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Energy levels of a charged particle in the field of a spherically symmetric uniform charge distribution

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The non-relativistic energy spectrum for a negatively charged particle in the field of a spherically symmetric charge distribution is obtained. A complete solution is found which includes a representation of the wavefunction exterior to the charge distribution. The energy eigenvalues are found from the matching condition at the boundary of the nucleus and the energy levels for a mu meson in the field of a lead nucleus are calculated.

I. INTRODUCTION

The distribution of positive charge in the nucleus has negligible effect on the electron energy level structure of hydrogen-like atoms. This is not true when a heavier particle, in particular a mu meson, finds itself in the presence of a heavy nucleus. The mu meson is about 210 times heavier than the electron and thus will find itself closer to the positive charge distribution on the average; in fact, in the 1*s* state, the meson has roughly a fifty percent chance of being found within the charge distribution. Since the energy levels are readily measured, the mu meson can be used as a probe for examining the detailed charge distritution in the nucleus. In recent years, this experimental tool has yielded great quantities of information about the structure of various nuclei.

The mu mesic atom has been subject to extensive theoretical investigation,[1-3] and complete relativistic treatments of the problem are available for various charge distributions. The aim of this paper is not to repeat these solutions, but rather to examine the mu mesic atom in the simple case of a uniform spherical charge distribution, by means of the Schrödinger equation. Because the nucleus is no longer represented by a point charge, the form of the wave function on the interior of the charge distribution is different from that outside. The problem is pedagogically interesting because the energy eigenvalues are found from the condition that the wave function and its derivative are continuous at the boundary of the nucleus. This is a somewhat different development than is usually encountered in the textbook treatment of the hydrogenic atom,[4] where the wave function is chosen to be zero at the origin and forced to be zero at infinity.

Although a form for the wave function exterior to the charge distribution can be found which is zero at infinity, it is rather difficult to express in the region of the boundary of the nucleus. When the principal quantum number, n, and the angular momentum quantum number, l, are related by $n = l + 1$, the exterior wave function can be represented by an asymptotic expansion of the Whittaker function.[5,6] However, for $n = l + 1$, the expansion fails and another representation must be found.

The following sections present a complete solution to the problem including the required representation of the exterior wave function in the region of interest. The results are used to calculate the mu meson ground state energies over values of atomic number between $Z = 20$ and $Z = 92$, and the character of these energies for small and large values of nuclear radius is discussed. In addition, the mu mesic energy level structure for lead is calculated.

II. ANALYSIS

A. Solution of the Wave Equation

The potential energy of a negatively charged particle in the field of a spherically symmetric charge distribution is given by

$$V = -\frac{Ze^2}{R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), \quad r \leqslant R \tag{1a}$$

$$V = -Z_e{}^2/r \qquad r \geqslant R \tag{1b}$$

and is shown in Fig. 1. Since the functional form of the potential differs from the interior to the exterior of the nucleus, the form of the Schrödinger equation

$$-(\hbar^2/2m)\nabla^2\psi + V\psi = E\psi \tag{2}$$

will also differ in these regions. Consequently, the wave function has different forms in these regions.

Because of spherical symmetry, the wave equation separates into its radial and angular components as usual, and

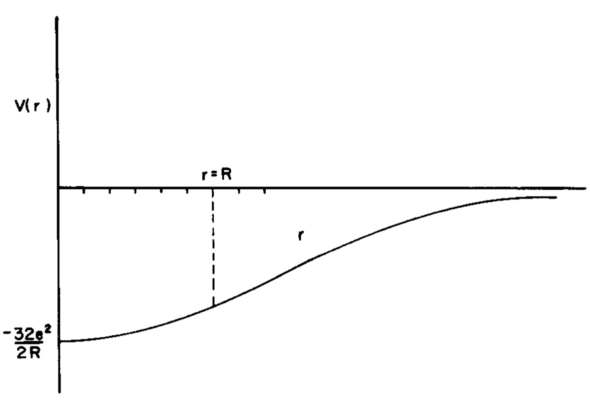

Fig. 1. The potential energy of a negatively charged particle in the field of a uniform spherically symmetric charge distribution of radius R.

we need only consider the differential equations for the radial part.

B. Interior Solution

For $r \leqslant R$ the radial component of the wave function is found from

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}+2\mu\left(E+\frac{3Ze^2}{2R}-\frac{Ze^2r^2}{2R^3}\right)\right]\times R(r)=0 \quad (3)$$

where the potential (la) has been substituted in Eq. (2). Letting

$$\mu = m/\hbar^2,$$
$$\epsilon = -E$$
$$\rho = (8\epsilon\mu)^{1/2}r,$$
$$\Lambda = (3Ze^2/8\epsilon R) - 1/4,$$
$$\beta^4 = Ze^2/64\mu R^3\epsilon^2,$$

Eq. (3) becomes

$$\left\{\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right)+\left[\Lambda-\beta^4\rho^2-\frac{l(l+1)}{\rho^2}\right]\right\}R(\rho)=0. \quad (4)$$

This equation has the same form as that of the three dimensional harmonic oscillator. Thus it is sensible to assume solutions of that type:

$$R(\rho)=\exp(-\beta^2\rho^2/2)\,t^{l/2}V(t) \quad (5)$$

where $t=\rho^2\beta^2$. Putting Eq. (5) into Eq. (4) we get for V

$$tV''(t)+V'(t)(l+\tfrac{3}{2}-t)+\gamma V(t)=0$$

where

$$\gamma=\tfrac{1}{2}\left\{\left[(\Lambda/2\beta^2)-\tfrac{1}{2}\;-(l+1)\right]\right\}.$$

The form of Eq. (5) is the same as that of the confluent hypergeometric equation. The solution to this equation which satisfies the requirement that the wave function be finite at $r=0$ is written

$$V(t)={}_1F_1(-\gamma,\, l+\tfrac{3}{2},\, t) \quad (6)$$

where ${}_1F_1$ can be expressed in series form as

$${}_1F_1(-\gamma,l+\tfrac{3}{2},t)=1+\frac{(-\gamma)t}{(l+\frac{3}{2})}+\frac{(-\gamma)(-\gamma+1)}{(l+\frac{3}{2})(l+\frac{5}{2})}\frac{t^2}{2}+\cdots+\frac{(-\gamma)(-\gamma+1)\cdots(-\gamma+u-1)t^u}{(l+\frac{3}{2})(l+\frac{5}{2})\cdots(l+\frac{3}{2}+u-1)u!}+\cdots. \quad (7)$$

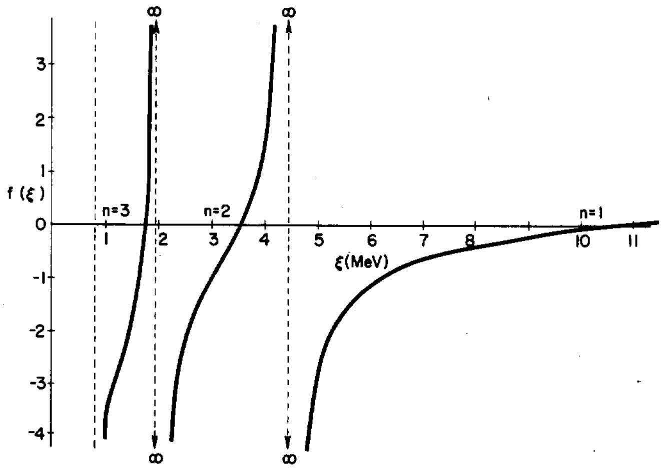

Fig. 2. A plot of Eq. (14) for $l=0$. As n increases, ϵ becomes progressively smaller. Between $\epsilon = 1.3$ ($n=3$) and $\epsilon = 0$ there are an infinity of roots. In the usual notation these energies would correspond to the 1*s*, 2*s*, 1*s*, 2*s*, . . . , *ns* levels.

It can be shown that this series converges for all values of t.

The solution to the radial part of the Schöedinger equation on the interior of the charge distribution is then

$$R(\rho)=A\exp(-\beta^2\rho^2/2)(\rho\beta)^l\,{}_1F_1(-\gamma,\,l+\tfrac{3}{2},\,\rho^2\beta^2) \quad (8)$$

where A is the normalization constant.

C. Exterior Solution

Substituting Eq. (1b) into Eq. (2), the radial component of the wave equation is

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}+\frac{2m}{h^2}\left(E+\frac{Ze^2}{r}\right)\right]R(r)=0. \quad (9)$$

As before, we set $\rho=(8\mu\epsilon)^{1/2}r$ and $k=Ze^2(\mu/2\epsilon)^{1/2}$ with $E=-\epsilon$. The wave equation becomes

$$\left[\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right)-\frac{l(l+1)}{\rho}+\frac{k}{\rho}-\tfrac{1}{4}\right]R(\rho)=0. \quad (10)$$

Assuming a solution of the form

$$R(\rho)=\rho^{-1}W(\rho), \quad (11)$$

Eq. (10) becomes

$$W''(\rho)+\left[-\tfrac{1}{4}+\frac{k}{\rho}-\frac{l(l+1)+\frac{1}{4}}{\rho^2}\right]W(\rho)=0. \quad (12)$$

Equation (12) is Whittaker's differential equation. The linearly independent solutions of this equation are written $W_{k,m}(\rho)$ and $W_{k,-m}(\rho)$, where $m=l+1$. Only the first of these has the required property of vanishing at infinity.

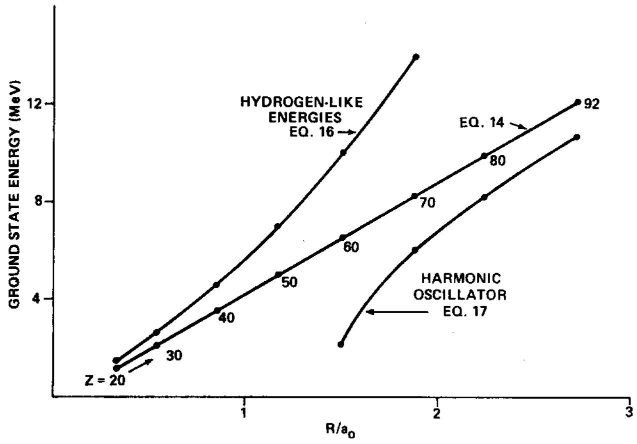

Fig. 3. The ground state energy levels of the mu meson as a function of Z calculated from Eq. (14). For small values of Z, the ground state energy can be estimated by the Bohr levels, Eq. (16). For large values of Z it approaches the value estimated from Eq. (17).

Hence the radial part of the wave function exterior to the charge distribution is

$$R(\rho) = BW_{k,\ell+1/2}(\rho). \tag{13}$$

The $W_{k,m}$ are known as Whittaker functions. In order to determine the energy eigenvalues, a suitable representation of this function at the boundary of the nucleus must be found.

D. The Energy Eigenvalues

The energy eigenvalues for this system are obtained by matching the logarithmic derivatives of the solutions [Eqs. (8) and (13)] at the boundary of the charge distribution. The matching condition is

$$f(\epsilon) = \frac{l+1}{\rho} - \rho\beta^2 - \frac{2\rho\gamma\beta^2}{l+\frac{3}{2}} \frac{{}_1F_1(1-\gamma, l+\frac{5}{2}, \rho^2\beta^2)}{{}_1F_1(-\gamma, l+\frac{3}{2}, \rho^2\beta^2)} - \left.\frac{W_{k}{}',_{\ell+1/2}}{W_{k,\ell+1/2}}\right|_{r=R} = 0. \tag{14}$$

The roots of this transcendental equation will yield the energy eigenvalues for the system.

When the radius of the charge distribution shrinks to zero, the wave function and its derivative on the interior go to zero. The complete wave function is then given by Eq. (13), and must be zero at the origin. The solution which meets this requirement is written

$$R(\rho) = \rho^{\ell+1}\exp(-\rho/2)\,{}_1F_1(l+1+k, 2l+2, \rho) \tag{15}$$

which is the usual form of the hydrogenic wave function in the field of a point charge. In order for it to be zero at infinity, κ must be an integer n, so that the series, ${}_1F_1$, terminates. The energy eigenvalues are obtained from:

$$k = n = (Ze^2/\hbar)(m/2\epsilon)^{1/2} \text{ or } \epsilon = Z^2e^4m/2\hbar^2n^2, \tag{16}$$

which are the same as those obtained by the Bohr theory. The quantity, n, is the principal quantum number.

As the radius of the charge distribution grows, a progressively larger portion of the wave function is contained within the nuclear volume. In the limit of large radii, the wave function is almost entirely within the nucleus and can be represented by Eq. (8). The energy eigenvalues are obtained by forcing this solution to be zero at infinity. This requires that the parameter, γ, be equal to an integer, j. This gives the energy eigenvalues

$$\epsilon = \tfrac{3}{2}(Ze^2/R) - (Ze^2/\mu R^3)^{1/2}(n+\tfrac{3}{2}) \tag{17}$$

where $n = 2j + l$. These eigenvalues are those of the usual three dimensional harmonic oscillator whose energy levels are displaced by a constant.

For intermediate radii of the extended nucleus, Eq. (14) is used to obtain the energy spectrum and κ is no longer an integer. The energy eigenvalues can still be labeled by the principal quantum number and associated with the orbital angular momentum quantum number, l, as in the simpler system. The Whittaker function in Eq. (14) is evaluated by means of the asymptotic expansion[6]

$$W_{k,\ell+1/2}(z) = \exp[-z/2]z^k \times \left[1 + \sum_{n=1}^{\infty} \frac{(l+\frac{1}{2})^2 - (k-\frac{1}{2})^2 \cdots (l+\frac{1}{2})^2 - (k-n-\frac{1}{2})^2}{n!\,z^n}\right]. \tag{18}$$

This form is only valid when

$$\mathrm{Re}(k - l - 1) \leq 0 \tag{19}$$

and

$$|\arg z| \leq \pi\alpha \leq \pi. \tag{20}$$

When $l = 0$, the energy eigenvalues are found by choosing a sufficiently large value for ϵ, and reducing it until condition (14) is met. Only one eigenvalue is found, for with smaller ϵ, k increases until condition (19) is violated. This root is assigned the principal quantum number $n = 1$. For other values of l, roots are found in the same manner corresponding to the principal quantum number given by $n = l + 1$. When $n \neq l + 1$, the above expansion fails and another representation of the Whittaker function must be used.

The required form, due to Hartree,[7] is written

$$W_{k,\ell+1/2} = \Gamma(k+l+1)(-k)^{-(\ell+1)}[G_\ell(z)\cos\pi k + H_\ell(z)\sin\pi k] \tag{21}$$

Table I. Energy levels for a mu meson in the field of a lead nucleus (MeV).

n	$l = 0$	$l = 1$	$l = 2$
1	10.52		
2	3.52	4.65	
3	1.76	2.08	2.13

where

$$G_\ell(z)=k^{\ell+1}\exp(-z/2)\,z^{\ell+1}\,{}_1F_1[(1+l-k),\,(2+2l),\,z]$$

and

$$H_\ell(Z)=\frac{1}{\pi(2l+1)!}\left(\exp(-z/2)z^{\ell+1}z^{-\ell}\left\{\sum_{m=0}^{2\ell}\frac{\Gamma(k-l)(2l-m)!\,(2l+1)!}{\Gamma(k+l+1-m)m!}\right.\right.$$

$$\left.+\sum_{m=2\ell+1}^{\infty}\frac{(-1)^{m+1}\Gamma(k-l)z^m}{\Gamma(k+l+1-m)m!\,(m-2+1)!}\,[\psi(m+1)+\psi(m-2l-\psi(k-l+1-m)]\right\}$$

$$\left.+(2l+1)!\left\}k^{\ell+1}\exp(-z/2)z^{1+\ell}{}_1F_1[(1+l-k),\,(2+2l),\,z\log z]\right\{\right)$$

where the ψs are the derivatives of the gamma function. With this form, the complete energy level structure can be extracted.

The roots for a given value of l can be obtained graphically. A typical plot of $f(\epsilon)$ is shown in Fig. 2 for $l=0$. The zeros of $f(\epsilon)$ are labeled by increasing values of the principal quantum number for decreasing values of the energy corresponding to roots of Eq. (14). Only the first three principal quantum numbers are shown in the plot. The remaining energy eigenvalues will be on the energy axis between the origin and the dashed line which denotes the asymptotic value of $f(\epsilon)$ on the lower side of the region bounding the third eigenvalue. Similar graphs can be plotted for other values of l to complete the determination of the energy spectrum.

The ground state energy levels for values of atomic number, Z, between 20 and 92 are computed from Eq. (14) using the techniques above. The radius of the charge distribution in all cases is given by

$$R=1.2A^{1/3}\times 10^{-13}\,\text{cm} \tag{22}$$

and the mass of the muon is taken to be 207 electron masses. These energy values are plotted in Fig. 3 as a function of Z and R/a_O, where a_O is the muonic Bohr radius of an atom with charge Z. The extent to which Eqs. (16) and (17) approximate those ground state energies is also shown. The curve representing the energies extracted from Eq. (16) approach those obtained from Eq. (14) for lower values of Z and R/a_O. This implies that most of the ground state wave function lies outside of the charge distribution but within the Bohr radius, as in a simple hydrogenic system. The ground state energies given by Eq. (17) approach those of Eq. (14) for large values of Z and R/a_O, indicating that most of the wave function lies within nuclear volume. This feature of the energy level structure for heavy nuclei clearly suggests that the mu meson can be used as a nuclear probe.

Table II. Transition energies for a mu meson in the field of a lead nuclues.

Transition	Energies (MeV) Experimental	Theoretical
$2p$–$1s$	5.87	5.96
$3d$–$2p$	2.57	2.52

III. THE NONRELATIVISTIC MU MESIC ENERGY LEVEL STRUCTURE FOR LEAD

The mu meson energy level structure for lead is extracted from Eq. (14). The mass of the mu meson is again 207 electron masses, $Z=82$, and the radius of the charge distribution is given by Eq. (22) where $A=207$ for lead. The roots of Eq. (14) are extracted by a computer aided search technique. The level structure is presented in Table I and transition energies are compared with those experimentally observed in Table II: Agreement is good.[8,9]

A plot of Eq. (14) for $l=0,1,2$ is shown in Fig. 4. For each value of l, the roots are labeled by increasing values of n subject to the restriction $n>l+1$. As a consequence, the $2s$ level is greater than the $2p$ level. (Recall that $\epsilon=-E$.) In the simple one electron hydrogenic system, the reverse is true. Captured electrons are likely to fall into the $2s$ state, which is metastable since parity

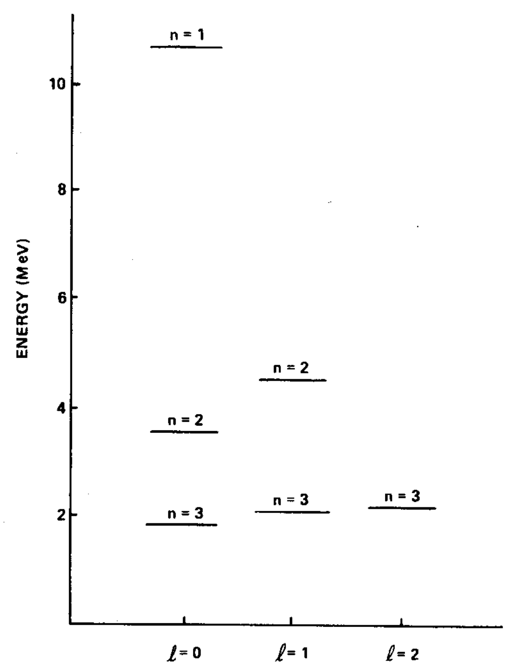

Fig. 4. The energy levels of a mu meson in the field of a lead nucleus. The plot shows the zeros of $f(\epsilon)$ for $l=0,1,2$. Note that the $2p$ state lies above the $1s$ state (since $\epsilon=-E$).

considerations make the 2s–1s transition "forbidden." In the mesonic system, the meson has a very high probability of cascading from the 2s to the 2p level and then into the 1s state. Gamma rays corresponding to this transition are easily detected and it is from these that a good deal of information about the charge distribution is obtained.

When the nucleus is represented by a point charge, the energies are degenerate with respect to the orbital angular momentum quantum number l. The removal of this degeneracy is due to the coupling of the spin-orbit angular momenta in more detailed treatments of this system. It has been shown that degeneracy is also removed for a particle in the field of an extended nucleus. This can be understood in terms of the perturbation calculation which determines these energies for small values of Z. It is found that the matrix element specifying the perturbation energies connects states of the point charge system of l and $l + 2$. Although these calculations are valid for small values of Z only, the notion that the degeneracy is removed due to coupling of the orbital angular momentum via the charge distribution provides an explanation of level splitting with intuitive appeal.

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THE PAULI EFFECT

The title "Pauli Effect" was given to the phenomenon that whenever Pauli entered a laboratory, the apparatus broke or stopped working. The physicist Ehrenfest said that he could reduce the Pauli effect to a more general law, namely that one mishap never comes alone.

Once Pauli visited a laboratory in Milano, Italy. People there wanted to make a little joke with him and arranged a Pauli effect—they prepared it so the moment the door opened, an electric contact was made which would cause a little explosion and noise and many things would fall apart. They arranged it all with great care. Pauli opened the door. Nothing happened.

—Victor Weisskopf