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As we see, the leading term is the same, and the following terms are correction terms. For large Z , however, the correction terms are negligible. Thus for uranium, Eq. (28) yields a correction factor 0.99748. Even for Z as small as 8, the "correction" is very small ($\sim 1.5\%$) and 2.5% for $Z = 1$. Thus, interestingly enough, the errors made in using the simplifying approximations Eqs. (19) and (20) cancel almost exactly!

It must be particularly noted that this method leads easily and simply to the peculiar 7/3-power dependence found from the Thomas-Fermi-Dirac model in a much more laborious way.

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Two-Dimensional Analog to the Hydrogen Atom

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The Schrödinger equation is applied to a two-dimensional problem of two mass points, one with the charge and mass of an electron and the other with the charge and mass of a proton. Solution exists in closed form, and the result is compared with that for a three-dimensional hydrogen atom.

INTRODUCTION

A "two-dimensional hydrogen atom" can be defined within the context of wave mechanics if the motion of the electron around the proton is constrained to be planar. Our objective in formulating this situation was to gain insight into the electronic structure that an atom might have in a two-dimensional space continuum. The problem was not designed to be practical in terms of three-dimensional reality. The atom might be one, for example, that an inhabitant of Flatland¹ would encounter. To have a direct application to reality, certain forces would have to exist which impose the planarity restriction on the electron and the proton; and the Schrödinger equation, as we apply it, would have to be separable from the complete equation of the total system. Nevertheless, the results provide an interesting comparison to the classic hydrogen atom in three dimensions and its one-dimensional counterpart,² and test the effect of the loss of a single space dimension.

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¹ E. A. Abbott, *Flatland* (Dover Publications, Inc., New York; Barnes and Noble, Inc., New York, 1963).

² R. Loudon, *Am. J. Phys.* **27**, 649 (1959).

I. SOLUTION

Solution of $H\Psi = E\Psi$ can be obtained in closed form using the polar coordinates, r and ϕ . Both particles are treated as mass points with reduced mass μ and the differential equation becomes separable if $\Psi(r, \phi)$ is assumed to be of the form $R(r)\Phi(\phi)$. A potential energy of $-\epsilon^2/r$ is taken to apply, as was done in the one-dimensional system.² Angular wavefunctions are

$$\Phi(\phi) = e^{il\phi}/(2\pi)^{1/2},$$

where $l = 0, \pm 1, \pm 2, \pm 3 \dots$. The development of the radial equation,

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \left(\frac{8\pi^2\mu}{\hbar^2} \left(E + \frac{\epsilon^2}{r} \right) - \frac{l^2}{r^2} \right) R(r) = 0,$$

parallels a standard treatment of the hydrogen atom in three dimensions.³ Introducing $E = -2\pi^2\mu\epsilon^4/N^2\hbar^2$ and the variable x , defined by $r = N\hbar^2x/8\pi^2\mu\epsilon^2$, into the radial equation, solution can be found in the form $R(r) =$

³ H. Eyring, J. Walter, and G. E. Kimball, *Quantum Chemistry* (John Wiley & Sons, Inc., New York, 1944), p. 82.

$G(x)x^{|l|}e^{-x/2}$, where $G(x)$ must satisfy the differential equation

$$x[d^2G(x)/dx^2] + (2|l| + 1 - x) [dG(x)/dx] + (N - |l| - 1/2)G(x) = 0.$$

This latter equation can be identified with the differential equation that is satisfied by associated Laguerre polynomials $L_k^p(x)$, where $p = 2|l|$ and $k = |l| + N - 1/2$. Since p and k are both integral, with $p \leq k$, N is restricted to values $1/2, 3/2, 5/2 \dots$.

To facilitate comparison with the results from the hydrogen-atom problem in three dimensions, N can be redefined by $n = N + 1/2$. Then

$$\Psi_{n,l} = \beta \left\{ \frac{(n-1-|l|)!}{[(|l|+n-1)!]^3 (2n-1)} \right\}^{1/2} e^{-\beta r/2} (\beta r)^{|l|} L_{n+|l|-1}^{2|l|}(\beta r) \frac{e^{i\phi}}{(2\pi)^{1/2}}$$

with $\beta = x/r$. Limitations on the quantum numbers become $n = 1, 2, 3 \dots$ and $l = 0, \pm 1, \pm 2, \dots, \pm (n-1)$. The dependence of energy on n is

$$E_n = [-2\pi^2\mu\epsilon^4 / (n - 1/2)^2 h^2].$$

Table I is a listing of normalized radial functions, $R_{n,l}(r)$.

II. DISCUSSION

Denoting l' and m as the azimuthal and magnetic quantum numbers associated with the wavefunctions of a three-dimensional hydrogen atom, the angular wavefunctions Φ_l of a planar hydrogen atom correlate with the three-dimensional angular functions, when $m = \pm l'$. Angular-momentum vectors are perpendicular to the plane of motion in the two-dimensional case.

The radial wavefunctions $R_{n,l}$ indicate that, using the three-dimensional hydrogen atom as a reference, the electron distribution is displaced towards the nucleus. For the ground state, the radial distribution function $2\pi r \Psi_{1,0}^2$ has its maximum at r equal to $1/4$ of the Bohr radius, while the three-dimensional radial distribution function vs distance peaks at the Bohr radius. Ex-

TABLE I. Two-dimensional radial wavefunctions.

n	l	$R_{n,l}$
1	0	$\beta e^{-\beta r/2}$
2	0	$[\beta/(3)^{1/2}]e^{-\beta r/2} (1 - \beta r)$
2	+1, -1	$[\beta^2 r/(6)^{1/2}]e^{-\beta r/2}$
3	0	$[\beta/2(5)^{1/2}]e^{-\beta r/2} (2 - 4\beta r - \beta^2 r^2)$
3	+1, -1	$[\beta^2 r/(30)^{1/2}]e^{-\beta r/2} (18 - 6\beta r)$
3	+2, -2	$[\beta^3 r^2/(70)^{1/2}]e^{-\beta r/2}$
4	0	$[\beta/6(7)^{1/2}]e^{-\beta r/2} (6 - 18\beta r - 9\beta^2 r^2 - \beta^3 r^3)$
4	+1, -1	$[\beta^2 r/14]e^{-\beta r/2} (12 - 8\beta r + \beta^2 r^2)$
4	+2, -2	$[\beta^3 r^2/2(210)^{1/2}]e^{-\beta r/2} (5 - \beta r)$
4	+3, -3	$[\beta^4 r^3/6(35)^{1/2}]e^{-\beta r/2}$

cept for the shift in the electron cloud towards the nucleus, radial wavefunctions are otherwise very similar, since both are described by associated Laguerre polynomials.

Coincident with the shift in the radial distribution of the electron, energy levels are lower in the planar case. The energy of the ground state of the planar atom is four times that of the three-dimensional atom, each value being negative. For a one-dimensional hydrogen atom, the ground-state energy becomes negatively infinite, and the wavefunction concentrates sharply at the origin.² Other quantum mechanical systems, e.g., particle in a three-dimensional box, the three-dimensional harmonic oscillator, also exhibit lower ground-state energy levels when space dimensions are removed.

Further, it can be noted for a two-dimensional continuum, that Gauss' theorem is not satisfied when the potential energy of the proton and the electron is $-\epsilon^2/r$. Thus, with respect to electrodynamics, the special two-dimensional space, which has been formulated and studied above, is not analogous to real space. This consideration does not affect the angular electronic structure which is described by the Φ_l , since angular solutions are independent of the electrodynamics that operate, provided that the potential energy depends only on r .