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Quantum Theory of a Square Well Plus Delta Function Potential

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The s-wave Schrödinger equation is solved for a square well potential with an attractive delta function at the well edge. This potential provides a number of soluble illustrations of the techniques of quantum mechanics. The bound state eigenvalue problem is solved as is the scattering problem for the cross section. The potential is capable of producing a sharp low energy resonance. The resonances are discussed using both R-matrix and S-matrix methods. Numerical examples are presented.

I. INTRODUCTION

Exact solutions of the Schrödinger equation are known for only a few potential forms. For the most part, the soluble problems are used to illustrate either bound state or scattering calculations. For some potentials, e.g., the square well, both types of solutions may be obtained with ease.

One may approach the scattering problem from a more sophisticated viewpoint by using R-matrix theory or S-matrix methods. Either of these provides a convenient framework with which to discuss resonances. Of particular interest are the narrow resonances which are characteristic of low energy neutron scattering from heavy nuclei. In addition, bound states of a system can be related to certain poles of the S matrix continued into the complex energy plane.

We present here a simple potential which illustrates all of the features above. The bound state and scattering wavefunctions can be determined exactly. The potential is capable of producing sharp resonances in the cross section. The R-matrix theory can be applied explicitly. The analytical continuation of the S matrix can be accomplished allowing the identification of bound states and resonances of the system.

It is thought that this potential form has its importance in the rich variety of soluble examples it offers for a number of techniques in quantum theory. Each type of calculation is presented in a separate section beginning with a specification of the general properties of the potential in Sec. II. The bound state problem is presented in Sec. III, and Sec. IV contains the scattering calculation. The R-matrix description is presented in Sec. V and the S-matrix theory in Sec. VI. Finally, some sample calculations are given in Sec. VII.

All of the calculations are limited to the consideration of s states or s waves. The extension to higher angular momentum values is possible and direct. No references are given in the sections to follow as the methods and concepts employed are well known to students of quantum mechanics. A concise yet readable account of the analytical properties of the S matrix and of the Jost function is given in the book by Taylor.[1] A discussion of the R matrix may be found in the book by Wu and Ohmura.[2]

II. PROPERTIES OF THE POTENTIAL

The three-parameter potential is taken to be

$$V(r) = -[q_0^2\epsilon(r-a) + B\delta(r-a)]\hbar^2/2m, \quad (1)$$

where

$$\epsilon(r-a) = 1, \qquad r<a$$
$$= 0, \qquad r>a$$

and $\delta(r-a)$ is the delta function. Thus the potential consists of a square well with an attractive delta function at the well edge. We shall consider only the case for which both q_0^2 and B are positive. Alternative cases could also be discussed.

The s wave radial equation to be solved is

$$(d^2\psi/dr^2) + [k^2 + q_0^2\epsilon(r-a) + B\delta(r-a)]\psi = 0, \qquad \psi(0) = 0, \quad (2)$$

with $E=\hbar^2k^2/2m$ being the system energy (which may be negative or even complex for later considerations).

The effect of the delta function is to produce a discontinuity in the derivative of the wavefunction at $r=a$. Letting a^+ and a^- denote the result of taking the limit as r approaches a from the right and left, respectively, we find on integrating Eq. (2) over a small interval containing the point $r=a$,

$$d\psi/dr\,|_{a^+}-d\psi/dr\,|_{a^-}=-B\psi(a). \qquad (3)$$

This equation, together with the continuity of ψ at $r=a$, allows us to obtain the solution of the differential equation.

III. BOUND STATES

We first investigate bound state solutions, letting $E=-\hbar^2k_2^2/2m$. There exists the possibility of two types of bound states—one with energy below the square well depth, $k_2^2>q_0^2$, and the other with energy above the well depth, $0<k_2^2<q_0^2$. Considering the former case, let $q_2^2=k_2^2-q_0^2>0$. The solution of the radial equation which is continuous at $r=a$ is

$$\begin{aligned}\psi&=A\sinh(q_2r), && r<a\\ &=A\sinh(q_2a)\exp[-k_2(r-a)], && r>a \qquad (4)\end{aligned}$$

with A a normalization constant which can be determined if desired. On evaluating the required discontinuity in the derivative of ψ at $r=a$, we obtain

$$k_2A\sinh(q_2a)+q_2A\cosh(q_2a)=BA\sinh(q_2a)$$

or

$$B-q_2\coth(q_2a)=k_2=(q_0^2+q_2^2)^{1/2}. \qquad (5)$$

Considered as a function of q_2, the left-hand side of this transcendental equation is monotonically decreasing from the value $B-1/a$ at $q_2=0$. The right-hand side is monotonically increasing from the value q_0 at $q_2=0$. Thus at most, a single solution for q_2 (or k_2) exists if and only if $(B-q_0)a>1$. We see that the attractive delta function potential must be sufficiently strong in order to bind a single state with energy below the square well depth.

We now look for solutions above the well bottom with $0<k_2^2<q_0^2$, letting $q_1^2=q_0^2-k_2^2$. The continuous solution with the bound state form is

$$\begin{aligned}&A\sin(q_1r), && r<a,\\ &A\sin(q_1a)\exp[-k_2(r-a)], && r>a. \qquad (6)\end{aligned}$$

Evaluation of the discontinuity in the derivative of the wave function at $r=a$ gives

$$q_1\cot(q_1a)=B-(q_0^2-q_1^2)^{1/2}, \qquad 0<q_1<q_0. \qquad (7)$$

As the cotangent function is periodic with period π, it is convenient to discuss possible solutions of Eq. (7) for q_1 in ranges such that

$$(n-1)\pi<q_1a<n\pi, \qquad n=1,2,3,\cdots,$$

always with the understanding that $q_1<q_0$.

The $n=1$ case $(q_1a<\pi)$ requires special attention. We have seen that if $(B-q_0)a>1$, one bound state exists below the well bottom. Then no bound state solution of Eq. (7) exists for $q_1a<\pi$. For $q_1a<\pi$, the left-hand side of Eq. (7) is monotonically decreasing from the value $1/a$ at $q_1=0$. Thus no solution exists if $(B-q_0)a>1$.

If $(B-q_0)a<1$, then no solution exists below the well bottom, but a solution may exist above the well bottom in the range $q_1a<\pi$ if the well is deep enough. For $q_0a>\pi/2$, the monotonic behavior of the two sides of Eq. (7) yields a solution for $q_1a<\pi$ if and only if $(B-q_0)a<1$. This condition is also necessary for a more shallow well $(q_0a<\pi/2)$. There is for this case, however, the additional sufficiency requirement, $q_0\cot(q_0a)<B$.

Solutions of Eq. (7) may exist for $q_1a>\pi$ for either type of value of $(B-q_0)a$ if the well is sufficiently deep. These solutions will be of the form

$$q_1a=n\pi+\Delta_n, \qquad 0<\Delta_n<\pi, \qquad q_0a>n\pi+\Delta_n$$

and are determined by the solutions for Δ_n of

$$(n\pi+\Delta_n)\cot\Delta_n=Ba-[q_0^2a^2-(n\pi+\Delta_n^2]^{1/2}. \qquad (8)$$

In the extreme case of $(B-q_0)a\gg1$, solutions occur near $q_1a=n\pi\ll Ba$.

Summarizing, we find that if $(B-q_0)a>1$, then a bound state exists with energy below the square well depth. If so, then no solution exists above the well depth in the range $q_1a=(q_0^2-k_2^2)^{1/2}a<\pi$. If $(B-q_0)a<1$, then no solution exists with energy below the well depth, but a solution may exist for $q_1a<\pi$. In either case, solutions may exist in the well with $q_1a=n\pi+\Delta_n<q_0a$.

IV. SCATTERING STATES

We now turn to the description of scattering states (s wave only) for this potential setting $E=\hbar^2k_1^2/2m>0$, $q_1^2=q_0^2+k_1^2$. For $r>a$, the wavefunction must be of the form $\psi=C\sin(k_1r+\delta)$ where the phase shift δ is to be determined. The scattering cross section is expressed in terms of the phase shift as

$$\sigma=4\pi\sin^2\delta/k_1^2.$$

The solution of Eq. (2) which is continuous at $r=a$ is

$$\begin{aligned}\psi&=A\sin(q_1r), && r<a\\ &=A\sin(q_1a)\sin(k_1r+\delta)/\sin(k_1a+\delta), && r>a. \quad (9)\end{aligned}$$

The discontinuity in the derivative of ψ at $r=a$ leads to

$$[B\sin(q_1a)-q_1\cos(q_1a)]\sin(k_1a+\delta)=-k_1\sin(q_1a)\cos(k_1a+\delta).$$

This may be rewritten as

$$\tan\delta=-\frac{[B\sin(q_1a)-q_1\cos(q_1a)]\sin(k_1a)+k_1\sin(q_1a)\cos(k_1a)}{[B\sin(q_1a)-q_1\cos(q_1a)]\cos(k_1a)-k_1\sin(q_1a)\sin(k_1a)}$$

or equivalently,

$$\sin^2\delta=\frac{\{[B\sin(q_1a)-q_1\cos(q_1a)]\sin(k_1a)+k_1\sin(q_1a)\cos(k_1a)\}^2}{[B\sin(q_1a)-q_1\cos(q_1a)]^2+k_1^2\sin^2(q_1a)}. \quad (10)$$

The cross section may now be calculated as a function of energy, $E=\hbar^2k_1^2/2m$.

Inspection of Eq. (10) reveals that a low energy ($k_1a\ll1$) resonance may be expected to occur if $\tan(q_1a)\approx q_1/B$. Then $\sigma\approx4\pi/k_1^2$. To investigate the possibility of resonances further, we set $q_1a=da+\epsilon$ with $\tan(da)=d/B$. Then with ϵ considered to be small,

$$\tan(q_1a)\approx d/B+\epsilon(B^2+d^2)/B^2+\cdots,$$
$$\cot(q_1a)\approx B/d-\epsilon(B^2+d^2)/d^2+\cdots.$$

Substitution into Eq. (10) gives

$$\sin^2\delta=\frac{\{k_1\cos(k_1a)+\sin(k_1a)[(B^2+d^2)-B/a]\epsilon/d\}^2}{k_1^2+[(B^2+d^2)-B/a]^2\epsilon^2/d^2}.$$

We introduce a resonance energy $E_d=\hbar^2k_d^2/2m$ with $d^2=q_0^2+k_d^2$. For E close to E_d we have

$$\epsilon=q_1a-da\approx\tfrac{1}{2}a(k_1^2-k_d^2)/d.$$

In this approximation, we have if $|\sin(k_1a)|\ll1$,

$$\sigma\approx\frac{4\pi}{k_1^2}\left(\frac{(\Gamma/2)^2}{(E-E_d)^2+(\Gamma/2)^2}\right), \quad (11)$$

which is a Breit–Wigner resonance form with width

$$\Gamma=(\hbar^2/2m)\{4d^2k_d/[(B^2+d^2)a-B]\},$$

and resonance energy $E_d=\hbar^2k_d^2/2m$ with k_d determined by

$$k_d=(d^2-q_0^2)^{1/2}, \qquad \tan(da)=d/B.$$

V. R-MATRIX FORMULATION

The R-matrix description of scattering provides a convenient basis for discussing resonances. It is defined in terms of the wavefunction and its derivative at $r=a^+$,

$$\psi(a)=Ra(d\psi/dr)|_{a^+}. \quad (12)$$

To express the S matrix, $S=\exp(2i\delta)$, in terms of the R matrix, we note that for $r>a$,

$$\psi=A[S\exp(ikr)-\exp(-ikr)].$$

Employing Eq. (12), we have

$$S=\exp(-2ika)(1+ikaR)/(1-ikaR). \quad (13)$$

The R matrix can be expressed in terms of a complete set of functions defined in the interior region, $0<r<a^+$. Thus we consider the interior wavefunction which is a solution of

$$(d^2\psi/dr^2)+[k^2+q_0{}^2\epsilon(r-a)+B\delta(r-a)]\psi=0, \qquad \psi(0)=0 \quad (14)$$

and the associated eigenvalue problem,

$$(d^2y_n/dr^2)+[k_n{}^2+q_0{}^2\epsilon(r-a)+B\delta(r-a)]y_n=0, \quad (15)$$

with boundary conditions,

$$y_n(0)=0,$$

$$dy_n/dr\,|_{a^+}=0=dy_n/dr\,|_{a^-}-By_n(a).$$

Setting $q_n{}^2=q_0{}^2+k_n{}^2$, we find that the normalized solution of the eigenvalue equation is

$$y_n=\left(\frac{2}{a}\frac{q_n{}^2+B^2}{q_n{}^2+B(B-1/a)}\right)^{1/2}\sin(q_nr), \qquad r<a, \quad (16)$$

with q_n such that $\tan(q_na)=q_n/B$ determines the eigenvalue $k_n{}^2=(q_n{}^2-q_0{}^2)$. There may also be solutions with $k_n{}^2$ negative. These are to be included in the set.

We now expand the interior wavefunction ψ in the orthornormal set $\{y_n\}$.

$$\psi(r)=\sum_n c_ny_n(r),$$

$$c_n=\int_0^a y_n(r)\psi(r)dr. \quad (17)$$

The coefficient c_n can be evaluated by applying Green's theorem to Eqs. (14) and (15) and using the boundary conditions. Thus,

$$y_n(a)\left.\frac{d\psi}{dr}\right|_{a^+}=-(k^2-k_n{}^2)\int_0^a y_n(r)\psi(r)dr.$$

Recalling the definition of the R matrix Eq. (12), we have

$$c_n=-y_n(a)\psi(a)/[aR(k^2-k_n{}^2)].$$

The interior wavefunction then has the expansion,

$$\psi(r)=-[\psi(a)/aR]\sum_n[y_n(a)y_n(r)/(k^2-k_n{}^2)].$$

This equation may be solved for R by evaluating at $r=a$:

$$R=-\sum_n[y_n(a)^2/a(k^2-k_n{}^2)]$$

$$=-\sum_n[\gamma_n{}^2/(E-E_n)], \quad (18)$$

with

$$\gamma_n{}^2=y_n(a)^2\hbar^2/(2ma),$$

$$E_n=\hbar^2k_n{}^2/(2m).$$

Since $y_n(a)$ is known from Eq. (16),

$$\gamma_n{}^2=\left(\frac{\hbar^2}{ma^2}\right)\left(\frac{q_n{}^2}{q_n{}^2+B(B-1/a)}\right). \quad (19)$$

From Eq. (13) for the S matrix and Eq. (18) for the R matrix, we can write the scattering amplitude,

$$e^{i\delta}\sin\delta=(S-1)/(2i)$$

$$=e^{-ika}\{-\sin(ka)+[(e^{-ika}kaR)/(1-ikaR)]\}$$

$$=e^{-ika}\left(-\sin(ka)-\frac{e^{-ika}ka\sum_n[\gamma_n{}^2/(E-E_n)]}{1+ika\sum_n[\gamma_n{}^2/(E-E_n)]}\right). \quad (20)$$

The conventional separation has been performed here so that the first term in Eq. (20) represents hard sphere scattering.

We concentrate on resonance effects by allowing E to be close to one of the eigenvalues, $E=E_n+\Delta$, with Δ small compared with the eigenvalue spacing. Then

$$R=-\sum_m [\gamma_m^2/(E-E_m)]=-[\gamma_n^2/(E-E_n)]-P_n(E),$$

with

$$P_n(E)=\sum_{m\neq n}\gamma_m^2/(E_n-E_m+\Delta)$$

slowly varying for E close to E_n.

After some algebraic manipulation, the amplitude has the form,

$$e^{i\delta}\sin\delta=e^{-ika}(-\sin(ka)-e^{-ika}\{[\tfrac{1}{2}\Gamma_n+(E-E_n)P_nQ_n](1+ikaP_n)/(E-E_r+i\tfrac{1}{2}\Gamma_n)\}) \qquad (21)$$

with

$$\Gamma_n=2ka\gamma_n^2/(1+k^2a^2P_n^2),$$
$$E_r=E_n-kaP_n\Gamma_n/2,$$
$$Q_n=ka/(1+k^2a^2P_n^2). \qquad (22)$$

To this point, no approximations have been made. In order to obtain the usual resonance form, the slowly varying but energy dependent quantities, P_n, Q_n, k, Γ_n, and E_r are evaluated at $E=E_n$. Thus, for example, Γ_n becomes

$$\Gamma_n=2k_na\gamma_n^2/(1+k_n^2a^2P_n(E_n)^2).$$

If a resonance occurs at low energy $(k_na\ll1)$, then a Breit–Wigner resonance is obtained.

$$\sigma\approx(4\pi/k^2)(\tfrac{1}{2}\Gamma_n)^2/[(E-E_r)^2+(\tfrac{1}{2}\Gamma_n)^2].$$

A detailed analysis shows that the resonance is very sharp if $q_0\ll B$ and if $Ba\gg1$.

VI. S-MATRIX THEORY

We now turn to the analytic continuation of the S matrix into the complex E or k planes ($E=\hbar^2k^2/2m$). For k real, $q=(q_0^2+k^2)^{1/2}$, the solution of the radial equation is

$$\psi=A\sin(qr), \qquad r<a$$
$$=A\sin(qa)(Se^{ikr}-e^{-ikr})(Se^{ika}-e^{-ika})^{-1}, \qquad r>a \qquad (23)$$

and

$$d\psi/dr\,|_{a^+}-d\psi/dr\,|_{a^-}=-B\psi(a).$$

Thus,

$$S=f(-k)/f(k)=e^{-2ika}\{[B\sin(qa)-q\cos(qa)-ik\sin(qa)]/[B\sin(qa)-q\cos(qa)+ik\sin(qa)]\}, \qquad (24)$$

where

$$f(k)=-e^{ika}[B\sin(qa)-q\cos(qa)+ik\sin(qa)]/q$$

is the Jost function.[3]

To continue $f(k)$ [and $S(k)$] into the complex k plane, we choose q to be the branch of $(q_0^2+k^2)^{1/2}$ which is real positive for k real. Explicitly, with $k=k_1+ik_2$, $q=q_1+iq_2$,

$$q_1=\{\tfrac{1}{2}(q_0^2+k_1^2-k_2^2)+\tfrac{1}{2}[|q_0^2+k_1^2-k_2^2|^2+4k_1^2k_2^2]^{1/2}\}^{1/2},$$
$$q_2=k_1k_2/q_1. \qquad (25)$$

Then,

$$f(k)=-\{[B+i(k-q)]\exp[i(k+q)a]-[B+i(k+q)]\exp[i(k-q)a]\}(2iq)^{-1}, \qquad (26)$$

which is analytic for all k.

The S matrix $S(k)$ is then analytic except for poles corresponding to the zeros of $f(k)$. We first consider poles of the S matrix corresponding to bound states of which there may be two types. For $k=k_1+ik_2$, $q=q_1+iq_2$, the Jost function may have a zero for $k_1=0$ and either $q_1=0$ or $q_2=0$. In the former case, we have

$$f(ik_2)=0=-\{[B-k_2-q_2]\exp[-(k_2-q_2)a]-[B-k_2+q_2]\exp[-(k_2+q_2)a]\}(2q_2)^{-1}$$

or, after rearranging,

$$B-q_2\coth(q_2a)=k_2=(q_0^2+q_2^2)^{1/2},$$

which is Eq. (5) for a bound state below the square well depth and can only be satisfied if $(B-q_0)a>1$.

In the latter case, $q_2=0$, the Jost function has zeros at the solutions of

$$q_1\cot(q_1a)=B-k_2=B-(q_0^2-q_1^2)^{1/2}$$

which is Eq. (7) for bound states above the well depth.

To deal with poles of the S matrix which may correspond to resonances, we let $k=k_1-ik_2$, $q=q_1-iq_2$, with k_1, k_2, q_1, q_2 all nonnegative. This portion of the complex k plane maps onto the lower half of the second sheet of the Riemann surface for $S(E)$ which is cut along the positive real axis. On setting $f(k)=0$, we obtain the complex transcendental equation,

$$\exp(q_2a)[B-q_2+k_2-i(q_1-k_1)]\exp(iq_1a)=\exp(-q_2a)[B+q_2+k_2+i(q_1+k_1)]\exp(-iq_1a).$$

We rewrite this in polar form,

$$\exp(q_2a)[(B-q_2+k_2)^2+(q_1-k_1)^2]^{1/2}\exp[i(q_1a-\theta)]$$
$$=\exp(-q_2a)[(B+q_2+k_2)^2+(q_1+k_1)^2]^{1/2}\exp[-i(q_1a-\phi)],$$

with

$$\tan\theta=(q_1-k_1)/(B-q_2+k_2), \qquad \tan\phi=(q_1+k_1)/(B+q_2+k_2).$$

A set of real coupled transcendental equations is obtained on equating both modulus and phase of each side of this equation. They are

$$4q_2a=\ln\{[(B+q_2+k_2)^2+(q_1+k_1)^2]/[(B-q_2+k_2)^2+(q_1-k_1)^2]\},$$
$$2q_1a=2n\pi+\tan^{-1}[(q_1-k_1)/(B-q_2+k_2)]+\tan^{-1}[(q_1+k_1)/(B+q_2+k_2)]. \qquad (27)$$

These equations may be solved for k_1, k_2 with q_1 and q_2 determined by Eqs. (25). Alternatively, they may be solved for q_1, q_2 with

$$k_1=\{\tfrac{1}{2}(q_1^2-q_0^2-q_2^2)+\tfrac{1}{2}[\,|\,q_1^2-q_0^2-q_2^2\,|^2+4q_1^2q_2^2]^{1/2}\}^{1/2},$$
$$k_2=q_1q_2/k_1.$$

If we denote a zero of $f(k)$ by $\bar{k}_1-i\bar{k}_2$, then $S(E)$ has a pole at $E=E_r-i\Gamma/2$ with

$$E_r=\hbar^2(\bar{k}_1^2-\bar{k}_2^2)/(2m), \qquad \Gamma=4\hbar^2\bar{k}_1\bar{k}_2/(2m).$$

An iterative scheme for determining the solutions $\bar{q}_1$, $\bar{q}_2$ of Eqs. (27) which has been found successful is as follows: An initial estimate of $q_2a=0.001$, $q_1=q_0$, n equal to the largest integer not exceeding q_0a/π is substituted into the right-hand side of Eqs. (27) to calculate a second estimate for q_1a, q_2a. This process is repeated until convergence occurs—very rapidly in practice.

To obtain a next pole, the first estimate for q_1a is taken to be the value for the preceding pole plus π, for q_2a, the value for the preceding pole, and n is increased by unity. In a number of sample calculations, this procedure has always been found to converge rapidly.

VII. NUMERICAL EXAMPLES

We present here the results of calculations for two sets of potential parameters corresponding to the conditions $(B-q_0)a<1$ and $(B-q_0)a>1$, respectively. We use a system of units for which $\hbar^2/(2m)=1$, $a=1$. The square well depth q_0^2 will be renamed $V_0=q_0^2$. The parameters are chosen in each case so that a low energy resonance is present.

For the first case, $q_0=10.145$, $B=11.045$, $a=1$ so that $(B-q_0)a=0.9<1$, and the well depth $V_0=102.921$. The potential supports three bound s states which from Eq. (7) have energy eigenvalues

$$E_0=-102.7, \qquad E_1=-84.39, \qquad E_2=-48.90.$$

The scattering cross section (s-wave) in arbitrary units is shown in Fig. 1 as a function of energy. Note that a low energy resonance appears in the vicinity of $E=0.3$. This resonance is rather broad as might be expected since B and q_0 differ by a small amount.

The S matrix has poles at $E=E_1-iE_2$. The first eight are given in Table I.

For the second calculation, $q_0=9.51$, $V_0=90.44$, $B=100$, $a=1$, and $(B-q_0)a\gg1$. While the well depth is little changed from the previous case, the much stronger delta function strength produces substantial changes in both the ground state eigenvalue and the resonance widths.

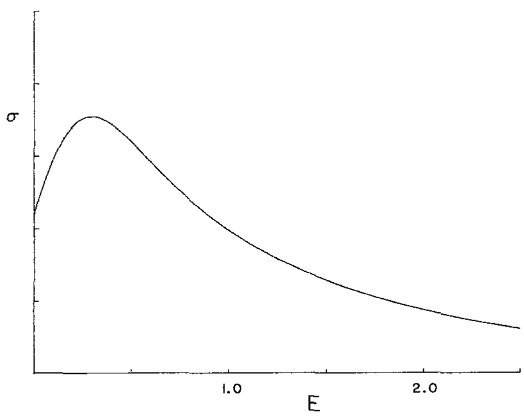

FIG. 1. The s-wave cross section in arbitrary units is shown as a function of energy for potential parameters $V_0=102.921$, $B=11.045$, $a=1$. The resonance peak occurs near $E=0.3$.

TABLE I. Location of resonance poles $E=E_1-iE_2$.

E_1	E_2
0.038	0.459
73.83	9.544
167.4	16.44
280.7	23.63
413.8	31.21
566.6	39.15
739.2	47.44
931.5	56.05

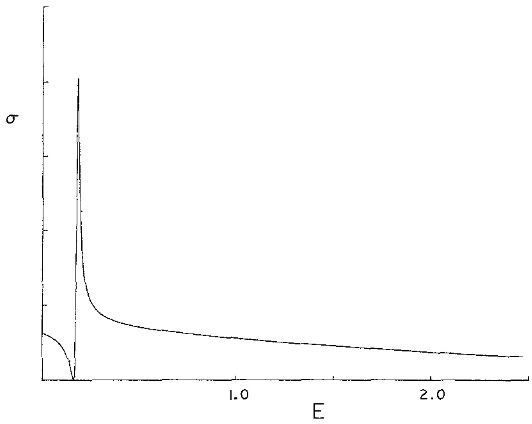

FIG. 2. The s-wave cross section in arbitrary units is shown as a function of energy for potential parameters $V_0 = 90.44$, $B = 100$, $a = 1$. The resonance peak occurs near $E = 0.18$.

The bound state eigenvalues are calculated from Eqs. (5) and (7) and have the values

$$E_0 = -2545.0, \qquad E_1 = -80.35, \qquad E_2 = -50.10.$$

The s-wave scattering cross section in arbitrary units is shown in Fig. 2 as a function of energy. The resonance is seen to be very sharp with a peak at $E = 0.18$. The dip in the cross section on the low energy side of the resonance is characteristic of the interference between the hard sphere scattering amplitude and the Breit–Wigner resonance amplitude.

The first eight resonance poles of the S matrix are given in Table II.

In this second example, the cross section has its peak at an energy very close to the real part of the resonance pole energy while the imaginary part is small in comparison. This feature contrasts with the first example. There the pole location is such that the imaginary part is in fact considerably larger than the real part. As a result of the pole being relatively far from the real E axis, interference between the pole term and the background shifts the cross section peak away from the value of the real part of the resonance pole energy.

TABLE II. Location of resonance poles, $E = E_1 - iE_2$.

E_1	E_2
0.1841	0.0077
70.64	0.2671
161.2	0.6195
271.8	1.134
402.5	1.835
553.1	2.736
723.8	3.849
914.5	5.181

SUMMARY

The Schrödinger equation has been solved for the bound s-state and scattering s-waves for a square well potential with a delta function potential at the well edge. The nature of low energy resonances has been investigated in both R-matrix and S-matrix formulations. Numerical examples have been presented to illustrate the features of this potential.

[1] John R. Taylor, *Scattering Theory* (Wiley, New York, 1972).

[2] T. Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Englewood Cliffs, N. J., 1962).

[3] The Jost function is defined here so that $f(k) = 1$ if the potential is identically zero.