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And John is older by

$$A_j - A_m = (1-0.66)20 = 6.8 \text{ yr}.$$

Several of the important features of this discussion are illustrated in Fig. 1, which is a plot of the relationship between A_j and A_m as determined in John's frame and Mary's frame for the above example. Notice the cross-over point for the two curves; this is where Mary has come to a halt (so that she is in the same inertial frame as John) and has not yet reversed her initial velocity. For a moment John and she are in the same inertial frame, and during that moment their measurements agree (but not their ages).

The twin paradox, and this method of resolving it, illustrates in a dramatic way one of the revolutionary aspects of Einstein's theory: there is no universal time system that all observers can agree to and the time at which an event occurs depends on the Lorentz frame in which it is observed.

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Hydrogenic Wave Functions for an Extended, Uniformly Charged Nucleus

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An exact, closed solution for the hydrogenic wave functions of an extended, uniformly-charged nucleus is presented.

1. INTRODUCTION

In calculations on muon and pion capture by complex nuclei, a hydrogenic wave function is needed for calculation of the relevant matrix elements. As is well known, a wave function corresponding to a point charge nucleus will not suffice in these cases. Typically the charge distribution of the nucleus is taken to be uniform or of a Saxon–Woods type. The hydrogenic wave functions are then obtained by a numerical integration of the Schrödinger equation.[1] Surprisingly, it does not seem to be generally recognized that in the case of an uniform charge distribution, an exact, closed solution exists for the wave function. In this note we present such a solution and explicitly display the ground state muonic wave function for the nucleus ^{40}Ca.

2. RADIAL WAVE FUNCTION

For a uniformly charged nucleus of radius R, the potential is given by

$$V=(-Ze^2/R)[\tfrac{3}{2}-\tfrac{1}{2}(r^2/R^2)], \qquad 0\leq r\leq R,$$
$$=-Ze^2/r, \qquad r\geq R.$$

The radial equation for the outside part is

$$-(2m)^{-1}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{Ze^2}{r}R+\frac{l(l+1)}{2mr^2}R=ER.$$

Defining in the usual way,

$$\rho=\alpha r,$$
$$\alpha^2=8m\,|E|,$$
$$\lambda=2mZe^2/\alpha,$$

and letting $R(\rho)=\chi(\rho)/\rho$, one obtains the Whittaker equation

$$\frac{d^2\chi}{d\rho^2}+\left(-\tfrac{1}{4}+\frac{\lambda}{\rho}+\frac{\tfrac{1}{4}-(l+\tfrac{1}{2})^2}{\rho^2}\right)\chi=0 \qquad (1)$$

with solutions $M_{\lambda,l+\frac{1}{2}}(\rho)$, $W_{\lambda,l+\frac{1}{2}}(\rho)$. $M_{\lambda,l+\frac{1}{2}}(\rho)$ must be rejected since it does not have the correct asymptotic behavior at infinity. The outside solution is therefore,

$$R_l^{\text{out}}(\rho)=\rho^{-1}W_{\lambda,l+\frac{1}{2}}(\rho)$$
$$=e^{-\rho/2}\rho^l U(l+1-\lambda,\,2l+2,\,\rho), \qquad (2)$$

where $U(l+1-\lambda,\,2l+2,\,\rho)$ is a Kummer function.[2] Using a standard integral representation for U, a closed-form solution for $R_l^{\text{out}}(\rho)$ is

$$R_0^{\text{out}}(\rho)=[\Gamma(2-\lambda)]^{-1}e^{-\rho/2}$$
$$\times\int_0^\infty e^{-\rho t}t^{1-\lambda}(1+t)^{\lambda-1}[\rho(1+t)-\lambda]dt, \qquad (3a)$$
$$R_l^{\text{out}}(\rho)=[\Gamma(l+1-\lambda)]^{-1}e^{-\rho/2}\rho^l$$
$$\times\int_0^\infty e^{-\rho t}t^{l-\lambda}(1+t)^{l+\lambda}dt, \qquad l\geq 1. \qquad (3b)$$

For the inside part, the modified radial equation is

$$-(2m)^{-1}\frac{d^2\chi}{dr^2}+\left(|E|-\frac{3}{2}\frac{Ze^2}{R}+\frac{1}{2}\frac{Ze^2}{R^3}r^2+\frac{l(l+1)}{2mr^2}\right)\chi=0. \qquad (4)$$

If we define

$$\beta^4=mZe^2/R^3,$$
$$\xi^2=2[\tfrac{3}{2}(Ze^2/R)-|E|](mR^3/Ze^2)^{1/2},$$
$$y=\beta^2r^2,$$

then

$$y(d^2\chi/dy^2)+\tfrac{1}{2}(d\chi/dy)+\tfrac{1}{4}\{\xi^2-y-[l(l+1)/y]\}\chi=0.$$

If we further make the substitution

$$\chi(y)=y^{1/2(l+1)}\exp(-\tfrac{1}{2}y)\omega(y),$$

then

$$y(d^2\omega/dy^2)+(l+\tfrac{3}{2}-y)(d\omega/dy)-\tfrac{1}{4}(2l+3-\xi^2)\omega=0. \qquad (5)$$

This is Kummer's equation[2] with solutions $M(a,b,y)$ and $U(a,b,y)$ where,

$$a=\tfrac{1}{4}(2l+3-\xi^2)$$

and

$$b=l+3/2.$$

The behavior at $y=0$ is

$$M(a,b,y)\rightarrow 1,$$
$$U(a,b,y)\rightarrow\Gamma(l+\tfrac{1}{2})y^{-1/2(2l+1)}/\Gamma(a).$$

Since we demand $\chi(0)=0$, the permissible solution is $M(a,b,y)$. Therefore, the inside solution is

$$R_l^{\text{in}}(r)=\exp(-\tfrac{1}{2}\beta^2r^2)M(a,\,l+\tfrac{3}{2},\,\beta^2r^2).$$

Using an integral representation for M, we can also write the above as

$$R_l^{\text{in}}(r) = \exp(-\tfrac{1}{2}\beta^2 r^2)\,\Gamma(b)[\Gamma(b-a)\Gamma(a)]^{-1} \times \int_0^1 \exp(\beta^2 r^2 t)\, t^{a-1}(1-t)^{l+(1/2)-a} dt. \quad (6)$$

However, in practical applications, it is more convenient to write $M(a, b, z)$ in terms of an expansion

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n}{(b)_n}\frac{z^n}{n!} + \cdots, \quad (7)$$

where $(a)_n$ is the Pochhammer symbol.[3] Typically, only a few terms of the expansion are necessary.

The complete wave function can now be written

$$R_l(r) = (1/N_l) R_l^{\text{in}}(r), \qquad r \leq R, \quad (8a)$$

$$= (1/N_l)\epsilon_l R_l^{\text{out}}(r), \qquad r \geq R, \quad (8b)$$

where

$$\epsilon_l = R_l^{\text{in}}(R)/R_l^{\text{out}}(R). \quad (9)$$

3. ENERGY EIGENVALUE

The energy eigenvalue E can be found by matching derivatives of the inside and outside solutions at $r=R$. A straightforward calculation leads to the result

$$-\beta^2 R^2 + \frac{2\beta^2 R^2 a}{b}\frac{M(a+1, b+1, \beta^2 R^2)}{M(a, b, \beta^2 R^2)} = l - \tfrac{1}{2}\alpha R - \alpha R(l+1-\lambda) \times \frac{U(l+2-\lambda, 2l+3, \alpha R)}{U(l+1-\lambda, 2l+2, \alpha R)}. \quad (10)$$

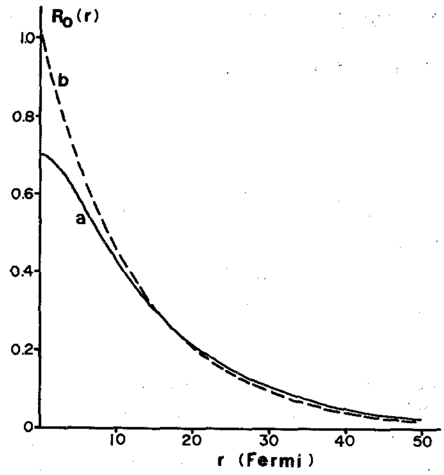

FIG. 1. Ground state wave function (not normalized) for ^{40}Ca muonic atom. a: Uniformly charged nucleus with radius 4.615 F; b: point charge nucleus.

Equation (10) is an algebraic solution for E. Unfortunately, we cannot give an explicit expression for E which must be obtained numerically.

In some cases, for example the ground state, it may be more convenient (and accurate) to find E separately using perturbation theory through second order. We have performed such a perturbation calculation for the ground state energy of a ^{40}Ca muonic hydrogen atom. The corresponding wave function is shown in Fig. 1. We have also included the point charge wave function for comparison.

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