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# One-Dimensional Hydrogen Atom

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The time-independent Schrödinger equation is solved for the bound state solutions of the one-dimensional Coulomb potential, $-e^2/|x|$. The wave functions obtained are normalizable and continuous. The energy spectrum consists of a set of discrete levels with energies equal to the Bohr energies of hydrogen, and a set of continuum levels with energies lying strictly between the discrete levels. The odd wave functions associated with the discrete levels are differentiable everywhere, but the even wave functions associated with the continuum levels have a cusp with infinite slope at the origin. *The energy levels are not degenerate.* Following the method of Loudon, the bound state solutions of a truncated Coulomb potential $-e^2/(|x|+a)$, $a>0$, are also obtained. For small $a$, the discrete spectrum of this potential contains: (1) energy levels which approach arbitrarily close to the discrete levels of the true Coulomb potential as $a\to 0$, with *odd* wave functions identical to the odd wave functions of the true Coulomb potential at $a=0$; (2) energy levels which approach arbitrarily close to the discrete levels of the true Coulomb potential as $a\to 0$, with *even* wave functions which *are not* the even wave functions of the true Coulomb potential at $a=0$; and (3) a ground state level belonging to an even wave function with an arbitrarily large binding energy as $a\to 0$. The ground state energy is quite sensitive to the actual value of $a$ chosen. It is concluded that the solution to the one-dimensional hydrogen atom shows a critical dependence on the detailed behavior of the Coulomb potential at small distances.

## INTRODUCTION

We solve the time-independent Schrödinger equation (SE) for the bound state energies of the one-dimensional "hydrogen atom" with Coulomb potential energy

$$V(x) = -e^2/|x|. \qquad (1)$$

Such a potential comes to mind rather naturally when one tries to think of a one-dimensional problem (other than the simple harmonic oscillator) which is a suitable exercise in solving the SE in series. It turns out, however, that a complete discussion and interpretation of the solutions of the SE for the Coulomb potential in one dimension is in some respects more difficult than for the three-dimensional hydrogen atom.

The same problem has been considered by Loudon.[1] In particular, he finds that the energy spectrum is *degenerate*, and this idea seems to have gained some currency.[2] However, we find that the energy spectrum of the one-dimensional hydrogen atom is *not degenerate* (although it does have certain interesting and unusual features). The arguments leading to this conclusion are presented in the following sections.

Insofar as possible, we use Loudon's notation and solutions[1] in order to facilitate comparison of our results with his. In Sec. I we quote Loudon's solutions to the SE for the potential [Eq. (1)]. The boundary conditions are imposed in Sec. II, determining the energy spectrum and the acceptable wave functions. In Sec. III we give Loudon's solution for the related problem of the truncated Coulomb potential

$$V(x) = -e^2/(|x|+a); \qquad a>0, \qquad (2)$$

with explicit formulas for the energy levels for small $a$. This enables us to interpret the spectrum and solutions of the true Coulomb potential ($a=0$) in Sec. IV. In Sec. V we suggest a one-dimensional potential which is a suitable exercise for solving the SE in series.

* Work supported in part by the National Science Foundation Undergraduate Research Participation Program.

[1] R. Loudon, Amer. J. Phys. **27**, 649 (1959). For a comment on Loudon's paper see M. Andrews, Amer. J. Phys. **34**, 1194 (1966). The two-dimensional hydrogen atom has been discussed by B. Zaslow and M. E. Zandler, Amer. J. Phys. **35**, 1118 (1967).

[2] Book review by G. Sposito, Phys. Today **21**, 81 (1968).

## I. SOLUTIONS OF THE SCHRÖDINGER EQUATION

The time-independent Schrödinger equation in one dimension for a particle of mass $m$ with Coulomb potential energy given by

$$V(x) = -e^2/|x| \tag{3}$$

is

$$-(\hbar^2/2m)(d^2\Psi/dx^2) - (e^2/|x|)\Psi = E\Psi. \tag{4}$$

We wish to find the negative energy solutions of this equation.

Following Loudon,[1] we introduce dimensionless variables $\alpha$ and $z$ for energy and length by writing

$$E = -\hbar^2/(2ma_0^2\alpha^2) \tag{5}$$

and

$$x = \tfrac{1}{2}\alpha a_0 z, \tag{6}$$

where $a_0$ is the Bohr radius:

$$a_0 = \hbar^2/me^2. \tag{7}$$

Then the SE becomes

$$d^2\Psi/dz^2 - \tfrac{1}{4}\Psi + (\alpha/|z|)\Psi = 0. \tag{8}$$

For $z \geq 0$ this equation has two linearly independent solutions, one diverging as $e^{z/2}$, and the other converging as $e^{-z/2}$ as $z \to \infty$. Only the latter solution leads to a normalizable wave function. This solution (which we call $\Psi_+$) is

$$\Psi_+(z) = B_+ W_\alpha(z) \tag{9}$$

where $B_+$ is an arbitrary constant and $W_\alpha$ is given by[1,3–5]

$$W_\alpha(z) = \frac{e^{-z/2}}{\Gamma(-\alpha)}\left(-\frac{1}{\alpha} + \sum_{r=1}^{\infty}\frac{(1-\alpha)_r}{r!(r+1)!}A_r z^{r+1} + [\ln z + \psi(1-\alpha) - \psi(1) - \psi(2)] \times \sum_{r=0}^{\infty}\frac{(1-\alpha)_r}{r!(r+1)!}z^{r+1}\right). \tag{10}$$

In Eq. (10), $A_r$ is given by

$$A_r = \sum_{n=0}^{r-1}[(n+1-\alpha)^{-1} - (n+1)^{-1} - (n+2)^{-1}], \tag{11}$$

and $(1-\alpha)_r$ stands for

$$(1-\alpha)_r = (1-\alpha)(2-\alpha)\cdots(r-\alpha) \tag{12}$$

with $(1-\alpha)_0 \equiv 1$. The function $\psi(s)$ is the logarithmic derivative of the gamma function $\Gamma(s)$,

$$\psi(s) = \Gamma'(s)/\Gamma(s), \tag{13}$$

and like the gamma function it has simple poles at $s = 0, -1, -2, \cdots$.

For $z \leq 0$, the normalizable solution of Eq. (8), which we call $\Psi_-$, and which converges as $e^{z/2}$ as $z \to -\infty$ is[6]

$$\Psi_-(z) = B_- W_\alpha(-z), \tag{14}$$

where $B_-$ is an arbitrary constant and $W_\alpha(-z)$ can be evaluated from Eq. (10).

The solution for $W_\alpha$ quoted in Eq. (10) is very elegant and compact, and it is the most convenient form of the solution to Eq. (8) for discussing the energy spectrum and wave functions of the one-dimensional hydrogen atom. In Appendices A and B we indicate how the solutions to Eq. (8) can be obtained by the more familiar approach of solving this equation by power series, using the method of Frobenius.

## II. ENERGY SPECTRUM AND WAVE FUNCTIONS

In the previous section we quoted the solution to the Schrödinger equation for the one-dimensional Coulomb potential which was normalizable as $|z| \to \infty$. We now require that $\Psi(z)$ be continuous at $z = 0$. Because the potential is singular at the origin, no such requirement is imposed on $\Psi'$ there, and indeed it would be incorrect to do so.

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[3] Erdélyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 261.

[4] *Handbook of Mathematical Functions*, Nat'l. Bur. Std., Appl. Math. Ser. 55 (U. S. Gov't. Printing Office, Washington, D.C., 1966), p. 504.

[5] In order to obtain Eq. (10) from Ref. 3 or 4, it is useful (for $z \geq 0$) to transform Eq. (8) by writing $\Psi = e^{-z/2}w$. Then Eq. (8) becomes $zw'' - zw' + \alpha w = 0$ which is Kummer's equation [Ref. 4, Eq. (13.1.1)] with $b = 0$ and $a = -\alpha$. The solution which is well-behaved as $z \to \infty$ is $U(-\alpha, 0, z)$ [Ref. 4, Eq. (13.1.8)]. By using $U(-\alpha, 0, z) = zU(1-\alpha, 2, z)$ [Ref. 4, Eq. (13.1.29)], the solution of Eq. (10) quoted in the text of the paper can be obtained by algebraic rearrangement of the explicit form for $U(1-\alpha, 2, z)$ given in Ref. 4, Eq. (13.1.6). Formulas (6.3.5) and (6.3.6) of Ref. 4 for $\psi(s)$ are useful in making this rearrangement.

[6] This result is most easily obtained from Eq. (8) for $z \leq 0$ by making the change of variable $z \to -z$.

Boundary conditions on $\Psi'$ at $z=0$, if needed, will be derived from the Schrödinger equation.[7]

Continuity of $\Psi$ at $z=0$ requires, from Eqs. (9), (10), and (14), that

$$-(1/\alpha)[B_+/\Gamma(-\alpha)]=-(1/\alpha)[B_-/\Gamma(-\alpha)]. \tag{15}$$

Assuming $\alpha\neq 0$ ($\alpha=0$ would correspond to infinite binding energy), Eq. (15) has the solution

$$B_+=B_-\equiv B, \tag{16}$$

*provided* that $\alpha$ is not equal to a positive integer, $N$. [In the latter case, where $\alpha=N$, Eq. (15) is identically satisfied since $1/\Gamma(-N)=0$.] The wave functions so obtained have a continuous energy spectrum

$$E=-\hbar^2/(2ma_0^2\alpha^2), \tag{17}$$

with all positive values of $\alpha$ allowed *except* $\alpha=1, 2, \cdots$. We label these (continuum) wave functions by the two integers between which the acceptable values of $\alpha$ lie. Thus for

$$N-1<\alpha<N; \qquad N=1, 2, \cdots, \tag{18}$$

the wave functions are

$$\Psi_{N-1,N}(z)=B_{N-1,N}W_\alpha(|z|). \tag{19}$$

They are even functions of $z$.

We note that these continuum wave functions are continuous and differentiable everywhere except at the origin where they are continuous but not differentiable. In fact, the slope of $\Psi_{N-1,N}$ is infinite at the origin, since, using Eq. (10), we have

$$\Psi_{N-1,N}(z)\to[B_{N-1,N}/\Gamma(-\alpha)]\times[-(1/\alpha)+|z|\ln|z|+O(z)]; \qquad |z|\to 0. \tag{20}$$

[7] *No* boundary condition (BC) is imposed on the derivative of $\Psi$. This is because the BC (if any) on $\Psi'$ should *always* be derived from the SE. For a potential which is finite everywhere, it follows from the SE that $\Psi''$ is finite so that $\Psi'$ is continuous. For a singular potential, the nature of the BC (if any) on $\Psi'$ depends on the details of the singularity of the potential and is always derived from the SE for the particular potential in question. Examples are the infinite square well (no BC on $\Psi'$, but $\Psi=0$), and the $\delta$-function potential (the SE determines the discontinuity in $\Psi'$).

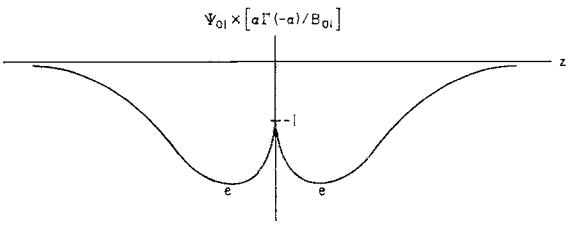


FIG. 1. The first continuum wave function $\Psi_{01}$ corresponding to any energy with $\alpha<1$. The shape of the function $\Psi_{01}$ follows from Appendix B.

Therefore,

$$\Psi'_{N-1,N}(z)\to[B_{N-1,N}/\Gamma(-\alpha)]\,\mathrm{sgn}(z)\ln|z|; \qquad |z|\to 0, \tag{21}$$

[where $\mathrm{sgn}(z)$ is the sign of $z$], or, assuming $B_{N-1,N}>0$,

$$\Psi'_{N-1,N}(0\pm)=\mp\infty. \tag{22}$$

In spite of the singular slope at $z=0$, $\Psi_{N-1,N}$ is normalizable and $B$ is determined by the normalization. The first two continuum wave functions are shown in Figs. 1 and 2.

We now consider $\Psi$ when $\alpha$ is a positive integer $N$. Following Loudon,[1] when $\alpha=N$ all terms in Eq. (10) are finite except for $\Gamma(-\alpha)$ and $\psi(1-\alpha)$, both of which have simple poles. Because $1/\Gamma(-N)=0$, only the term in Eq. (10) involving $\psi(1-\alpha)$ contributes, and we find

$$W_N(z)=\lim_{\alpha\to N}\left[\frac{\psi(1-\alpha)}{\Gamma(-\alpha)}\right]e^{-z/2}\sum_{r=0}^{\infty}\frac{(1-N)_r}{r!(r+1)!}z^{r+1}. \tag{23}$$

Since $\psi(1-\alpha)$ and $\Gamma(-\alpha)$ both have simple poles at $\alpha=N$, the limit in Eq. (23) is finite [actually[1,8] it is equal to $(-1)^{N+1}N!$]. Furthermore, $(1-N)_r$ is zero for $r\geq N$ and the series in Eq. (23) terminates. Thus[9]

$$W_N(z)=(-1)^{N+1}N!e^{-z/2}\sum_{r=0}^{N-1}\frac{(1-N)_r}{r!(r+1)!}z^{r+1}. \tag{24}$$

Continuity of $\Psi$ as given in Eqs. (9) and (14) at $z=0$ for $\alpha=N$ is already assured by Eq. (15). Nevertheless, we can check this directly from

[8] Reference 4, Eqs. (6.3.7) and (6.1.17).

[9] The $W_N(z)$ are related to the associated Laguerre polynomials $L^1_N$. See Ref. 1.

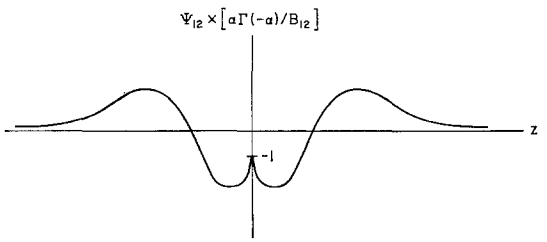


FIG. 2. The second continuum wave function $\Psi_{12}$ corresponding to any energy with $1<\alpha<2$.

Eq. (24). Writing

$$\Psi_+(z)=B_+W_N(z); \quad \Psi_-(z)=B_-W_N(-z), \quad (25)$$

we see that

$$\Psi_+(0)=\Psi_-(0)=0. \quad (26)$$

Thus continuity of $\Psi$ at $z=0$ does not give a relation between $B_+$ and $B_-$. We must, however, find such a relation since only one of these constants can be determined by normalization. To do so we derive a subsidiary boundary condition directly from the SE for the case $\alpha=N$; we integrate the SE across an infinitesimal region about the origin to find a condition on the derivative of $\Psi$ at $z=0$. (This procedure is identical to that used to find the boundary condition on $\Psi'$ for a one-dimensional delta function potential.) From Eq. (24) we see that (if $\alpha=N$)

$$\Psi_\pm(z)\to O(z), \qquad (|z|\to 0). \quad (27)$$

Using this in Eq. (8), integrating from $-\epsilon$ to $\epsilon$, and taking the limit $\epsilon\to 0$, we find that the derivative of $\Psi$ is continuous at the origin if $\alpha=N$:

$$\Psi'_+(0)=\Psi'_-(0). \quad (28)$$

From Eqs. (24) and (25) we have immediately that

$$\Psi'_\pm(0)=\pm(-1)^{N+1}N!B_\pm, \quad (29)$$

so that

$$B_+=-B_-\equiv B. \quad (30)$$

Thus for $\alpha=N$ the wave functions corresponding to the discrete energy levels

$$E_N=-me^4/(2\hbar^2N^2); \qquad N=1, 2, \cdots \quad (31)$$

are

$$\Psi_N(z)=B_N\,\mathrm{sgn}(z)\,W_N(|z|) \quad (32)$$

where we have labeled these (discrete) wave functions by the energy quantum number $N$, and where $W_N$ is given by Eq. (24). The constant $B_N$ can be determined by normalization. The discrete wave functions are odd functions of $z$ and are continuous and differentiable everywhere. The first three discrete wave functions are shown in Fig. 3.

The energy spectrum for the one-dimensional hydrogen atom is shown in Fig. 4. *The energy levels are not degenerate.* Furthermore, there are both *odd* (discrete energy) and *even* (continuous energy) *wave functions.*

## III. TRUNCATED COULOMB POTENTIAL

The solution to the SE for the bound states of the one-dimensional Coulomb potential found in the previous sections includes wave functions belonging to a continuous energy spectrum and having infinite slope at the origin. Neither of these properties is a normal feature of solutions of quantum mechanical bound state problems. However, the Coulomb potential is singular at the origin; this is clearly why the one-dimensional Coulomb problem has bound state solutions with such unusual properties.

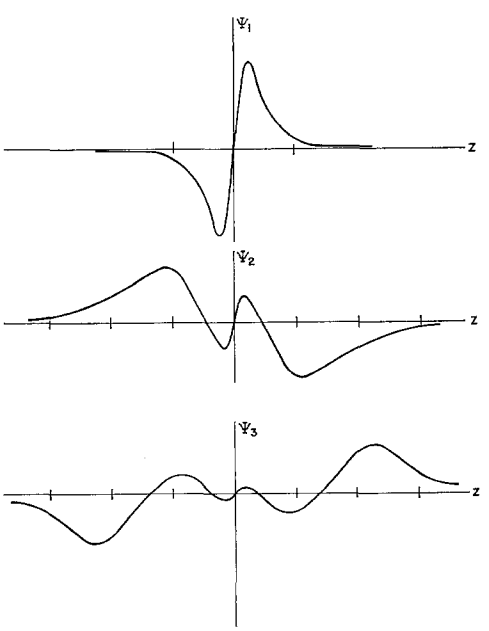


FIG. 3. The first three discrete wave functions (unnormalized) of the one-dimensional hydrogen atom.

It is therefore of considerable interest to study a related problem where the potential is finite everywhere. We will consider the truncated Coulomb potential where the potential energy is given by

$$V(x) = -e^2/(|x|+a); \qquad a>0, \tag{33}$$

and is shown in Fig. 5. Following essentially the discussion given by Loudon,[1] we observe that the SE for this potential is still of the form Eq. (8), provided the independent variable is chosen as $z_\pm$ where

$$z_\pm = 2(a\pm x)/\alpha a_0, \tag{34}$$

with the plus-or-minus sign chosen according as $x$ is positive or negative. Then the SE becomes

$$\Psi'' - \tfrac{1}{4}\Psi + (\alpha/z_\pm)\Psi = 0. \tag{35}$$

The normalizable solutions of Eq. (35) with the correct behavior as $|x|\to\infty$ are

$$\Psi(z_\pm) = B_\pm W_\alpha(z_\pm). \tag{36}$$

For $a\neq 0$, the appropriate boundary conditions for the wave functions at $x=0$ are continuity of $\Psi$ *and* continuity of $\Psi'$ since for $a\neq 0$, the truncated Coulomb potential is never infinite.[7] These conditions require

$$(B_+ - B_-)W_\alpha(z') = 0 \tag{37a}$$

$$(B_+ + B_-)dW_\alpha(z')/dz' = 0, \tag{37b}$$

where $z'$ is the (common) value of $z_\pm$ at $x=0$:

$$z' = 2a/\alpha a_0. \tag{38}$$

If $B_+ = B_-$ the truncated Coulomb potential has even wave functions with energies corresponding to $\alpha$ determined (as a function of the truncation distance $a$) by

$$dW_\alpha(z')/dz' = 0. \tag{39}$$

Correspondingly, if $B_+ = -B_-$, the truncated Coulomb potential has odd wave functions with energies determined by

$$W_\alpha(z') = 0. \tag{40}$$

In the case where the truncation distance $a$ becomes small compared with the Bohr radius $a_0$ (*and* provided that $\alpha$ is not also small compared with one), we have $z'\ll 1$ and Eqs. (39) and (40) can be solved explicitly for $\alpha$ by using Eq. (10). Equation (40) becomes

$$(2a/a_0\alpha)\psi(1-\alpha) - 1/\alpha = 0, \tag{41}$$

and Eq. (39) becomes

$$\ln(2a/a_0\alpha) + \psi(1-\alpha) = 0, \tag{42}$$

as found by Loudon.[1] For small $a/a_0$ these equations have a solution only for $\alpha$ near a positive integer $N$. Since[1,10]

$$\psi(1-\alpha)\to 1/(\alpha-N); \qquad \alpha\to N, \tag{43}$$

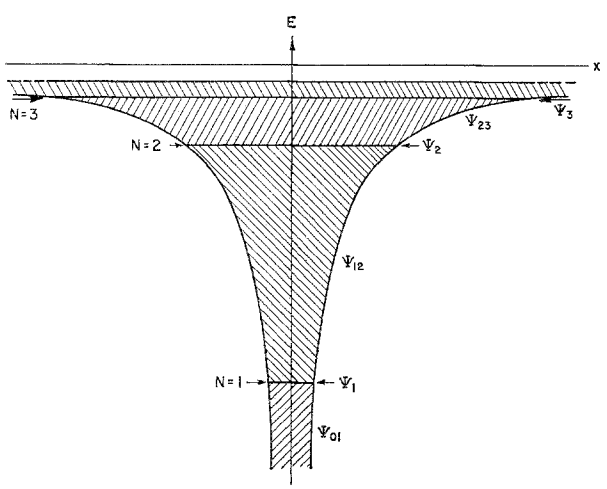


FIG. 4. The energy spectrum of the one-dimensional hydrogen atom. The energies are given by $E=-\hbar^2/(2ma_0^2\alpha^2)$. The odd wave functions $\Psi_N$ have discrete energies given by $\alpha=N$ $(N=1, 2, \cdots)$. The even wave functions $\Psi_{N-1,N}$ have a continuous spectrum with energies corresponding to $\alpha$ anywhere within the range $N-1<\alpha<N (N=1, 2, \cdots)$. The energies are not degenerate.

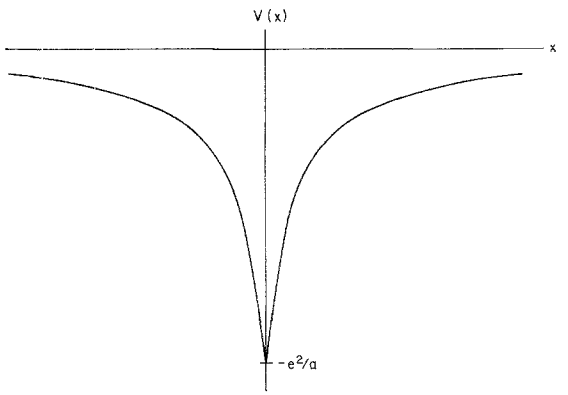


FIG. 5. The truncated Coulomb potential of Eq. (33).

[10] Reference 4, Eq. (6.3.6).

Eq. (41) becomes

$$\alpha = N + 2a/a_0 \tag{44}$$

and Eq. (42) becomes

$$\alpha = N + 1/\ln(a_0/a), \tag{45}$$

so that the energies are

$$E_N = -\frac{\hbar^2}{2ma_0^2N^2}\begin{pmatrix} 1-4a/Na_0; & \text{odd states} \\ 1-2/N\ln(a_0/a); & \text{even states} \end{pmatrix} \quad (a/a_0 \ll 1). \tag{46}$$

The odd state energy levels of the truncated Coulomb potential lie slightly above the discrete energies of the true Coulomb potential; the even state energy levels of the truncated potential in turn lie slightly above the odd energy levels of the truncated potential. (The last statement follows since $s > 2\ln s$ for large $s$.)

Equation (46) gives all energy levels of the truncated Coulomb potential for small $a/a_0$ with the exception of those for which $\alpha$ is also small. If both $a/a_0$ and $\alpha$ are small, $z'$ is not necessarily small, and the procedure of using the lowest order terms in Eq. (10) to obtain equations like Eqs. (41) and (42) is open to question. However, if $\alpha$ is small ($\alpha < 1$, say) we may use the explicit representation of $\Psi$ (namely $\Psi_{01}$) found in Eq. (B9). We showed in Appendix B that $W_\alpha(|z|)$ is never zero (see also Fig. 1), and it follows immediately that Eq. (40) can never be satisfied for $\alpha < 1$. Thus we have only an even wave function (and *only* one, obtained by matching at the two $e$'s in Fig. 1) with energy determined by Eq. (39).

We assume that $\alpha$ for this state, while small, is still much larger than $a/a_0$; i.e.,

$$a/a_0 \ll 1, \qquad \alpha \ll 1, \quad \text{but such that} \quad z' \ll 1. \tag{47}$$

Then we can keep the lowest order terms in $z'$ in Eq. (10) to evaluate Eq. (39) and we find

$$1/2\alpha + \ln(2a/a_0\alpha) = 0, \tag{48}$$

as given by Loudon.[1] For small $\alpha$, $1/\alpha \gg \ln(1/\alpha)$, so that Eq. (48) can be written

$$\alpha = 1/[2\ln(a_0/a)]. \tag{49}$$

For $\alpha$ as given in Eq. (49) it is easy to see that $z' \ll 1$ so that the assumptions of Eq. (47) are satisfied. Therefore, we have found the only energy eigenstate when $\alpha$ is small (for small $a/a_0$). This is the ground state of the truncated Coulomb potential; it has an even wave function with energy

$$E = -(\hbar^2/2ma_0^2)[2\ln(a_0/a)]^2 \quad (a/a_0 \ll 1). \tag{50}$$

## IV. CONCLUSIONS

### A. Comparison of Solutions

Having obtained the complete solution for the one-dimensional hydrogen atom for both the true and truncated Coulomb potentials, we wish to discuss to what extent the solutions for the truncated Coulomb potential approach those for the true Coulomb potential as the truncation distance $a$ goes to zero.

We consider first the odd wave functions of the truncated potential with energies given by Eq. (46). As $a/a_0$ becomes arbitrarily small, the energies approach arbitrarily close to the discrete energies of the odd wave functions of the true Coulomb potential. At $a=0$, it follows from Eqs. (36) and (44) that the odd wave functions of the truncated Coulomb potential are identical to the odd wave functions of the true Coulomb potential. Thus, the odd solutions of the true Coulomb potential can be obtained as the limit of the odd solutions of the truncated potential as $a \to 0$. This is to be expected since both solutions satisfy boundary conditions requiring continuity of $\Psi$ *and* of $\Psi'$.

Next we consider the even wave functions of the truncated potential with energies given by Eq. (46). As $a/a_0$ becomes arbitrarily small, the energies approach arbitrarily close to the discrete energies of the *odd* wave functions of the true Coulomb potential. At $a=0$, however, the even wave functions of the truncated potential are *not* the even wave functions of the true Coulomb potential. This is neither serious nor surprising; it simply means that the even wave functions of the true potential *cannot* be obtained from those of the truncated potential by the limiting process $a \to 0$. Indeed this is to be expected since the even wave functions for the true Coulomb potential do not have a continuous derivative at the origin,

and one would not expect them to be derivable from the even wave functions of the truncated Coulomb potential which *must* have a continuous derivative at $x=0$. It is easy to see how this behavior arises by considering the SE [Eq. (4)] for the truncated potential [Eq. (33)]. Considering only even wave functions, and integrating Eq. (4) from $-\epsilon$ to $\epsilon$ across $x=0$, we obtain

$$-(\hbar^2/2m)[\Psi'(+\epsilon)-\Psi'(-\epsilon)]$$
$$-e^2\int_{-\epsilon}^{\epsilon}\frac{\Psi(x)dx}{|x|+a}=E\int_{-\epsilon}^{\epsilon}dx\Psi(x). \quad (51)$$

For small $\epsilon$, it is sufficient to consider the (nonzero) value of $\Psi$ at $x=0$, and we find

$$\Psi'(+\epsilon)-\Psi'(-\epsilon)=-(4m/\hbar^2)\Psi(0)$$
$$\times[\epsilon E+e^2\ln(1+\epsilon/a)]. \quad (52)$$

For the truncated potential ($a>0$), the limit $\epsilon\to 0$ in Eq. (52) implies that $\Psi'$ is continuous at $x=0$. For the true Coulomb potential ($a=0$), the limit $\epsilon\to 0$ in Eq. (52) requires that $\Psi'$ have a cusp with infinite slope at the origin

$$\Psi'(0+)-\Psi'(0-)=(-\infty)\cdot\mathrm{sgn}[\Psi(0)], \quad (53)$$

which is precisely the behavior for $\Psi'$ found in Eq. (22).

### B. Physical Interpretation of Solutions

The unusual feature of the solution to the SE for the one-dimensional Coulomb potential is, of course, the presence of wave functions with continuous spectra and infinite slope at the origin. One might argue that the infinite slope of the functions is so undesirable that they simply cannot be considered as acceptable wave functions. If this rather arbitrary point of view is adopted, the wave functions are easily discarded by setting $B_{N-1,N}$ equal to zero. This amounts to imposing, *ad hoc*, a boundary condition requiring continuity of the derivative of $\Psi$ even in the presence of a singular potential. A great deal of insight is lost by this approach, as we shall show below, and we are not in sympathy with it. Even if one insists on discarding the continuum solutions, the conclusion is inescapable that *the energy spectrum of the one-dimensional hydrogen atom is not degenerate.*

We prefer to keep the continuum solutions to the SE as acceptable wave functions at least until we answer the question: What do they mean? Let us suppose for the sake of discussion that an object such as the one-dimensional hydrogen atom[11] (composed of a "proton" and an "electron") actually existed in nature. Then one would not expect the potential to be Coulomb at arbitrarily small distances, but only down to the "proton radius", say a fermi or so. However, this is essentially the problem of the truncated Coulomb potential which was discussed in Sec. IV. It follows that such a "real" one-dimensional hydrogen atom has a spectrum with discrete energy levels of two types: (1) those belonging to odd wave functions with energies quite close to the discrete levels of the true Coulomb potential, and (2) those belonging to even wave functions with discrete energies having no correspondence to the continuum of energies of the even states of the true Coulomb potential. The ground state is an example of the latter type of level, and its energy would be quite sensitive to the "proton" size, as shown by Eq. (50). In fact, the binding energy of the "real" one-dimensional hydrogen atom would be about 5000 eV! If "real" one-dimensional hydrogen atoms existed, one could, in principle, measure this binding energy experimentally and thus obtain a value for the truncation distance appropriate to the problem.

It is certainly clear that keeping only the *discrete* levels in the solution of the true Coulomb potential *would not* provide a satisfactory description of the physical problem. For then one would have found only the levels near the odd levels of a "real" one-dimensional hydrogen atom. The even half of the spectrum would be missing, and in particular one would predict a binding energy of 13.6 eV rather than one on the order of 5000 eV.

The proper interpretation of the continuum solutions for the true Coulomb potential is as follows: They arise because it is not "physical" to solve the SE in one dimension for the *singular*

[11] The term "one-dimensional hydrogen atom" is slightly ambiguous. We mean a system with potential proportional to $1/|x|$, not a system with potential proportional to the solution of the one-dimensional Poisson equation for a point charge at the origin, $V(x)\propto|x|$.

potential $1/|x|$; indeed, they serve as a *warning* that the actual discrete spectrum for a "real" one-dimensional hydrogen atom, where the appropriate potential is $1/(|x|+a)$, will contain discrete even levels whose location depends on $a$. The location of one of the levels, the ground state, will be quite sensitive to the choice of the truncation distance $a$.

In view of the preceding remarks about the nature of the spectrum of the one-dimensional hydrogen atom (and in particular the dependence of the location of the even levels on the behavior of the Coulomb potential at small distances) it is really rather remarkable that one can, for the hydrogen atom in three dimensions, solve the SE for the singular potential $1/r$ and find solutions which are not affected by the detailed behavior of the potential for small $r$ (especially for zero angular momentum where there is no centrifugal barrier term to keep the electron away from the proton).

To see how this happens we consider zero angular momentum so that the wave function is

$$\Psi(\mathbf{r}) = (4\pi)^{-1/2} R(r), \tag{54}$$

where $u$, defined by

$$u(r) = rR(r), \tag{55}$$

satisfies the SE

$$-(\hbar^2/2m)(d^2u/dr^2) - (e^2/r)u = Eu, \tag{56}$$

identical to Eq. (4) for $r \geq 0$, and with identical solutions. However, in three dimensions the appropriate boundary condition[12] at $r=0$ is $u(0)=0$ and this immediately excludes solutions for $u$ which are "even" in $r$. Thus the continuum solutions of the true Coulomb potential and the even solutions of the truncated Coulomb potential (which are the ones affected by the behavior of the potential at small distances) are not acceptable wave functions in three dimensions.

[12] The boundary condition $u(0)=0$ is usually obtained by arguing that the radial wave function $R(r)$ should be finite at $r=0$. This argument is not at all convincing; more persuasive is the following: We certainly want the transition amplitude $T$ for electromagnetic radiation to be well-defined. For dipole radiation, $T$ is proportional to the matrix element of the momentum between initial and final states [J. L. Powell and B. Crasemann, *Quantum Mechanics* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1965), p. 422]. Thus the integral

$$\int_0^\infty r^2 dr \frac{u}{r} \frac{d}{dr}\left(\frac{u}{r}\right)$$

must exist at $r=0$ for all acceptable wave functions. For either the true or truncated Coulomb potential, this will only be true if $u(0)=0$.

## V. PEDAGOGICAL COMMENT

Solution of the SE for the following one-dimensional potential

$$\begin{aligned} V(x) &= -e^2/x; \qquad x>0 \\ &= +\infty; \qquad x \leq 0 \end{aligned} \tag{57}$$

would be a straightforward exercise for a student familiar with the solution of the one-dimensional simple harmonic oscillator by power series. Making $V$ infinite at $x=0$ is equivalent to requiring that $\Psi$ be zero there so that one finds only the discrete spectrum with well behaved wave functions (Appendix A). The continuum solutions (Appendix B) need not be considered since they are even functions, linearly independent of the odd discrete functions, and therefore not zero at $x=0$. This problem is not completely artificial since its solution is the zero angular momentum solution of the three-dimensional hydrogen atom.

## APPENDIX A

### Series Solution of Schrödinger Equation for Discrete Wave Functions

In this appendix we indicate how the solution [Eq. (24)] for the discrete wave functions of the one-dimensional hydrogen atom can be obtained by the more familiar procedure of solving the SE in series. We consider Eq. (8) for $z \geq 0$ and remove the asymptotic dependence by defining $w$ as

$$\Psi_+(z) = e^{-z/2} w(z). \tag{A1}$$

Then Eq. (8) becomes

$$w'' - w' + (\alpha/z)w = 0, \tag{A2}$$

which we solve in series by writing

$$w(z) = \sum_{r=0}^{\infty} b_r z^{r+\delta} \tag{A3}$$

where $\delta$ and the $b_r$ are to be determined. Sub-

stituting (A3) into (A2) and equating coefficients of like powers of $z$ gives

$$b_0\delta(\delta-1)=0, \tag{A4}$$

$$b_{r+1}(r+\delta)(r+\delta+1)+b_r(\alpha-r-\delta)=0; \qquad r=0, 1, \cdots. \tag{A5}$$

Equation (A4) can be satisfied by choosing $\delta=0$ or $\delta=1$. The choice $\delta=1$ gives, from Eq. (A5),

$$b_r=b_0[(1-\alpha)_r/r!(r+1)!]; \quad r=0, 1, \cdots, \tag{A6}$$

where $b_0$ is an arbitrary constant. Thus

$$\Psi_+(z)=b_0e^{-z/2}\sum_{r=0}^{\infty}\frac{(1-\alpha)_r}{r!(r+1)!}z^{r+1}. \tag{A7}$$

It is easy to see that the choice $\delta=0$ leads once again to the solution [Eq. (A7)] so that only one of the two linearly independent solutions of the SE has been found from the series [Eq. (A3).] In fact, Eq. (A7) is the solution of the SE which gives rise to the odd wave functions with discrete energy levels. This may be seen by examining the convergence of the series in Eq. (A7) for large $z$. By comparing it with the series $e^z$ we see that $\Psi_+$ in Eq. (A7) goes as $e^{z/2}$ as $z\to\infty$ unless the series terminates. The series terminates (and $\Psi_+\to e^{-z/2}$ as $z\to\infty$) only if $\alpha=N$ where $N$ is a positive integer. Thus we find

$$\Psi_+(z)=B_+W_N(z) \tag{A8}$$

where the polynomial $W_N(z)$ is given by Eq. (24), and where we have chosen $b_0$ as

$$b_0=B_+(-1)^{N+1}N!. \tag{A9}$$

## APPENDIX B

### Continuum Wave Functions Obtained from Series Solution of Schrödinger Equation

In this appendix we indicate how the solution [Eq. (19)] for the continuum wave functions of the one-dimensional hydrogen atom can be obtained from the series solution of the SE found in Appendix A. It is easy to show for any two solutions $\Psi^{(1)}$ and $\Psi^{(2)}$ of the SE [Eq. (18)] belonging to the same energy that

$$(\Psi^{(1)}\Psi^{(2)\prime}-\Psi^{(2)}\Psi^{(1)\prime})'=0. \tag{B1}$$

Equation (B.1) is equivalent to the statement that the Wronskian of $\Psi^{(1)}$ and $\Psi^{(2)}$ is independent of $z$:

$$Wr(\Psi^{(1)}, \Psi^{(2)})\equiv\Psi^{(1)}\Psi^{(2)\prime}-\Psi^{(2)}\Psi^{(1)\prime}=C. \tag{B2}$$

Furthermore, provided $C$ is not zero, $\Psi^{(1)}$ and $\Psi^{(2)}$ are linearly independent. We use Eq. (B2) with $C$ not zero to find a second solution to the SE, $\Psi^{(2)}$, linearly independent of the solution [Eq. (A7) found in Appendix A.

We consider the region $z\geq 0$ and define $v$ by

$$\Psi_+^{(2)}(z)=e^{-z/2}v(z). \tag{B3}$$

Then using $\Psi_+$ in Eq. (A7) for $\Psi^{(1)}$, the Wronskian (B2) can be written

$$wv'-vw'=De^z \tag{B4}$$

where $D$ is a non-zero constant and $w$ is defined (without the $b_0$ of Appendix A) as

$$w(z)=\sum_{r=0}^{\infty}\frac{(1-\alpha)_r}{r!(r+1)!}z^{r+1}. \tag{B5}$$

When multiplied by $1/w^2$, Eq. (B4) can be integrated immediately to give $v$ and thus $\Psi_+^{(2)}$ in Eq. (B.3) as

$$\Psi_+^{(2)}(z)=De^{-z/2}w(z)\int^z d\zeta e^{\zeta}[w(\zeta)]^{-2} + D_1e^{-z/2}w(z), \tag{B6}$$

where $D_1$ is an arbitrary constant.

We saw in Appendix A that $w\to e^z$ as $z\to\infty$ unless $\alpha$ equals a positive integer $N$, in which case $w$ is a polynomial. Suppose first that $\alpha=N$ and consider the integral in Eq. (B6). For large $z$ it diverges as $e^z$, the first term in Eq. (B6) diverges as $e^{z/2}$, and we must take $D=0$ to obtain a normalizable wave function. Then Eq. (B6) reduces to the solution for $\Psi_+$ found in Appendix A. Now suppose that $\alpha\neq N$. Then the integral in Eq. (B6) *converges* as $e^{-z}$ and the first term converges as $e^{-z/2}$. Choosing the lower limit of the integral in Eq. (B6) as $\infty$, we must take $D_1=0$ to obtain a normalizable wave function. Thus the continuum wave functions are of the form

$(\alpha \neq N)$[13]

$$\Psi_+(z) = De^{-z/2}w(z)\int_\infty^z d\zeta e^{\zeta}[w(\zeta)]^{-2}, \quad \text{(B7)}$$

where $w$ is given in Eq. (B5) and where we have suppressed the superscript.

If we consider the region $\alpha<1$, it follows from Eq. (B5) that $w>0$ for $z>0$. Thus $\Psi_+/D$ in Eq. (B7) is always negative for $z>0$. Furthermore, as $z\to 0$, $\Psi_+$ is given by

$$\Psi_+(z)\to D(-1+\alpha z\ln z); \qquad z\to 0. \quad \text{(B8)}$$

It follows then, from Eq. (20), that the continuum wave function $\Psi_{01}$ is given by[14]

$$\Psi_{01}(z) = \frac{B_{01}e^{-|z|/2}}{\alpha\Gamma(-\alpha)}\, w(|z|)\int_\infty^{|z|}\frac{d\zeta e^{\zeta}}{[w(\zeta)]^2}. \quad \text{(B9)}$$

Thus in the notation of Sec. III [see Eq. (19)],

$$W_\alpha(z) = \frac{e^{-z/2}w(z)}{\alpha\Gamma(-\alpha)}\int_\infty^z \frac{d\zeta e^{\zeta}}{[w(\zeta)]^2}; \qquad \alpha<1. \quad \text{(B10)}$$

[13] Equation (B7) is valid only for $z$ greater than the largest zero of $w(z)$. For $\alpha<1$, since $w$ has no zeros if $z>0$, it is valid for all $z>0$. For $\alpha\neq N, \alpha>1$, $w$ has zeros and the expression (B7) must (and can) be continued across the zeros of $w$ to obtain an expression valid for all $z>0$.

[14] We have made use of Eqs. (14) and (16) to write Eq. (B9) for $z\leq 0$ as well as for $z\geq 0$.

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# A Pedagogical Measurement of the Velocity of Light*

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An inexpensive, simple, easily constructed and convincing technique is described both for demonstrating that the speed of light is finite, and for measuring its value. All that is required is about $40 and a good oscilloscope. Detailed instructions and experimental results are given, and it is found that $c=(29.8\pm0.6)$ cm/nsec.

## INTRODUCTION

An inexpensive, simple, easily constructed and convincing technique will be described both for demonstrating that the speed of visible light is finite, and for measuring its value. The vocabulary has been chosen so that the apparatus described can be constructed by people with little or no working knowledge of high-speed electrical or optical pulse techniques.

The measurement is performed in the following way. A light pulser is placed at one end of an optical bench. It simultaneously emits a pulse of light which travels down the bench to a detector and an electrical pulse which travels down a cable to the trigger input of an oscilloscope, starting its sweep. The detector receives the light pulse and sends a corresponding electrical pulse to the signal input of the CRT, so that a pulse is shown on the CRT face. As the detector is slid along the

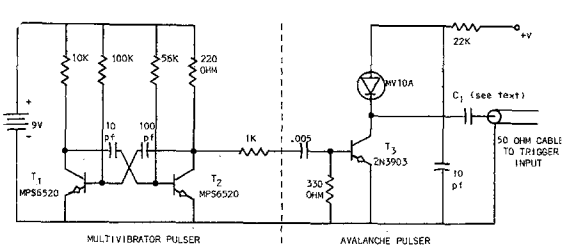


FIG. 1. Light pulser schematic.

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