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where the frequency ω is given by $\hbar\omega = E_0 - E_e$. It is this time-dependent expression that is most interesting to display in a computer movie. At first, the probability is greatest over one well. Later, the probability begins to build up over the other well while decreasing over the original well. This physically oscillates with a tunneling frequency of $\omega/2$ and is shown in Fig. 3 for $x_0 = 0.4$ fm.

* This work was supported by a faculty summer fellowship from the graduate school of the University of Idaho.

[1] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1967).

[2] D. Park, *Introduction to Quantum Theory* (McGraw-Hill, New York, 1964).

[3] I. R. Lapidus, Amer. J. Phys. **38**, 905 (1970).

[4] J. L. Powell and B. Crasemann, *Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1962), pp. 109–113.

Schrödinger Particle in a Gravitational Well

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Introductory quantum mechanics textbooks[1] generally include treatments of the one-dimensional Schrödinger equation for motion in a piece-wise constant potential (finite and infinite square wells, double wells, barrier scattering, etc.) and in the familiar harmonic oscillator potential. The pedagogically intermediate case of a potential linear in position arises in the interesting problem of Schrödinger particle dynamics in the uniform gravitational field,[2] in related connection with the equivalence principle,[3] and in simple WKBW-type treatments of the Schrödinger equation based on piece-wise linear approximations to smooth potentials.[4] In this note we remind the reader that the Schrödinger eigenvalues and eigenfunctions for motion in a *bounded* uniform gravitational field (gravitational well)[5] can be determined directly from the well-known Airy rainbow function[6,7] solution for the corresponding case of *unbounded* motion.[8] The problem is an excellent illustration of how boundary conditions determine allowed eigenfunctions and eigenvalues from the general regular solution of a second-order differential equation. Moreover, the resulting eigenfunctions and eigenvalues are explicitly mass dependent, suggesting that perhaps a suitably designed experiment can discriminate mass in a bounded uniform gravitational field.[9] To investigate this possibility, we consider the closely related problem of one-dimensional scattering from a gravitational barrier (or well), embedded in a region of otherwise constant potential, and find that the transmission coefficient is indeed mass dependent. Consequently, for sufficiently low energies, Schrödinger particles in a bounded uniform gravitational field do not necessarily satisfy the equivalence principle.[10] As an illustrative example, we construct a barrier that can provide resonance transmission of electrons at a given velocity but which is opaque to more massive particles at the same velocity. We conclude with the observation that the Ehrenfest equations predict mass-independent motion in the classical limit of a well-defined packet moving with mean energy large compared to the gravitational level spacings.

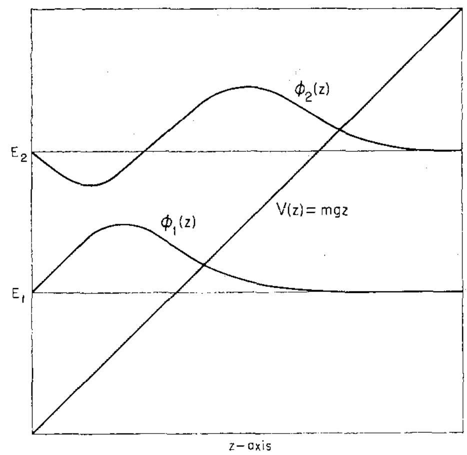

FIG. 1. General forms of the two lowest eigenfunctions for a particle moving in one dimension in a uniform gravitational field above an impenetrable plane.

Let us consider a single mass point executing one-dimensional motion along the z axis in a uniform gravitational field above an impenetrable xy plane.[11] The potential and the expected form of the first two eigenfunctions and their associated eigenvalues are shown in Fig. 1. The Schrödinger equation is

$$[-(\hbar^2/2m)(d^2/dz^2) + mgz]\phi(z) = E\phi(z), \quad (1)$$

where the allowable eigenfunctions satisfy the boundary conditions

$$\phi(z) = 0, \qquad z \leq 0, \quad (2a)$$

$$\phi(z) \to 0, \qquad E/mg < z \to \infty, \quad (2b)$$

with the latter specifying the requirement for a decaying solution in the classically forbidden region. Introducing

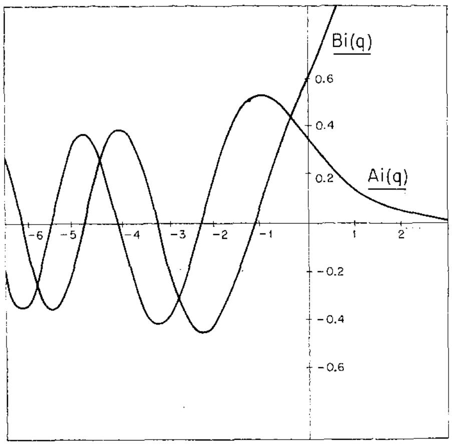

FIG. 2. The Airy functions $Ai(q)$ and $Bi(q)$. From Ref. 6.

the variable change

$$q=a(mgz-E), \qquad a=[2/mg^2\hbar^2]^{1/3}, \tag{3}$$

we find that Eq. (1) becomes

$$d^2\phi(q)/dq^2=q\phi(q). \tag{4}$$

The regular solution of Eq. (4) is the well-known Airy rainbow function, which has the integral representation[6]

$$Ai(q)=\pi^{-1}\int_0^\infty \cos(qu+\tfrac{1}{3}u^3)du, \tag{5a}$$

and is shown for convenience in Fig. 2. The boundary condition of Eq. (2b) is evidently satisfied by this general solution [Eq. (5a)], which decays for

$$q=a(mgz-E)>0.$$

The second linearly independent solution,

$$Bi(q)=\pi^{-1}\int_0^\infty [\exp(qu-\tfrac{1}{3}u^3)+\sin(qu+\tfrac{1}{3}u^3)]du, \tag{5b}$$

also shown in Fig. 2, does not satisfy the boundary condition of Eq. (2b), although it can be employed in the scattering problem treated below. Figures (1) and (2) demonstrate that when we also satisfy the boundary condition of Eq. (2a), the zeroes of the $Ai(q)$ function, $-q_n$, determine the allowable eigenvalues directly from Eq. (3) for $z=0$, at which point q must equal $-q_n$. Therefore, we have

$$E_n=q_n/a=q_n(\tfrac{1}{2}mg^2\hbar^2)^{1/3} \tag{6}$$

for the allowable eigenvalues, and the associated eigenfunctions are

$$\phi_n(z)=\frac{A_n}{\pi}\int_0^\infty \cos[(amgz-q_n)u+\tfrac{1}{3}u^3]du, \tag{7}$$

where[12]

$$A_n=\left((2g)^{1/2}\frac{m}{\hbar}\right)^{1/3}\Big/\int_{-q_n}^\infty Ai(q)^2dq. \tag{8}$$

Figures (1) and (2) show graphically how the boundary conditions determine the allowable eigensolutions from the general solution of Eq. (5) through the requirement that the classical turning points z_n $(=E_n/mg)$ equal q_n/amg.

Evidently, the eigenfunctions and eigenvalues for a Schrödinger particle in a gravitational well are explicitly mass dependent. We might suspect that this result could lead to mass-dependent scattering of Schrödinger particles from a bounded uniform gravitational field. To investigate such a possibility, we consider the closely related problem of scattering from a gravitational barrier (or well; let $m\rightarrow -m$) embedded in a region of otherwise constant potential:

$$V(z)=mgz, \qquad 0\leq z\leq \Delta z, \tag{9a}$$

$$V(z)=0, \qquad \text{otherwise.} \tag{9b}$$

Following the familiar procedure,[13] we find that the transmission coefficient is

$$\begin{aligned}T=(2/\pi)^2(\{(amg/k)[Ai'(q_0)Bi'(\Delta q)-Ai'(\Delta q)Bi'(q_0)]\\+(k/amg)[Ai(q_0)Bi(\Delta q)-Ai(\Delta q)Bi(q_0)]\}^2\\+[Ai(q_0)Bi'(\Delta q)+Ai'(\Delta q)Bi(q_0)-Ai'(q_0)Bi(\Delta q)\\-Ai(\Delta q)Bi'(q_0)]^2)^{-1},\end{aligned} \tag{10}$$

where

$$q_0=q(z=0)=-aE, \tag{11a}$$

$$\Delta q=q(z=\Delta z)=a(mg\Delta z-E), \tag{11b}$$

k is the magnitude of the wave vector, and the primes indicate differentiation with respect to the variable q [Eq. (3)].

To demonstrate the explicit mass dependence of the transmission coefficient [Eq. (10)], it is sufficient here to present a specific illustrative example. We consider incident particles of velocity slightly less than that required classically to surmount the barrier ($\frac{1}{2}v^2\cong g\Delta z$). Choosing $\Delta z\cong 0.4$ cm, we find that Eq. (10) predicts a resonance transmission $T\cong 1$ for an electron mass, while for protons of the same velocity we find the classical result $T\rightarrow 0$.[14] Consequently, we must conclude that Schrödinger particle kinematics in a uniform gravitational field is not necessarily mass independent for sufficiently low energy particles.[9,10] Experiments designed to measure the gravitational force on individual electrons and positrons apparently produce particle energies as low as 10^{-10} eV.[2] Evidently, some three orders of magnitude in energy lowering must be achieved before the mass dependence of the transmission coefficient for a gravitational barrier [Eq. (10)] can possibly be investigated by direct electron scattering experiments.[15]

To demonstrate that the mean position and momentum of a well-defined packet in a gravitational well will follow Newtonian mechanics, with its motion mass independent in agreement with the equivalence principle, we consider the Ehrenfest equations

$$(d/dt)\langle\Psi\,|\,-i\hbar(d/dz)\,|\,\Psi\rangle = -\langle\Psi\,|\,dV/dz\,|\,\Psi\rangle = -mg, \quad (12a)$$

$$(d/dt)\langle\Psi\,|\,z\,|\,\Psi\rangle = m^{-1}\langle\Psi\,|\,-i\hbar(d/dz)\,|\,\Psi\rangle, \quad (12b)$$

where the wave function $\Psi(z, t)$ evolves from an initial wave function $\Psi(z, t_0)$ in the form of a localized packet of mean energy large compared to the level spacing. Equation (12a) evidently provides the correct linear dependence on mass and time for the mean momentum and employing this result in Eq. (12b) shows trivially that the mean velocity is mass independent. Perhaps the reader will respond to a customary suggestion and investigate as an exercise the explicit construction of the Green's function for the time-dependent Schrödinger equation in the case of a gravitational well and examine the conditions under which packet dispersion can possibly become appreciable.[16]

The author is grateful to S. T. Epstein for bringing pertinent references to his attention, to R. G. Newton for comments, and to W. M. Fairbank for suggesting that a wider class of students than those of the author's acquaintance might possibly be interested in the foregoing material. That inertial and gravitational masses do not appear as a ratio in the Schrödinger equation was first drawn to the author's attention some years ago by E. H. Kerner, to whom a note of thanks is due. The referee has kindly drawn my attention to the work of I. I. Gol'dman *et al.* cited in Ref. 5.

[1] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968), 3rd ed., Chaps. 2 and 4.

[2] For discussion of experiments designed to measure the force of gravity on electrons and positrons, see F. C. Witteborn and W. M. Fairbank, Nature **220**, 436 (1968).

[3] The connection is not entirely trivial since gravitational and inertial masses do not cancel out directly in the Schrödinger equation, unlike the Newtonian case. Recent discussion of the equivalence principle in quantum mechanics is given by D. M. Greenberger, Ann. Phys. (N. Y.) **47**, 116 (1968); J. Math. Phys. **11**, 2329, 2341 (1970).

[4] See C. Eckart, Rev. Mod. Phys. **20**, 399 (1948) and the more recent work of R. G. Gordon, J. Chem. Phys. **51**, 14 (1969). The closely related problem of an infinite well with slanted base has been considered recently by T. Dymski, Amer. J. Phys. **36**, 54 (1968), and J. N. Churchill and F. O. Arntz, *ibid.* **37**, 693 (1969).

[5] The solution of this problem is described at length by I. I. Gol'dman, V. D. Krivchenkov, V. I. Kogan, and V. M. Galitshii, *Problems in Quantum Mechanics* (Academic, New York, 1960), pp. 10 and 75. The reader is well advised to note that the foregoing volume is distinct in content from the closely related collection of problems by I. I. Gol'dman and V. C. Krivchenkov, *Problems in Quantum Mechanics* (Pergamon, London, 1961), which does not discuss Schrödinger particle motion in a bounded gravitational field.

[6] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[7] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw–Hill, New York, 1966), p. 79.

[8] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics—Non-Relativistic Theory* (Pergamon, London, 1965), 2nd ed., p. 73.

[9] The possibility of mass analyzing a mixture of particles by gravitational means alone has been discussed on several occasions; see, S. T. Epstein, Phys. Letters **11**, 233 (1964), and references cited therein.

[10] It can be argued that boundary conditions are part of the force on a particle and, consequently, other than gravitational forces are present in the scattering of Schrödinger particles from a bounded gravitational barrier or well. However, the regions of constant potential immediately preceding and following the barrier or well are perhaps legitimately regarded as parts of the preparation and analysis apparatus, respectively, providing the counterparts to Galileo's restraining hands and the subsequently impacted earth below in the classic gedanken experiment.

[11] Our development [Eqs. (1)–(8)] follows closely that given by I. I. Gol'dman *et al.* in Ref. 5. The boundary condition $[V(z=0)\to\infty]$ gives rise to the discrete energy levels whereas unbounded motion leads to a continuum of allowed energy eigenvalues, as is well known. See Ref. 8.

[12] The reader should recognize that the Airy rainbow function is well studied and tabulated (Ref. 6), and, consequently, the solution of Eqs. (6)–(8) is complete and practical in a computational sense, exhibiting all the dependence upon the parameters of the problem, m, $\hbar$, and g. Moreover, the form of the eigenfunctions in the momentum representation is evident from Eq. (7).

[13] L. I. Schiff, Ref. 1, p. 101. We have employed the value of the Wronskian for $Ai(q)$ and $Bi(q)$ in obtaining Eq. (10); see Ref. 6.

[14] This choice of Δz gives $mg\Delta z \cong 10^{-13}$ eV for an electron, which is comparable to its largest level spacing in a gravitational well [Eq. (6), for $n=1$ and 2]. For a proton, however, $Mg\Delta z \cong 10^{-10}$ eV, which is two orders-of-magnitude larger than its level spacing in a gravitational well. Moreover, for an electron, $q_e = a_e mg\Delta z \cong 4$, while for a proton $q_p = a_p Mg\Delta z \cong 600$ and, consequently, the Airy functions for an electron oscillate only once in the barrier width while those for a proton oscillate many times, resulting in classical behavior. Specific forms for the Airy

functions, $Ai(q)$ and $Bi(q)$, for large q, required in obtaining the classical limit for Eq. (10), are given in Ref. 6.

[15] Discussion of specific experimental considerations and the optimal choice of particles and parameters which can be employed to test the predictions of Eq. (10) are clearly beyond the scope of this Note. Order-of-magnitude criteria for the quantum range of Eq. (10), however, are $amg\Delta z \cong 10$ and $\hbar^2 k^2/2m \cong mg\Delta z$.

[16] For construction of the Green's function in the case of unbounded motion where dispersion does not occur, see I. I. Gol'dman and V. D. Krivchenkov, Ref. 5, pp. 11 and 99. A somewhat complementary question arises with regard to state preparation. It has been argued that it is probably impossible in principle to find two normalizable wave functions associated with particles of different masses which provide identical coordinate and velocity distributions. See Ref. 9.

Undergraduate Experiment on rms Values

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In the study of electric circuits, it seems pedagogically justifiable to perform an experiment which includes both the calibration and experimental verification of the rms value of a waveform, mathematically written as

$$Y_{\rm rms} = T^{-1/2}\left(\int_0^T Y^2(t)dt\right)^{1/2}. \quad (1)$$

Details are presented on the construction, calibration, and utilization of a simple home-made calorimeter consisting of a resistor in thermal contact with a thermistor. The resistor is used to convert the electrical energy into heat, and the thermistor is used as a thermometer. Once the temperature characteristics of the calorimeter have been determined for a known power dissipation in the resistor, subsequent measurements can be made to verify the definition of rms values for any periodic waveform. The calorimeter is adequate for periodic waveforms, inasmuch as the equilibrium temperature is measured; that is, equilibrium between power input, temperature rise, and heat lost to the ambient system.

The calorimeter consists of a $\frac{1}{4}$-W resistor bonded with epoxy cement to a thermistor. The value of the resistor (R) is chosen so that the available laboratory power supplies and signal generators can supply up to $\frac{1}{4}$ W without excessive (say, several percent) waveform distortion. In our experiment, sinusoidal and square waves were to be obtained from an Eico model 377, sine-square generator (about 1 k output impedance) and a Hewlett–Packard model 427, (about 500-Ω output impedance). Both generators can develop about 5 V rms across 1 k. A simple calculation suggests a value for R of about 100 Ω (it is found that the loading does not lead to excessive waveform distortion).

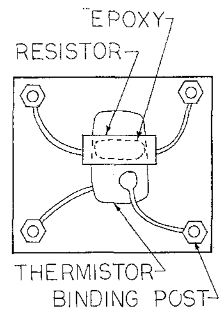

FIG. 1. General layout of the home-made calorimeter. The insulating base is approximately 2×2 in.

The thermistor was an ordinary disk type unit available from any local electronic supply store for about $1.00.[1] The disk shape is recommended because its dimensions can be easily reduced by judiciously applying pressure with ordinary diagonal cutters in such a way as to crack pieces away from the central region (where the thermistor leads are bonded by the manufacturer). Once the maximum amount of thermistor material has been cut away, the rough edges may then be smoothened on an ordinary grinding wheel. The thermistor now has a sufficiently small time-constant and thermal mass, and a resistance in the order of 1 kΩ. Of course, thermistors designed specifically for temperature measurement would be better, but they are more expensive and more difficult to procure easily.

Bonding is achieved simply and adequately by mixing enough epoxy so that there will be a large-area thermal contact between the resistor R and the thermistor.[2] To avoid electrical contact between the two leave enough epoxy between the two units to form an insulating layer. Curing time of the epoxy may be shortened by heating near an incandescent lamp.

Once the epoxy has cured, the entire unit is placed on a piece of insulating material (wood, plastic, circuit board, etc.) on which four binding posts have been installed. The wire leads from the resistors are then soldered to the binding posts as in Fig. 1.