outlined below.

II. ONE-DIMENSIONAL QUANTUM BOUNCER

The potential energy of a point mass, m, a distance y above the ground is given as mgy. When placed in the time-independent Schrödinger equation

$$[-(\hbar^2/2m)(d^2/dy^2)+V(y)-E]\Psi(y)=0, \quad (1)$$

where

$$V(y)=\begin{cases} mgy & y>0 \\ \infty & y=0, \end{cases}$$

one can proceed in the unusual manner to get Ψ and E. It is a simple matter to scale and transform Eq. (1) to the Airy equation

$$(d^2/dx^2-x)\Psi(x)=0. \quad (2)$$

This is accomplished by defining $y=E/mg+lx$ where a characteristic length, l, may be defined as

$$l=(\hbar^2/2m^2g)^{1/3}.$$

The boundary conditions on y translate as follows:

$$\Psi(y=0)=0=\Psi(x=-E/mgl),$$

$$\Psi(y=\infty)=0=\Psi(x=\infty).$$

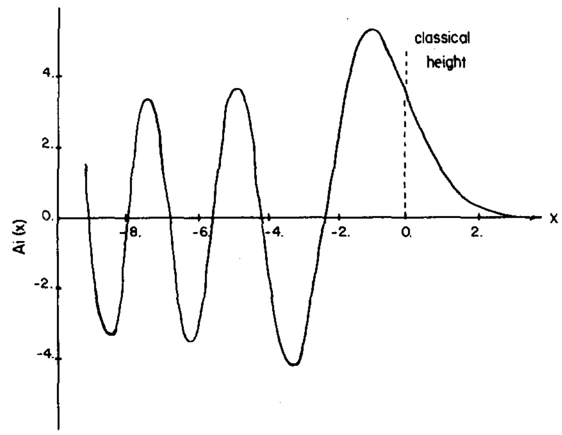

FIG. 1. Airy function vs x. The distance from $x=0$ to nodal points $n=1, 2, 3, \ldots$ represents the eigenvalue. Each eigenfunction is represented by the curve starting at the appropriate nodal point.

The solution to (2) is[1]

$$\Psi(x) = aAi(x) + bBi(x).$$

The second boundary condition requires $b=0$ since $Bi(x)$ shows unbounded growth with increasing x. The value of a is found from the normalization of the Airy function $Ai(x)$. The

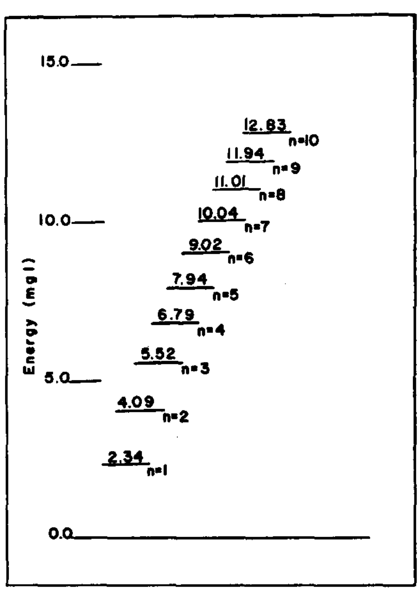

FIG. 2. Energy levels for $n=1$–10 in units of mgl.

eigenvalue spectrum is obtained from the first boundary condition. It is very convenient for presentation purposes that only one table or plot of $Ai(x)$ is required compared to the situation where different polynomials are required for different values of E.

Figure 1 shows a plot of $\Psi(x)$ which can be used for the form of all the wavefunctions by picking the appropriate nodal point for $\Psi=0$. The distance from $x=0$ to the nth nodal point x_n, is directly proportional to the energy of the nth state since

$$x_n = -E/mgl.$$

Thus, the energy level spacings are easily visualized by observing the node spacings of Ψ. Reference (1) shows a plot of $Ai(x)$, gives tables for $Ai(x)$ and the first few zeros of $Ai(x)$. We have used Ref. (1) to produce the energy-level diagram in Fig. 2. It could also be obtained by using a good metric ruler from a plot of Ψ.

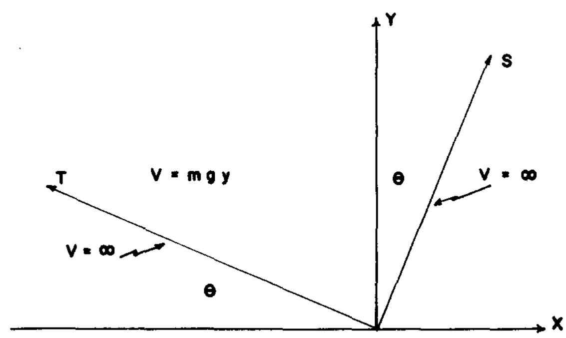

FIG. 3. Coordinates and potential energy for incline.

A similar scaling can be utilized in the particle-in-the-box problem to obtain a universal Ψ. However, the nodal points are not directly related to the energy level spacings as in this problem.

III. QUANTUM BOUNCER AND THE INCLINED PLANE

Figure 3 shows the arrangement to be considered in this section. The incline and stop at the end of the incline are considered as perfectly

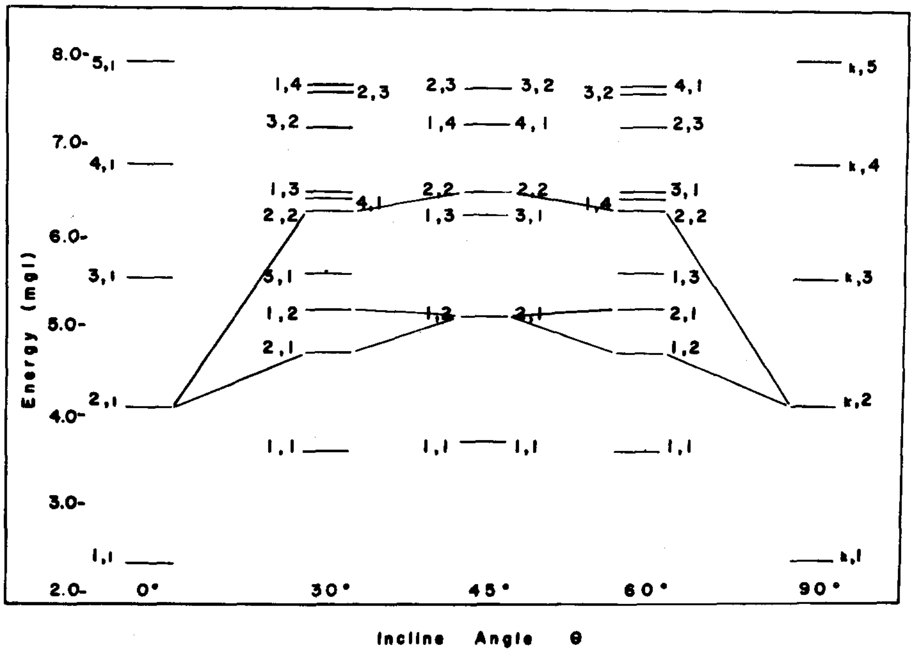

Fig. 4. Energy levels for the incline vs incline angle. The pairs of numbers (k, l) indentify the eigen energy state.

reflecting surfaces. We solve the quantum problem with the indicated potential by rotating axes so that $s=y\cos\theta+x\sin\theta$ and $t=-x\cos\theta+y\sin\theta$. Now, this results in ∇^2 becoming $\partial^2/\partial s^2+\partial^2/\partial t^2$ and V becomes ∞ if $t=0$ or $s=0$, otherwise V becomes $mg(s\cos\theta+t\sin\theta)$. By defining $E=E_s+E_t$, performing a separation of variables, defining

$$l_s=(\hbar^2/2m^2g\cos\theta)^{1/3},$$

$$l_t=(\hbar^2/2m^2g\sin\theta)^{1/3},$$

and scaling $(u=E_s/mg\cos\theta+l_s s,\ v=E_t/mg\sin\theta+l_t t)$, one finds

$$\Psi=Ai(u)Ai(v).$$

The energies are found as in Sec. II by locating the zeroes of the Airy functions by setting $Ai(u_k)=0$ and $Ai(v_l)=0$. Then, $E_t=(mgl_t\sin\theta)v_l$ and $E_s=(mgl_s\cos\theta)u_k$ for the (kl)th eigen energy. Figure 4 shows some of lower energy levels for various incline angles. The resulting symmetry at $\theta=0°$, $45°$, $90°$ is evident in Fig. 4 from the level degeneracies which are shown explicitly for the (1, 2) and (2, 1) levels.

IV. DISCUSSION

The results of Sec. II may be used to get $\Psi^*\Psi$ and discuss the meaning of the probability distribution. This is facilitated by the result that $E_n=mglx_n$; since, lx_n is the classical height a particle with mass m and energy E_n would bounce (l being the scale factor for distance units). The classical height is represented in Fig. 1 by the node-origin separation and always occurs at $x=0$ due to the choice of scale.

The results from Sec. III show the same features in two dimensions for $\Psi^*\Psi$. Now, however, the concept of sliding is not clear due to the nonzero energy of motion in the normal direction. However, we resolve the problem by noting that the size of the scale factor $\sim(\hbar^2/2m^2g)^{1/3}$ is so small that detecting the difference between sliding and the actual motion down the incline is not possible. Not only is this expected by the Bohr correspondence principle but it also shows the basic reason for the absence of quantum-gravitational effects in our surroundings due to the smallness of g.

The examples are good for showing the connection between wavefunction nodes and energy. The two dimensional example is also very good for illustrating how each degree of freedom permits a new quantum number. Extending further, one can see how to solve quite readily the problem of noninteracting systems by using the analogy between a degree of freedom and one of the systems under study.

[1] *Handbook of Mathematical Functions*, NBS-AMS 55 (1970), p. 446. May be obtained from Superintendent of Documents, U.S. Government Printing Office, Washington, DC 20402 for $9.00.

TRUTH FROM CONFUSION

If a new idea were to be admitted only when it had definitely proved its justification or even if we merely demanded that it must have a clear and definite meaning at the outset, then such a demand might gravely hamper the progress of science. We must never forget that ideas devoid of a clear meaning frequently gave the strongest impulse to the further development of science. The idea. . . of perpetual motion gave rise to an intelligent comprehension of energy; the idea of the absolute velocity of the earth gave rise to the theory of relativity; and the idea that the electronic movement resembled that of the planets was the origin of atomic physics. These are indisputable facts, and they give rise to thought, for they show clearly that in science or elsewhere fortune favors the brave.

—Max Planck