Phase Shifts for Scattering by a One-Dimensional Delta-Function Potential

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transition with a photomultiplier, he finds that a photon arrives at a particular place at a particular time. If the observer is a distance r from the emitting atom, he attributes his result to an instantaneous transition at a time r/c earlier than his detection of the photon. On the other hand, an observer may examine the emitted radiation from many atoms with an interferometer. In this case he infers that each excited atom continually emits a wave packet in all directions for a period of time on the order of the mean life of the excited state. This interferometer experiment is discussed below. It is familiar and straightforward, but the result is usually presented with different emphasis than that given here.

From the photon, or particle, point of view, the process of decay is instantaneous, and the mean life refers to the number of atoms which decay per unit time. From the wave packet point of view, however, the mean life is indeed a measure of the time over which the process of emission occurs.

The particle and wave duality for the emitted radiation corresponds to a duality in the way the emitting atom is described. A perturbation-theory treatment leads to a wavefunction for the excited atom which is continually changing with time, although we say that the atom can never be found in between states.

The experimental verification of the wave packet approach is obtained as follows: Let a beam of light produced by optical transitions of a particular kind be used as the light source for a Michelson interferometer. Doppler shifts due to the motion of the emitting atoms will be neglected so that each transition is essentially identical to every other. It will be assumed that the wave packets from different atoms are also essentially identical. These assumptions may be justified by the fact that the condition to be found for distinct fringes may be approached in practice. We further assume that the light from different atoms is not phase-coherent.

The situation may be analyzed in the usual way.[2] The fixed and the movable mirrors of the interferometer form two virtual images, L_1 and L_2, of the light source separated by a distance S (Fig. 1). Light from the far

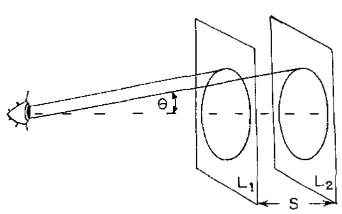

FIG. 1. The virtual images in the Michelson interferometer.

image, L_2, travels a distance $S \cos\theta$ further than light from L_1 before reaching the eye. Circular fringes are produced with maxima satisfying the equation

$$m\lambda = S \cos\theta.$$

Since the wave packets from different atoms are not phase-coherent, the fringes described above must be made up of contributions, due to the interference of each wave packet with itself. The beam splitter in the interferometer therefore divides each wave packet into two parts, one of which appears to come from L_1, the other of which appears to come from L_2. In order for a particular wave packet to contribute to the interference pattern, the part which comes from L_1 must reach the eye at the same time as the part which comes from L_2. Since the part which comes from L_2 has traveled a distance $S \cos\theta$ further, it must have originated in the atom at a time $S \cos\theta/c$ earlier. Thus the light source must be continually emitting a wave packet for a time $S \cos\theta/c$ in order for that packet to contribute to the fringes. It is found from experiment[3] that distinct fringes may be produced until $S \cos\theta/c$ approaches the mean life τ of the emitting state. We may therefore write an approximate condition for distinct fringes as $\tau > S \cos\theta/c$.

This result is well known. For a fuller discussion in which the same result is also presented in a different way, see Ref. 3.

[1] See, for example, D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1951), p. 412.

[2] See, for example, F. A. Jenkins and H. E. White, *Fundamentals of Optics* (McGraw-Hill Book Co., New York, 1957), 3rd ed., pp. 244–253.

[3] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959), pp. 315–322.

Phase Shifts for Scattering by a One-Dimensional Delta-Function Potential

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Several years ago, Eberly[1] showed that one-dimensional scattering problems can be solved in terms of phase shifts, and that the formalism is remarkably similar to the three-dimensional case. The results, of course, agree with those obtained by the conventional method of solving such problems.

The application of the partial-wave analysis to the problem of scattering from a one-dimensional delta-function potential is of particular interest from a pedagogic point of view because the determination of the phase shifts is quite straightforward in contrast to problems involving more complicated boundary conditions.

We consider first the "standard solution." For the potential given by $V(x) = A\delta(x)$, the Schrödinger equation is

$$-(\hbar^2/2m)(d^2\psi/dx^2) + A\delta(x)\psi = E\psi. \qquad (1)$$

If the incident wave approaches from the left, the solutions to the left and right of the origin are, respectively,

$$\psi_- = \exp(ikx) + r\exp(-ikx), \tag{2}$$

$$\psi_+ = t\exp(ikx). \tag{3}$$

From the boundary conditions at the origin

$$\psi_-(0) = \psi_+(0), \tag{4}$$

$$\psi_-'(0) - \psi_+'(0) = (2mA/\hbar^2)\psi(0), \tag{5}$$

we have

$$1 + r = t, \tag{6}$$

$$ik(1 - r - t) = 2\alpha t, \tag{7}$$

where $\alpha = mA/\hbar^2$.

We then obtain the reflection and transmission coefficients

$$R = |r|^2 = \alpha^2/(\alpha^2 + k^2), \tag{8}$$

$$T = |t|^2 = k^2/(\alpha^2 + k^2). \tag{9}$$

Now following Eberly we write

$$\psi = \exp(ikx) + f\exp(ik\epsilon x), \tag{10}$$

where f is the "scattering amplitude" and $\epsilon = \pm 1$ to the right and left of the origin. In terms of "phase shifts"

$$f_\epsilon = i\sum_{l=0}^{1} \epsilon^l \exp(i\delta_l)\sin\delta_l, \tag{11}$$

where δ_l is the phase shift of the lth partial wave.

The solutions to the right and left of the origin are then

$$\psi_+ = \exp(ikx) + f_+\exp(ikx), \tag{12}$$

$$\psi_- = \exp(ikx) + f_-\exp(-ikx). \tag{13}$$

Using Eqs. (4) and (5) we obtain

$$f_+ = f_- = i/(k - i\alpha). \tag{14}$$

From Eqs. (11), (14) we have

$$\delta_1 = 0, \tag{15}$$

$$\tan\delta_0 = \alpha/k. \tag{16}$$

The total cross section for scattering may be obtained from the form

$$\sigma_{\text{tot}} = 2\sum_{l=0}^{1} \sin^2\delta_l = 2\alpha^2/(\alpha^2 + k^2), \tag{17}$$

or by using the "optical theorem"

$$\sigma_{\text{tot}} = -2\,\text{Re}f_+, \tag{18}$$

which again yields Eq. (17).

It is of interest to note the relation between the total cross section for scattering and the reflection and transmission coefficients. Since

$$\sigma_{\text{tot}} = \sum_{i=\pm 1} |f_\epsilon|^2 = |f_+|^2 + |f_-|^2, \tag{19}$$

by comparing Eqs. (2), (3), (12) and (13) we obtain

$$\sigma_{\text{tot}} = |t - 1|^2 + |r|^2 = 2|r|^2 = 2R, \tag{20}$$

where we have used Eq. (6). This result is indeed confirmed by a comparison of Eqs. (8), (9), and (17).

From the point of view of partial waves, we note that for the problem of scattering from a one-dimensional delta-function potential there is only "s-wave" scattering. The delta function is, of course, the limiting case of the square-well potential discussed by Eberly. When $\alpha \to 0$, $\delta_0 \to 0$, and there is no scattering. When $k \to 0$, $\delta_0 \to \pi/2$, and there is no transmission.

[1] J. H. Eberly, Amer. J. Phys. **33**, 771 (1965).

Lattice Dynamics: Vibration Amplitudes in a Linear Chain

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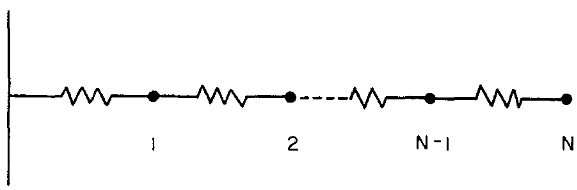
In doing lattice vibration problems, an important consideration is the stability of the model under consideration.

Dugdale and MacDonald[1] state that a linear chain is stable and contradict suggestions made by Peierls[2] and Domb[3] that a linear chain is inherently unstable due to thermal vibrations.

I wish to present an example where the mean square amplitude of vibration of an atom is proportional to m, where m means the mth atom, counting from one end of the chain. I assume harmonic and anharmonic forces between nearest neighbors only.



Total potential energy of chain is

$$U = \sum_{j=2}^{4}\sum_{m=0}^{N-1} \gamma_j(u_{m+1} - u_m)^j,$$

where $u_0 = 0$. Classical statistics shows that the average