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Coherent States and the Forced Quantum Oscillator*

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It is shown that the forced quantum oscillator subject to a transient classical force is easily described in terms of the "coherent states" recently found useful in the description of light from optical masers. The natural role these states play in understanding the notions of the phase of a quantum oscillator and the transition to the classical limit is also explained. Very simple derivations of the state vectors, energy transfer, and various transition probabilities are given.

I. INTRODUCTION

RECENT papers in this journal have applied powerful formal methods having their origin in quantum field theory to the study of the one-dimensional quantum mechanical oscillator subject to a transient classical force.[1,2] Although this problem arises in many contexts[3–5] and has been solved in many ways,[1–6] we wish to present a discussion which is not only extremely simple (on the level of a first year graduate course in quantum mechanics), but also provides insight into the concept of the phase of a quantum oscillator and the classical limit. The simplicity of our discussion rests on the use of the "coherent states" of an oscillator.[7] These states, which go over into coherent classical states for large quantum numbers, have recently been recognized to be of great utility in the quantum mechanical description of coherence of light from optical masers.

In Sec. II a summary of the harmonic oscillator variables and other conventions required is given. Section III is largely a summary of the properties of the coherent states, following Glauber.[7] In Sec. IV the forced oscillator problem is formulated in the Heisenberg picture in order to keep closely to the classical interpretation of the result. A simple Green's function method is used to construct the states of the system. In Sec. V the transition probability between arbitrary number states of the oscillator is computed in a completely elementary way, in contrast to the intricate methods used in Refs. 1 and 2.

II. HARMONIC OSCILLATOR IN QUANTUM MECHANICS

The position x of a classical oscillator of mass m, spring constant k subject to a uniform driving force $F(t)$ obeys Newton's equation of motion

$$m\ddot{x}+kx=F(t). \tag{2.1}$$

Until Sec. IV we shall be concerned only with the free oscillator, in which case $F(t)\equiv 0$. The solution of (2.1) is then

$$x=|A|\cos(\omega t-\phi), \tag{2.2}$$

where $\omega=(k/m)^{\frac{1}{2}}$ is the (circular) frequency, $|A|$ the amplitude and ϕ the phase angle of the oscillator. The point to be observed here is that the description of motion of a classical oscillator requires the specification of both amplitude and phase. It is not possible to ascribe such detailed information to the quantum oscillator.

To discuss the quantum oscillator we write

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[1] R. W. Fuller, S. M. Harris, and E. L. Slaggie, Am. J. Phys. **31**, 431 (1963).

[2] L. M. Scarfone, Am. J. Phys. **32**, 158 (1964).

[3] The most important example arises in the study of the infrared divergence problem, reviewed in Ref. 4.

[4] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Chap. 16.

[5] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

[6] R. P. Feynman, Rev. Mod. Phys. **20**, 267 (1948).

[7] See, especially, R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

down the classical energy, or Hamiltonian,

$$H=\frac{p^2}{2m}+\tfrac{1}{2}m\omega^2x^2, \tag{2.3}$$

where $p=m\dot{x}$ is the momentum conjugate to x,

$$xp-px\equiv[x,p]=i\hbar. \tag{2.4}$$

Let us work in the Heisenberg picture,[8] wherein state vectors are constants in time, the operators $\mathcal{O}(t)$ varying according to

$$i\hbar\dot{\mathcal{O}}(t)=[\mathcal{O}(t),H]. \tag{2.5}$$

From Eqs. (2.3)–(2.5) one finds

$$\begin{aligned}\dot{x}(t)&=p(t)/m,\\ \dot{p}(t)&=-m\omega^2x(t),\end{aligned} \tag{2.6}$$

which when combined show that the harmonic oscillator equation

$$\ddot{x}(t)+\omega^2x(t)=0 \tag{2.7}$$

is an operator identity satisfied by $x(t)$.

In order to discuss the eigenvalue spectrum and the relations among the eigenfunctions it is convenient to introduce two new dynamical variables, to be used in place of the old ones, p and x. To discover these let us note that the classical solution (2.2) can be written as

$$x_{\text{classical}}=\tfrac{1}{2}Ae^{-i\omega t}+\tfrac{1}{2}A^*e^{+i\omega t}, \tag{2.8}$$

where $A=|A|e^{i\phi}$. Here the asterisk denotes complex conjugate. In analogy we invent the time-independent dimensionless operators a and $a^\dagger$ ($a^\dagger$ is the Hermitian adjoint of a) by

$$x(t)=x_0[ae^{-i\omega t}+a^\dagger e^{i\omega t}], \tag{2.9}$$

where for convenience the length

$$x_0=(\hbar/2m\omega)^{\frac{1}{2}} \tag{2.10}$$

has been factored out. x_0 turns out to be the root-mean-square zero-point displacement. The momentum operator is accordingly

$$p(t)=-im\omega x_0[ae^{-i\omega t}-a^\dagger e^{i\omega t}]. \tag{2.11}$$

Alternatively, we can invert (2.10) and (2.11) to obtain a definition of a and $a^\dagger$ in terms of x and p

$$\begin{aligned}a(t)&=\frac{i}{(2m\hbar\omega)^{\frac{1}{2}}}[p(t)-im\omega x(t)],\\ a^\dagger(t)&=\frac{-i}{(2m\hbar\omega)^{\frac{1}{2}}}[p(t)+im\omega x(t)],\end{aligned} \tag{2.12}$$

where $a(t)$ and $a^\dagger(t)$ are given by

$$a(t)=ae^{-i\omega t},\quad a^\dagger(t)=a^\dagger e^{i\omega t}. \tag{2.13}$$

One notes the basic commutation rule

$$[a,a^\dagger]=1, \tag{2.14}$$

as follows from (2.12) and (2.4). In addition to (2.14) one has the trivial relations

$$[a,a]=[a^\dagger,a^\dagger]=0. \tag{2.15}$$

The Hamiltonian can now be expressed in the form

$$H=\tfrac{1}{2}\hbar\omega(a^\dagger a+aa^\dagger)=(a^\dagger a+\tfrac{1}{2})\hbar\omega. \tag{2.16}$$

Now we may verify from (2.14) and (2.5) that (2.13) is valid. $a(t)$ and $a^\dagger(t)$ are *normal mode* operators

$$i\hbar\dot{a}(t)=[a(t),H]=\hbar\omega a(t),\ \ddot{a}(t)+\omega^2a(t)=0. \tag{2.17}$$

As everybody knows, the eigenvalues of (2.16) are

$$E_n=(n+\tfrac{1}{2})\hbar\omega\quad n=0,1,2,\ldots, \tag{2.18}$$

so that the *number operator*

$$N_{\text{op}}\equiv a^\dagger a \tag{2.19}$$

has eigenvalues 0, 1, 2, ... The orthonormal eigenfunctions ψ_n belonging to E_n are given by

$$\psi_n=\frac{1}{(n!)^{\frac{1}{2}}}(a^\dagger)^n\psi_0, \tag{2.20}$$

where the ground state ψ_0 of the oscillator is defined by

$$a\psi_0=0. \tag{2.21}$$

The $a^\dagger$ and a operators raise or lower [with the exception of (2.21)] the degree of excitation of the oscillator by $\hbar\omega$

$$a^\dagger\psi_n=(n+1)^{\frac{1}{2}}\psi_{n+1}, \tag{2.22}$$

$$a\psi_n=(n)^{\frac{1}{2}}\psi_{n-1}, \tag{2.23}$$

[8] Heisenberg picture operators are labeled explicitly by the time variable t; otherwise the Schrödinger picture operator is meant.

from which we obtain the matrix elements

$$\langle m|a^\dagger|n\rangle=(n+1)^{\frac{1}{2}}\delta_{m,n+1},$$
$$\langle m|a|n\rangle=(n)^{\frac{1}{2}}\delta_{m,n-1}. \quad (2.24)$$

An oscillator excited to its nth quantum state behaves just as a collection of n indistinguishable Bose particles. One conventionally says that $a^\dagger$ creates (a destroys) a quantum $\hbar\omega$. Thus, the a's are called destruction, annihilation, or absorption operators according to the author's taste. The $a^\dagger$'s are accordingly called creation, or emission operators.

Let us now verify the claim that x_0 defined in Eq. (2.10) is the rms zero-point displacement, as claimed. We have

$$\langle x^2\rangle_0\equiv\langle 0|x^2|0\rangle=x_0^2\langle(a+a^\dagger)^2\rangle_0$$
$$=x_0^2=\hbar/2m\omega \quad (2.25)$$

on using the commutation rules and (2.21).

However, the mean position and momentum vanish in *any* number eigenstate, regardless of the size of n

$$\langle n|x|n\rangle=\langle n|p|n\rangle=0, \quad (2.26)$$

since x and p are nondiagonal operators in this so-called number representation in which we are working. Thus, the number states are definitely not the appropriate ones for a transition to the classical limit. The physical reason for this result is that the phase is completely undefined once the excitation number is specified, leading to the result (2.25). This "complementarity" of number and phase requires a careful discussion in order to make precise the meaning of the idea of the phase of a quantum oscillator.[9] The present problem is closely related to the subtle interpretation required of the energy–time uncertainty relation.

III. COHERENT STATES OF AN OSCILLATOR

What quantum mechanical states correspond to the kinds of motion encountered in "classical" circumstances? Besides having a high degree of mean excitation, they must satisfy, at least to a good approximation,

$$\langle\Psi_{\text{class}}|x(t)|\Psi_{\text{class}}\rangle=\text{const}\cos(\omega t-\phi). \quad (3.1)$$

As a matter of fact, we can find states satisfying (3.1) exactly, regardless of the number of oscillator quanta in the state.[7] Of course, from this set of states only those having $\langle\Psi|N_{\text{op}}|\Psi\rangle\gg1$ are necessary for the particular job of satisfying the correspondence principle. However, we wish to emphasize the utility of these states for a wide variety of problems not necessarily involving large quantum numbers.

Comparing Eq. (2.9) with (3.1) shows immediately that if we can find states $|\alpha\rangle$ which are eigenfunctions of the annihilation operator with complex eigenvalue α

$$a|\alpha\rangle=\alpha|\alpha\rangle, \quad (3.2)$$

then the mean value of $x(t)$ in these states

$$\langle\alpha|x(t)|\alpha\rangle=x_0(\alpha e^{-i\omega t}+\alpha^*e^{+i\omega t})\langle\alpha|\alpha\rangle \quad (3.3)$$

does indeed agree with (3.1) where the phase angle ϕ is given by

$$\alpha=|\alpha|e^{i\phi}. \quad (3.4)$$

The modulus of α is related to the mean excitation of the oscillator by

$$N=\frac{\langle\alpha|a^\dagger a|\alpha\rangle}{\langle\alpha|\alpha\rangle}=|\alpha|^2. \quad (3.5)$$

Thus, we can write the *eigenvalue* of a in the unsurprising form $(N)^{\frac{1}{2}}e^{i\phi}$. However, the corresponding decomposition of the *operator* a into $(N_{\text{op}})^{\frac{1}{2}}\exp(i\phi_{\text{op}})$ where ϕ_{op} is a Hermitian operator, is impossible.[9]

The states $|\alpha\rangle$ can be constructed out of number eigenstates

$$|\alpha\rangle=\sum_{n=0}^{\infty}|n\rangle\langle n|\alpha\rangle. \quad (3.6)$$

From (3.2) and (2.24) we find the expansion coefficient

$$\langle n|\alpha\rangle=\frac{\alpha}{(n)^{\frac{1}{2}}}\langle n-1|\alpha\rangle=\cdots=\frac{\alpha^n}{(n!)^{\frac{1}{2}}}\langle 0|\alpha\rangle, \quad (3.7)$$

whence

$$|\alpha\rangle=\langle 0|\alpha\rangle\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{\frac{1}{2}}}|n\rangle. \quad (3.8)$$

[9] L. Susskind and J. Glogower, Physics 1, 49 (1964) have given a careful and mathematically precise discussion of this question.

Normalizing $|\alpha\rangle$ determines $\langle 0|\alpha\rangle$ up to an arbitrary phase chosen as follows

$$1=\langle\alpha|\alpha\rangle=|\langle 0|\alpha\rangle|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}$$
$$=\exp(|\alpha|^2)|\langle 0|\alpha\rangle|^2,$$
$$\langle 0|\alpha\rangle=\exp(-\tfrac{1}{2}|\alpha|^2). \quad (3.9)$$

Finally,

$$|\alpha\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{\frac{1}{2}}}|n\rangle. \quad (3.10)$$

The probability of finding the oscillator in the nth level in the state $|\alpha\rangle$ is

$$P_n(\alpha)=e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}=\frac{e^{-N}N^n}{n!}. \quad (3.11)$$

We have therefore derived the familiar Poisson distribution which also expresses the photon distribution in classical waves.

The expression (3.10) for $|\alpha\rangle$ can be simplified further by introducing (2.20)

$$|\alpha\rangle=e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha a^\dagger)^n}{n!}|0\rangle$$
$$\equiv\exp[\alpha a^\dagger-\tfrac{1}{2}|\alpha|^2]|0\rangle. \quad (3.12)$$

Thus, by applying a very simple operator to the ground state, one generates a *coherent state* $|\alpha\rangle$. These states are the well-known minimum wave packets of a displaced harmonic-oscillator ground state, discussed very clearly in the book by Henley and Thirring.[10] In the Schrödinger picture the states evolve in time according to

$$|\alpha(t)\rangle\equiv e^{-iHt}|\alpha\rangle=\exp[\alpha a^\dagger e^{i\omega t}-\tfrac{1}{2}|\alpha|^2]|0\rangle. \quad (3.13)$$

This state describes a displaced ground-state wavefunction which vibrates back and forth with frequency ω without any change of shape or spreading.

The fact that $|\alpha\rangle$ and $|0\rangle$ are both normalized suggests that $\exp[\alpha a^\dagger-\frac{1}{2}|\alpha|^2]$ is equivalent to a unitary operator. Such an operator is expected to be of the form exp(ih) with h Hermitian. This suggests that we replace (3.12) by

$$|\alpha\rangle\equiv A(\alpha)|0\rangle, \quad A(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^*a). \quad (3.14)$$

[10] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, Inc., New York, 1962), Chap. 2.

That this is correct follows from the identity[11,12]

$$\exp A\,\exp B=\exp(A+B+\tfrac{1}{2}[A,B]), \quad (3.15)$$

valid when A and B commute with their commutator $[A,B]$. Specifically we get

$$\exp(\alpha a^\dagger)\exp(-\alpha a)$$
$$=\exp[\alpha a^\dagger-\alpha^*a]\exp(\tfrac{1}{2}|\alpha|^2), \quad (3.16)$$

which establishes the equivalence of (3.12) and (3.14).

The virtues of the operator $A(\alpha)$ are exhibited by the following properties. First, we note the relation

$$A(\alpha)=A^\dagger(-\alpha), \quad (3.17)$$

by means of which the unitarity statement

$$A^\dagger(\alpha)A(\alpha)=A(\alpha)A^\dagger(\alpha)=1 \quad (3.18)$$

takes the form[13]

$$A(-\alpha)A(\alpha)=A(\alpha)A(-\alpha)=1. \quad (3.19)$$

From these relations we see that $A(\alpha)$ and $A(-\alpha)$ are a kind of creation and annihilation operator for the coherent states. In particular,

$$|\alpha\rangle=A(\alpha)|0\rangle;\quad A(-\alpha)|\alpha\rangle=|0\rangle. \quad (3.20)$$

The basic origin of these properties lies in the fact that the $A(\alpha)$ are displacement operators of the normal coordinates a and $a^\dagger$. To show this we need to use the identity

$$[a,(a^\dagger)^n]=n(a^\dagger)^{n-1}, \quad (3.21)$$

which gives us

$$[a,\exp(\alpha a^\dagger)]=\sum_{n=0}^{\infty}[a,(a^\dagger)^n]\frac{\alpha^n}{n!}$$
$$=\alpha\exp(\alpha a^\dagger). \quad (3.22)$$

It follows that

$$[a,A(\alpha)]=e^{-\frac{1}{2}|\alpha|^2}[a,e^{\alpha a^\dagger}]e^{\alpha a}$$
$$=\alpha A(\alpha),$$

[11] This identity (Baker–Hausdorff) is proved in most modern books on quantum mechanics, for instance, Ref. 12.

[12] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1964), Vol. I, p. 442.

[13] The mathematically inclined reader can note that the unitary operators $A(\alpha)$ give a non-Abelian ray representation of the Abelian group of phase translations in the variable α of the coherent states $|\alpha\rangle$, where α may range over the complex α plane. The multiplication law is $A(\alpha_2)A(\alpha_1)=\exp[\frac{1}{2}(\alpha_2\alpha_1^*-\alpha_1\alpha_2^*)]A(\alpha_1+\alpha_2)$. The unimodular exponential factor vanishes only for real α_1, α_2 or for the special case $\alpha_2=\pm\alpha_1$.

so that rearrangement yields

$$A^\dagger(\alpha)aA(\alpha)=A^\dagger(\alpha)A(\alpha)a+A^\dagger(\alpha)[a,A(\alpha)]$$
$$=a+\alpha. \quad (3.23)$$

From this we find

$$A^\dagger(\alpha)a^\dagger A(\alpha)=a^\dagger+\alpha^*. \quad (3.24)$$

Let us now prove that the (nonspreading) states $|\alpha\rangle$ give rise to the minimum uncertainty product. First note that

$$\begin{aligned}(\bar{x})_\alpha{}^2&\equiv(\langle\alpha|x|\alpha\rangle)^2=x_0{}^2(\alpha+\alpha^*)^2,\\ (\overline{x^2})_\alpha&\equiv\langle\alpha|x^2|\alpha\rangle=x_0{}^2\langle\alpha|(a+a^\dagger)^2|\alpha\rangle\\ &=(\bar{x})_\alpha{}^2+x_0{}^2;\\ (\Delta x)^2&=\overline{x^2}-\bar{x}^2=x_0{}^2=\hbar/2m\omega.\end{aligned} \quad (3.25)$$

Hence, the rms displacement is the same as that in the ground state. A similar calculation shows that

$$(\Delta p)^2=m^2\omega^2x_0{}^2=\tfrac{1}{2}m\hbar\omega, \quad (3.26)$$

whence

$$\Delta p\Delta x=\tfrac{1}{2}\hbar. \quad (3.27)$$

Thus, the states $|\alpha\rangle$ are as "classical as possible" according to the principles of quantum mechanics, in the sense of (3.27).

Next, let us consider questions of orthogonality and completeness of the coherent states $|\alpha\rangle$. As these are eigenfunctions of a non-Hermitian operator there is no guarantee that states with differing phase are orthogonal. From Eq. (3.10) we compute the inner product $\langle\alpha|\beta\rangle$:

$$\begin{aligned}\langle\alpha|\beta\rangle&=\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]\sum_{mn}\frac{(\alpha^*)^m\beta^n}{(m!n!)^{\frac{1}{2}}}\langle n|m\rangle\\ &=\exp(\alpha^*\beta-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2);\\ |\langle\alpha|\beta\rangle|^2&=\exp(-|\alpha-\beta|^2).\end{aligned} \quad (3.28)$$

Despite this lack of orthogonality, the $|\alpha\rangle$ states are complete. More precisely, we prove the following identity[7,14]

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \quad (3.29)$$

where $d^2\alpha=d\ \mathrm{Re}\,\alpha\ d\ \mathrm{Im}\,\alpha$ and the integration is to be taken over the whole complex α plane.

[14] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

Transforming to polar coordinates $\alpha=re^{i\theta}$ and using (3.10) to express (3.29) in terms of number states converts Eq. (3.29) to

$$\begin{aligned}&\sum_{mn}\frac{m\rangle\langle n|}{(m!n!)^{\frac{1}{2}}}\frac{1}{\pi}\int_0^\infty dre^{-r^2}r^{m+n+1}\int_0^{2\pi}d\theta e^{i(m-n)\theta}\\ &=\sum_m|m\rangle\langle m|\frac{1}{m!}\int_0^\infty dr^2e^{-r^2}r^{2m}\\ &=\sum_m|m\rangle\langle m|=1.\end{aligned} \quad (3.30)$$

By means of Eq. (3.29) one can expand state vectors and matrix elements in terms of the coherent states. For more details on such procedures, and in particular for a discussion of special density matrices (coherent states, black-body radiation, etc.) the papers by Glauber[7] and Sudarshan[14] should be consulted.

IV. THE FORCED QUANTUM OSCILLATOR

We now show that the coherent states have a wider range of utility than simply to describe a highly excited oscillator. We show that the application of a "classical" driving force (i.e., one which is unaffected by the motion of the oscillator) generates coherent states regardless of the size of the average excitation of the oscillator.

We wish to solve the "scattering" problem of a quantum oscillator subject to a position independent classical force $F(t)$. The interaction energy of the oscillator with this force may be taken to be[1]

$$V=-xF(t)=-x_0(a+a^\dagger)F(t). \quad (4.1)$$

For orientation let us first consider the trivial problem of a constant force F_0. Then the oscillator is simply displaced to a different equilibrium position. The Hamiltonian

$$H=\hbar\omega(a^\dagger a+\tfrac{1}{2})-x_0(a+a^\dagger)F(t), \quad (4.2)$$

specialized to $F(t)=F_0$, can be diagonalized by "completing the square"

$$H=\hbar\omega(b^\dagger b+\tfrac{1}{2})-\frac{(x_0F_0)^2}{\hbar\omega}, \quad (4.3)$$

where the new normal coordinates b are

$$b=a-x_0F_0/\hbar\omega. \quad (4.4)$$

The b operators obey $[b,b^\dagger]=1$ and the eigenvalue spectrum is changed only by an over-all downward shift by an amount $-x_0{}^2F_0{}^2/\hbar\omega = -F_0{}^2/2m\omega^2$. The new ground state $|0\rangle'$ obeys

$$b|0\rangle'=0, \quad \text{or} \quad a|0\rangle'=(x_0F_0/\hbar\omega)|0\rangle', \tag{4.5}$$

so that in terms of the original coordinates and ground state $|0\rangle$

$$|0\rangle'=A\left(\frac{x_0F_0}{\hbar\omega}\right)|0\rangle=\exp\left[\frac{x_0F_0}{\hbar\omega}(a^\dagger-a)\right]|0\rangle. \tag{4.6}$$

The excited states of the displaced oscillator are found by applying $b^\dagger=a^\dagger-x_0F_0/\hbar\omega$ to (4.6).

Next, let us permit a time-dependent force to act on the oscillator. For simplicity suppose $F(\pm\infty)=0$, so that the oscillator is free at early and late times. As we wish to emphasize the similarities between the classical and quantum aspects, we use the Heisenberg picture, in which the displacement obeys the equation

$$\ddot{x}(t)+\omega_0{}^2x(t)=F(t)/m. \tag{4.7}$$

We solve (4.7) by means of a Green's function $G(t-t')$, defined by the differential equation

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)G(t-t')=\omega_0\delta(t-t') \tag{4.8}$$

and supplemented by appropriate boundary conditions. The delta function corresponds to an impulsive force applied at $t=t'$, and ω_0 has been included to make $G(t)$ dimensionless. Eq. (4.8) is solved in the conventional way by expanding $G(t)$ in a Fourier intergral. The solutions needed here are:

$$G_R(t)=\begin{cases}0 & t<0,\\ \sin\omega_0t & t>0,\end{cases} \tag{4.9}$$

$$G_A(t)=\begin{cases}-\sin\omega_0t & t<0,\\ 0 & t>0.\end{cases} \tag{4.10}$$

We may now write down solutions of Eq. (4.7) according to the alternative boundary conditions

$$\begin{aligned}x(t)&\to x_{\text{in}}(t), \quad t\to-\infty,\\ x(t)&\to x_{\text{out}}(t), \quad t\to+\infty,\end{aligned} \tag{4.11}$$

where x_{in} and x_{out} obey the free oscillator equation

$$\ddot{x}_{\text{in}}+\omega_0{}^2x_{\text{in}}=0, \quad \ddot{x}_{\text{out}}+\omega_0{}^2x_{\text{out}}=0. \tag{4.12}$$

The solutions clearly are

$$\begin{aligned}x(t)&=x_{in}(t)+\frac{1}{m}\int_{-\infty}^{\infty}G_R(t-t')F(t')dt',\\ x(t)&=x_{out}(t)+\frac{1}{m}\int_{-\infty}^{\infty}G_A(t-t')F(t')dt'.\end{aligned} \tag{4.13}$$

As a simple illustration let us find the state of the oscillator after the force has ceased to act. For this we need to express x_{out} in terms of x_{in}

$$x_{\text{out}}=x_{\text{in}}+\frac{1}{m}\int_{-\infty}^{\infty}G(t-t')F(t')dt', \tag{4.14}$$

$$G(t)=G_A(t)-G_R(t)=\sin\omega_0t. \tag{4.15}$$

Introducing the normal mode coordinates a and b by (cf. Eq. 2.9)

$$\begin{aligned}x_{\text{in}}(t)&=x_0(ae^{-i\omega_0t}+a^\dagger e^{i\omega_0t}),\\ x_{\text{out}}(t)&=x_0(be^{-i\omega_0t}+b^\dagger e^{i\omega_0t}),\end{aligned} \tag{4.16}$$

one finds from (4.14) that

$$b=a+iF(\omega_0)/(2m\hbar\omega_0)^{\frac{1}{2}}=a+i\alpha_0, \tag{4.17}$$

where $F(\omega)$ is defined by

$$F(\omega)=\int_{-\infty}^{\infty}e^{i\omega t}F(t)dt. \tag{4.15}$$

Hence, the total effect of the transient force is to displace the incoming normal mode a by an amount depending on that Fourier coefficient of $F(t)$ with the fundamental frequency of the oscillator. From the discussion of Sec. III we know that a unitary operator S exists which transforms a to b

$$a\to b=S^\dagger aS \tag{4.16}$$

and relates the in and out state vectors[15]

$$\Psi_{\text{in}}\to\Psi_{\text{out}}=S^\dagger\Psi_{\text{in}}. \tag{4.17}$$

(The operator S is often called the S matrix.)

[15] For a discussion of the significance of these (Heisenberg) state vectors and the S matrix, see Ref. 10, Chaps. 8–10, or Ref. 2. It is important to remember that Ψ_{out} and Ψ_{in} are constant state vectors; they are not $\psi(\pm\infty)$ as in the interaction picture.

Comparing Eqs. (3.23) and (4.16) indicates that

$$\exp\left\{i\frac{a^\dagger F(\omega_0)+aF^*(\omega_0)}{(2m\hbar\omega_0)^{\frac{1}{2}}}.\right\} \quad (4.18)$$

An interesting alternate way of writing (4.18) results on expressing the argument of the exponential as a time integral

$$\Psi_{\text{out}}=\exp\left\{\frac{-i}{(2m\hbar\omega_0)^{\frac{1}{2}}}\times\int_{-\infty}^{\infty}(a^\dagger e^{i\omega_0 t}+ae^{-i\omega_0 t})F(t)dt\right\}\Psi_{\text{in}};$$

$$\Psi_{\text{out}}=\exp\left[\frac{-i}{\hbar}\int_{-\infty}^{\infty}x_{\text{in}}(t)F(t)dt\right]\Psi_{\text{in}}. \quad (4.19)$$

These formulas express Ψ_{out} in terms of the "incoming" variables a and $a^\dagger$. Using (4.16) one can easily verify that S has the same form expressed in terms of b and $b^\dagger$ as it does in terms of a and $a^\dagger$.

After the force has ceased, the motion of the system is described by x_{out}, and measurements on the oscillator are appropriately described in terms of Ψ_{out}. In order to find how often a given state $\Psi_{\text{out},\nu}$ appears, we have to expand the state vector $\Psi_{\text{in},\mu}$ (which contains information about the preparation of the state μ of the oscillator) in terms of the $\Psi_{\text{out},\nu}$. Using the unitarity of S and labeling the states in (4.17) appropriately we have

$$\Psi_{\text{in},\mu}=\sum_\nu \Psi_{\text{out},\nu}\langle\Psi_{\text{out},\nu}|S|\Psi_{\text{out},\mu}\rangle\equiv\sum_\nu S_{\nu\mu}\Psi_{\text{out},\nu},$$

$$S_{\nu\mu}\equiv\langle\Psi_{\text{out},\nu}|S|\Psi_{\text{out},\mu}\rangle=\langle\Psi_{\text{in},\nu}|S|\Psi_{\text{in},\mu}\rangle. \quad (4.20)$$

In Eq. (4.20) the completeness of the Ψ_{out} was utilized. Eq. (4.17) was used again to establish the last equality.

The explicit form of the scattering matrix Eq. (4.18) completes the solution of the problem.

V. TRANSITION PROBABILITIES BETWEEN NUMBER EIGENSTATES

An interesting and important special case of Eq. (4.20) occurs when the μ, ν labels refer to number states. For example, the model of Sec. IV can with little effort be extended to describe the emission of photons by a classical current source.[5,7] Although it is the coherent states which are radiated by such a source, one often uses photon counters in conducting interesting experiments on the emitted radiation.

If the oscillator was initially in the number state n, then the probability amplitude that it is finally in the number state m is

$$S_{mn}=\langle m|S|n\rangle=\langle m|A(i\alpha_0)|n\rangle, \quad (5.1)$$

[see Eq. (4.18)]. In the simplest case of an oscillator initially in its ground state, the result follows directly from the analysis of Sec. II [see Eqs. (3.10) and (3.14)]

$$S_{m0}=\langle m|i\alpha_0\rangle=\frac{(i\alpha_0)^m}{(m!)^{\frac{1}{2}}}\exp(-\tfrac{1}{2}|\alpha_0|^2). \quad (5.2)$$

The probability of the transition $0\rightarrow m$ is therefore Poisson

$$P_{m0}=|S_{m0}|^2=\frac{|\alpha_0|^{2m}}{m!}\exp(-|\alpha_0|^2), \quad (5.3)$$

with a peak at $N=|\alpha_0|^2$

$$N=|\alpha_0|^2=|F(\omega_0)|^2/(2m\hbar\omega_0). \quad (5.4)$$

Hence, the most probable energy transferred to the oscillator,

$$\Delta E=N\hbar\omega_0=|F(\omega_0)|^2/2m, \quad (5.5)$$

also coincides with the average energy transfer

$$\sum_{m=0}^{\infty}P_{m0}(m\hbar\omega_0)=N\hbar\omega_0, \quad (5.6)$$

as well as with the expression obtained for a classical oscillator initially at rest.[1,16]

The general expression (5.1) for S_{mn} can also be evaluated explicitly in terms of known functions, as was shown by Fuller *et al.*[1] Our method is considerably shorter than theirs. First, we use Eq. (2.20) to write (in the following $m\geq n$)

$$(n!)^{\frac{1}{2}}S_{mn}=\langle m|A^\dagger(-i\alpha_0)(a^\dagger)^n|0\rangle. \quad (5.7)$$

Here we have written $A(\alpha)=A^\dagger(-\alpha)$. This allows us to exploit Eq. (3.24) and write the

[16] Note that in lowest order perturbation theory the energy transfer, $\hbar\omega|S_{10}^{(1)}|^2$ is also given by Eq. (5.5) (Here $S_{10}^{(1)}$ is the lowest order contribution to S_{10}).

right-hand side of (5.7) in the form

$$\langle m|(A^\dagger(-i\alpha_0)a^\dagger A(-i\alpha_0))^n A^\dagger(-i\alpha_0)|0\rangle$$
$$=\langle m|(a^\dagger+i\alpha_0^*)^n A(i\alpha_0)|0\rangle. \quad (5.8)$$

The binomial theorem is now employed to obtain

$$S_{mn}=\sum_{j=0}^{n}\frac{(n!)^{\frac{1}{2}}}{(n-j)!j!}(i\alpha_0^*)^{n-j} \times\langle m|(a^\dagger)^j A(i\alpha_0)|0\rangle. \quad (5.9)$$

Combining Eq. (3.21) with Eq. (2.20), one easily shows that

$$(a)^j|m\rangle=\left(\frac{m!}{(m-j)!}\right)^{\frac{1}{2}}|m-j\rangle,\quad m\geq j, \quad (5.10)$$

so that the matrix element in Eq. (5.9) is simply $S_{m-j,0}$. Inserting the explicit form of $S_{m-j,0}$ from Eq. (5.2) gives

$$S_{mn}=(m!n!)^{\frac{1}{2}}e^{-\frac{1}{2}|\alpha_0|^2}(i\alpha_0)^{m-n} \times\sum_{k=0}^{n}\frac{(-|\alpha_0|^2)^k}{k!(n-k)!(m+k-n)!}. \quad (5.11)$$

In obtaining the latter form the substitution $k=n-j$ has been made. It only remains to recognize the definition of the associated Laguerre polynomials[17]

$$L_m^{m-n}(x)=\sum_{k=0}^{n}\frac{(-x)^k m!}{k!(n-k)!(m-n+k)!} \quad (5.12)$$

to write (5.11) in the succinct form

$$S_{mn}=(n!/m!)^{\frac{1}{2}}e^{-\frac{1}{2}|\alpha_0|^2}(i\alpha_0)^{m-n} \times L_m^{m-n}(|\alpha_0|^2),\quad m\geq n, \quad (5.13)$$

where α_0 is $F(\omega_0)/(2m\hbar\omega_0)^{\frac{1}{2}}$ (see Eq. 4.17). One thus obtains for the transition probability $(m\geq n)$

$$P_{mn}=|S_{mn}|^2=\frac{n!}{m!}e^{-x}x^{m-n}[L_m^{m-n}(x)]^2;\quad x=|\alpha_0|^2. \quad (5.14)$$

If $m<n$ the sum in Eq. (5.9) actually cuts off at $j=m$. Repeating the calculation shows that

$$S_{mn}=(m!/n!)^{\frac{1}{2}}e^{-\frac{1}{2}|\alpha_0|^2}(i\alpha_0^*)^{n-m}L_n^{n-m}(|\alpha_0|^2),\quad m\leq n, \quad (5.15)$$

so that P_{mn} can be obtained from (5.14) by interchanging m and n.

In Ref. 1 it is shown by direct summation that the mean energy transferred to an oscillator initially in an arbitrary number state n is the same as for $n=0$. This result, too, can be obtained in a completely elementary manner. The energy shift is simply (the state Ψ_{in} is fixed)

$$\Delta E=\langle\Psi_{\text{in}}|(H_{\text{out}}-H_{\text{in}})|\Psi_{\text{in}}\rangle$$
$$=\hbar\omega[|\alpha_0|^2+i(\langle\Psi_{\text{in}}|a^\dagger|\Psi_{\text{in}}\rangle\alpha_0-\text{c.c.})], \quad (5.16)$$

using Eq. (4.17) to express b in terms of a. For number states the term in parentheses vanishes and (5.16) simplifies to

$$\Delta E=\hbar\omega|\alpha_0|^2=|F(\omega_0)|^2/2m. \quad (5.17)$$

The vanishing of the last terms in (5.16) for the number states can be regarded as due to the complete uncertainty in the phase of such states. If Ψ_{in} is a coherent state with phase parameter β, for instance, then a term $2\hbar\omega\,\text{Im}(\beta\alpha_0^*)$ has to be added to (5.17).

Finally, we mention another way that coherent states can be used to illuminate the concept of phase of a quantum mechanical oscillator. Using the operators appropriate[9] to describe the phase variable ϕ one can introduce a suitable "number-phase" uncertainty relation which reduces to the familiar $\Delta N\Delta\phi\geq\frac{1}{2}$ in the classical limit but is still meaningful for small quantum numbers. The coherent states are found to be very good minimum-uncertainty-product states. For further details the reader is referred to Ref. 18.

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[17] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 188.

[18] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **14**, 387 (1965).