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The Kepler Problem in Two-Dimensional Momentum Space*

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Fock studied the hydrogen atom problem in momentum space by projecting the space into a 4-dimensional hyperspherical space. He found that as a consequence of the symmetry of the problem in this space the eigenfunctions are the R_4 spherical harmonics and that the eigenvalues are determined only by the principal quantum number n. In this paper we note that if his method is applied to the 2-dimensional Kepler problem in momentum space, the eigenfunctions are R_3 spherical harmonics, Y_{lm}, and the eigenvalues are determined only by the quantum number l. These facts enable one to give a visualizable geometrical discussion of the dynamical degeneracy.

INTRODUCTION

A GEOMETRICAL illustration of a dynamical symmetry is often a great aid to the understanding. We have found that a 2-dimensional analog of Fock's treatment of the "accidental" degeneracy and "extra" symmetry of the hydrogen atom[1] is particularly helpful in this regard.

Fock began his discussion of the dynamical symmetry of the hydrogen atom with the momentum space Schrödinger (integral) equation for the atom. He made a change of the independent variables in this equation and then redefined his dependent variables to symmetrize the integral equation. We proceed analogously for a corresponding 2-dimensional problem.

I. FOCK TREATMENT OF THE TWO-DIMENSIONAL KEPLER PROBLEM

The Schrödinger wave equation for an electron in the potential, $V(\mathbf{r})$, is given in Hartree's atomic units by

$$(2E+\nabla^2)u(\mathbf{r})=2V(\mathbf{r})u(\mathbf{r}), \tag{1}$$

where $\mathbf{r}$ is the position vector of the electron, $u(\mathbf{r})$ the wavefunction, E the electronic energy associated with $u(\mathbf{r})$, and ∇^2 the Laplace operator for the electron.

This position space equation may be Fourier-transformed into a momentum space equation, which in the 2-dimensional case is

$$(p_0^2+p^2)\phi(\mathbf{p})=-2\int V'(\mathbf{p}-\mathbf{p}')\phi(\mathbf{p}')d^2p'. \tag{2}$$

The corresponding equation in the 3-dimensional case is formally the same if d^2p is changed to d^3p. In Eq. (2),

$$p_0^2=-2E; \tag{3}$$

$$\phi(\mathbf{p})=(2\pi)^{-1}\int e^{-i\mathbf{p}\cdot\mathbf{r}}u(\mathbf{r})d^2r,$$
$$u(\mathbf{r})=(2\pi)^{-1}\int e^{i\mathbf{p}\cdot\mathbf{r}}\phi(\mathbf{p})d^2p; \tag{4}$$

and

$$V'(\mathbf{p})=(2\pi)^{-2}\int e^{-i\mathbf{p}\cdot\mathbf{r}}u(\mathbf{r})d^2r,$$
$$V(\mathbf{r})=\int e^{i\mathbf{p}\cdot\mathbf{r}}V'(\mathbf{p})d^2p. \tag{5}$$

Analogous equations hold in the 3-dimensional problem if $2\pi\rightarrow(2\pi)^{\frac{3}{2}}$ and $d^2\rightarrow d^3$.

As shown in the Appendix, if the 2-dimensional potential is "Coulombic," that is,

$$V(\mathbf{r})=-Z/r, \tag{6}$$

where Z is the charge of the center, then

$$V'(\mathbf{p})=-Z/2\pi p. \tag{7}$$

Substitution of (7) into (2) gives

$$(p_0^2+p^2)\phi(\mathbf{p})=\frac{Z}{\pi}\int\frac{\phi(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|}d^2p'. \tag{8}$$

* This is taken from the introduction to the M.Sc. thesis of Tai-ichi Shibuya.
[1] V. Fock, Z. Physik 98, 145 (1935).

Now we may project the 2-dimensional momentum space onto the 3-dimensional sphere of radius p_0 in the same way that Fock did for the 3-dimensional momentum space problem. A sphere is drawn around the origin of the p space so that the center of the sphere coincides with the origin. The projective origin is taken at the south pole S. Then a vector $\mathbf{p}$, whose components are p_x, p_y, is projected onto the sphere where it may be described as $(p_0; \theta, \varphi)$ using ordinary spherical polar coordinates. From Fig. 1, we find that, for this stereographic projection,

$$p \rightarrow p_0 \tan\frac{\theta}{2}; \quad \varphi \rightarrow \varphi. \tag{9}$$

If the rectangular coordinates of the point P $(p_0; \theta, \varphi)$ are x, y, and z in a three-dimensional Cartesian space with origin at the center of the sphere, then we may define our variables so that

$$(x/p_0)^2 + (y/p_0)^2 + (z/p_0)^2 = 1, \tag{10}$$

$$\begin{cases} x/p_0 = \sin\theta \cos\varphi = 2p_0 p_x/(p_0^2+p^2), \\ y/p_0 = \sin\theta \sin\varphi = 2p_0 p_y/(p_0^2+p^2), \\ z/p_0 = \cos\theta = (p_0^2-p^2)/(p_0^2+p^2). \end{cases} \tag{11}$$

Let γ be the angle spanned between P $(p_0; \theta, \varphi)$ and P' $(p_0; \theta', \varphi')$ on the surface of the sphere. Then $2p_0 \sin\gamma/2$ is the distance between P and P', and

$$\left(2\sin\frac{\gamma}{2}\right)^2 = \left(\frac{x-x'}{p_0}\right)^2 + \left(\frac{y-y'}{p_0}\right)^2 + \left(\frac{z-z'}{p_0}\right)$$
$$= \frac{4p_0^2 |\mathbf{p}-\mathbf{p}'|^2}{(p_0^2+p^2)(p_0^2+p'^2)}. \tag{12}$$

The area element on the unit sphere is, by the third equation of (11),

$$d\Omega \equiv \sin\theta d\theta d\varphi$$
$$= \left(\frac{2p_0}{p_0^2+p^2}\right)^2 p dp d\varphi \equiv \left(\frac{2p_0}{p_0^2+p^2}\right)^2 d^2p. \tag{13}$$

Using (12) and (13) we can temporarily write Eq. (8) as

$$(p_0^2+p^2)^{\frac{3}{2}}\phi(\mathbf{p}) = \frac{Z/p_0}{2\pi}$$
$$\times \int \frac{1}{2\sin(\gamma/2)} (p_0^2+p'^2)^{\frac{3}{2}}\phi(\mathbf{p}')d\Omega'. \tag{14}$$

This is a symmetric integral equation, so that the function $(p_0^2+p^2)^{\frac{3}{2}}\phi(\mathbf{p})$ may be found as an eigenfunction of the kernel $(2\sin\gamma/2)^{-1}$. Introducing a new function,[2]

$$\psi(\Omega) \equiv [(8)^{\frac{1}{2}} p_0^2]^{-1} (p_0^2+p^2)^{\frac{3}{2}}\phi(\mathbf{p}), \tag{15}$$

into Eq. (14), we obtain

$$\psi(\Omega) = \frac{Z/p_0}{2\pi} \int \frac{1}{2\sin(\gamma/2)} \psi(\Omega')d\Omega', \tag{16}$$

where Ω means the collection of variables θ, φ.

It is a well-known result of the theory of homogeneous symmetric integral equations that if one can find an expansion of a kernel $K(\Omega,\Omega')$ as $\sum_n \lambda_n f_n^*(\Omega') f_n(\Omega)$ then the eigenfunctions and eigenvalues of the equation are $f_n(\Omega)$ and λ_n.[3] It is not difficult to find such an expansion for the kernel $(2\sin\gamma/2)^{-1}$: The common expansion of the 3-dimensional Coulomb potential

$$\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \equiv \frac{1}{r_{ij}} = \sum_{lm} \frac{r_<^l}{r_>^{l+1}} \frac{2\pi}{l+\frac{1}{2}} Y_{lm}(\Omega) Y_{lm}^*(\Omega'), \tag{17}$$

gives on the unit sphere

$$\left(\frac{1}{r_{ij}}\right)_{r_i=r_j=1} = \frac{1}{2\sin(\gamma/2)} = \sum_{lm} \frac{2\pi}{l+\frac{1}{2}}$$
$$\times Y_{lm}(\Omega) Y_{lm}^*(\Omega'). \tag{18}$$

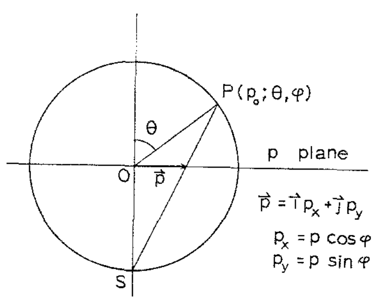

FIG. 1.

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[2] The factor $(\sqrt{8}p_0^2)^{-1}$ is inserted to normalize the function $\psi(\Omega)$ to 1.

[3] See, for example, S. G. Mikhlin, *Integral Equations* (Pergamon Press Inc., New York, 1957) (in translation).

Thus, if

$$\psi_{lm}(\Omega)=Y_{lm}(\Omega) \tag{19}$$

and

$$(Z/p_0)_{lm}=l+\tfrac{1}{2} \quad \text{or} \quad E_{lm}=-\tfrac{1}{2}\left(\frac{Z}{l+\frac{1}{2}}\right)^2, \tag{20}$$

then Eq. (16) is solved. This may be verified by substitution.

Now, in solutions (19) and (20), the quantum number m can take all the integral values from $-l$ to l, so that there exist $2l+1$ eigenfunctions which belong to the energy state l: in other words, the degeneracy of the state is $2l+1$. From (19) and (20) we see that the ground state wavefunction is

$$Y_{00}=\frac{1}{(4\pi)^{\frac{1}{2}}}, \tag{21}$$

and the first excited state wavefunctions are either

$$\begin{aligned} Y_{10}&=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\frac{z}{p_0};\\ Y_{1\pm1}&=\left(\frac{3}{8\pi}\right)^{\frac{1}{2}}\sin\theta e^{\pm i\varphi}; \end{aligned} \tag{22}$$

or any linear combinations, such as the real functions:

$$\left\{\begin{aligned} \frac{1}{\sqrt{2}}[Y_{11}+Y_{1-1}]&=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\sin\theta\cos\varphi\\ &=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\frac{x}{p_0},\\ \frac{-i}{\sqrt{2}}[Y_{11}-Y_{1-1}]&=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\sin\theta\sin\varphi\\ &=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\frac{y}{p_0},\\ Y_{10}&=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\frac{z}{p_0}. \end{aligned}\right. \tag{23}$$

The momentum-space wavefunctions corresponding to the Eqs. of (21) and (23) are, respectively,

$$\phi_{00}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}p_0^2(p_0^2+p^2)^{-\frac{3}{2}}, \tag{24}$$

and

$$\left\{\begin{aligned} \frac{1}{\sqrt{2}}[\phi_{11}+\phi_{1-1}]&=\left(\frac{6}{\pi}\right)^{\frac{1}{2}}p_0^2(p_0^2+p^2)^{-\frac{5}{2}}\cdot 2p_0p_x,\\ \frac{-i}{\sqrt{2}}[\phi_{11}-\phi_{1-1}]&=\left(\frac{6}{\pi}\right)^{\frac{1}{2}}p_0^2(p_0^2+p^2)^{-\frac{5}{2}}\cdot 2p_0p_y,\\ \phi_{10}&=\left(\frac{6}{\pi}\right)^{\frac{1}{2}}p_0^2(p_0^2+p^2)^{-\frac{5}{2}}\cdot(p_0^2-p^2). \end{aligned}\right. \tag{25}$$

The position-space wavefunctions corresponding to these functions in momentum space or in Fock's projective space may be obtained by Fourier transformation [using the second equation of (4)], or may be found directly as solutions of the position space Schrödinger Eq. (1). This is left as an exercise for the interested reader.

For the 3-dimensional Kepler problem, Fock obtained as the Schrödinger equation in his projective momentum space on the 4-dimensional hypersphere,

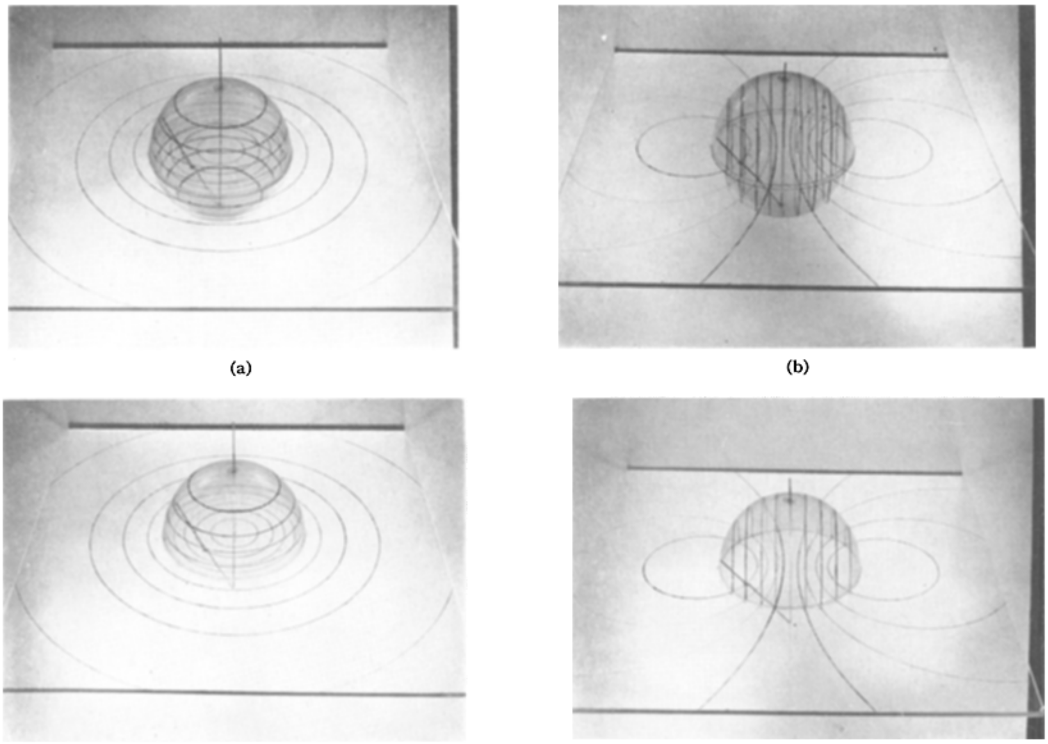
$$\psi(\Omega_4)=\frac{Z/p_0}{2\pi^2}\int\frac{1}{(2\sin\frac{1}{2}\omega)^2}\psi(\Omega_4')d\Omega_4'. \tag{26}$$

Here ω is the angle analogous to γ in (12) but now on the 4-dimensional hypersphere, so that $2\sin\frac{1}{2}\omega$ is the "distance" between two points on the unit hypersphere spanned by the angle ω; Ω_4 is the collection of angular variables α, θ, and φ ($p\rightarrow p_0\tan\alpha/2$, $\theta\rightarrow\theta$, $\varphi\rightarrow\varphi$) for the 4-dimensional hypersphere, and $d\Omega_4$ is the "area" element on the hypersphere. He solved this equation finding that the eigenfunctions are the R_4 spherical harmonics; that is,

$$\psi_{nlm}(\Omega_4)=Y_{nlm}(\Omega_4)\equiv A_{nl}(\alpha)Y_{lm}(\Omega), \tag{27}$$

where

$$\int Y_{nlm}^*(\Omega_4)Y_{n'l'm'}(\Omega_4)d\Omega_4=\delta_{nlm,n'l'm'} \tag{28}$$



(a) (b) (a′) (b′)

FIG. 2. Photograph (a): $|\phi_{10}|\leftrightarrow|Y_{10}|$; photograph (b): $\left|\frac{1}{\sqrt{2}}(\phi_{11}+\phi_{1-1})\right|\leftrightarrow\left|\frac{1}{\sqrt{2}}(Y_{11}+Y_{1-1})\right|$ $\left(\text{or } \left|\frac{-i}{\sqrt{2}}(\phi_{11}-\phi_{1-1})\right|\leftrightarrow\left|\frac{-i}{\sqrt{2}}(Y_{11}-Y_{1-1})\right|\right)$; photographs (a′) and (b′): same as (a) and (b), respectively, except that the southern hemispheres are removed.

and

$$A_{nl}(\alpha)=\frac{(\pi/2)^{\frac{1}{2}}\sin^l\alpha}{[n^2(n^2-1^2)\cdots(n^2-l^2)]^{\frac{1}{2}}}\times\frac{d^{l+1}}{d(\cos\alpha)^{l+1}}\cos n\alpha, \quad (29)$$

and the eigenvalues are

$$(Z/p_0)_{nlm}=n \quad \text{or} \quad E_{nlm}=-\tfrac{1}{2}\left(\frac{Z}{n}\right)^2. \quad (30)$$

Since the quantum number l can take all the integral values from 0 to $n-1$, and the quantum number m from $-l$ to l for each l state, the degeneracy of the energy state n is

$$\sum_{l=0}^{n-1}(2l+1)=n^2.$$

II. DISCUSSION OF THE DEGENERACIES

The degeneracies in the 2- and 3-dimensional Kepler problems arise because, in the 2-dimensional problem the momentum space "potential" has the symmetry of the 3-dimensional sphere, while, in the 3-dimensional problem, the momentum space "potential" has the symmetry of the 4-dimensional hypersphere. The way in which functions which are obviously degenerate on the sphere project into functions which are obviously different on the plane is illustrated for the first excited state wavefunctions of the 2-dimensional problem in Fig. 2.

In Fig. 2, photographs (a) and (b) correspond to the cases of the 2-dimensional first excited "s" state Y_{10}, and the 2-dimensional first excited "p" state $2^{-\frac{1}{2}}(Y_{11}+Y_{1-1})$ [or $-i2^{-\frac{1}{2}}(Y_{11}-Y_{1-1})$]. Pictures (a) and (b) indicate that in the angular space we can not distinguish one distribution on the sphere from another, since the distributions differ only in orientation. On the other hand the projected functions in the p plane are clearly different. The "s" function has a circular node at the equator of the sphere and the "p" function has a linear node parallel to the x or y axis. Photographs (a′) and (b′) indicate more clearly how the lines on the spheres project into those in the p planes. The lines drawn on the northern hemisphere correspond to those in the p plane inside of the equator of the sphere whose radius is p_0, and the lines on the southern hemisphere correspond to those in the p plane outside of the equator.

Exactly analgous relationships exist for the ordinary hydrogen–atom problem.

APPENDIX: THE FOURIER TRANSFORMATION OF THE 2-DIMENSIONAL COULOMBIC POTENTIAL

If the potential $V(\mathbf{r})$ is given, in 2-dimensional position space, by

$$V(\mathbf{r})=\frac{-Z}{r}e^{-\alpha r}, \tag{A1}$$

then its Fourier transform $V'(\mathbf{p})$ in the 2-dimensional momentum space is obtained as follows:

$$\begin{aligned}V'(\mathbf{p})&=(2\pi)^{-2}\int e^{-i\mathbf{p}\cdot\mathbf{r}}\frac{-Z}{r}e^{-\alpha r}d^2r\\&=(2\pi)^{-2}(-Z)\int_0^\infty dre^{-\alpha r}\\&\qquad\times\int_0^{2\pi}e^{-ipr\cos\theta}d\theta.\end{aligned} \tag{A2}$$

Now

$$\begin{aligned}\int_0^{2\pi}e^{-ipr\cos\theta}d\theta&=2\int_0^\pi\cos(pr\cos\theta)d\theta\\&=2\pi J_0(pr).^4\end{aligned} \tag{A3}$$

$$\begin{aligned}\therefore\quad V'(\mathbf{p})&=(2\pi)^{-1}(-Z)\int_0^\infty e^{-\alpha r}J_0(pr)dr\\&=\frac{-Z}{2\pi(p^2+\alpha^2)^{\frac{1}{2}}}.^5\end{aligned} \tag{A4}$$

When $\alpha\rightarrow 0$, we have

$$V(\mathbf{r})=-Z/r \tag{A5}$$

and the Fourier transform

$$V'(\mathbf{p})=-Z/2\pi p. \tag{A6}$$

[4] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1958), p. 10.
[5] Reference 4, p. 384.