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S-Matrix Solution for the Forced Harmonic Oscillator*

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The exact solution for a forced, undamped quantum harmonic oscillator is obtained by S-matrix techniques. The force $F(t)$ is assumed to vanish at $t=\pm\infty$. The only restriction is that the Fourier transform of $F(t)$ must exist. The transition probabilities are obtained in closed form in terms of Laguerre polynomials. The mean energy transfer to the oscillator is found to be independent of the initial state and is in agreement with the classical result for an oscillator originally at rest. This problem provides a good example of field theoretical procedures in an elementary context. Therefore, all field theoretical concepts are carefully defined and explained as they are introduced in order that the discussion may be self-contained.

I. INTRODUCTION

We consider the forced harmonic oscillator as an example for reviewing and demonstrating field theory techniques.[1] This problem bears a close analogy to the scattering situation for which the S-matrix formalism is commonly applied. At very early times ($t \to -\infty$) the system—i.e., the unperturbed oscillator—is in one of an infinite number of possible noninteracting ("free particle") states. The oscillator is then subjected to an interaction— the driving force $F(t)$—which cuts off at very late times ($t \to +\infty$). The system has then returned to one of the states possible for an unforced oscillator, and the question is asked: what is the probability that the interaction has induced a transition from an unperturbed oscillator state $|k\rangle$ to a final state $|j\rangle$? Modern field theory answers such questions by constructing a scattering operator S (see Sec. III) such that $|\langle j|S|k\rangle|^2$ gives the probability of a $k \to j$ transition.

In field theory, practical calculations are done with a series expansion for the S matrix [Eq. (20)], only a few terms of which are kept in the usual perturbation approach. The convergence of this series in most cases remains an open question. Even in perturbation theory of conventional quantum mechanics, the convergence of a calculation carried to any desired order is not well established.

We apply to the forced oscillator each step of a modern field theoretical perturbation calculation —introduction of the interaction representation and the S matrix (Sec. III); second-quantization formalism (IV); reordering in terms of "normal products" (V); and, finally, evaluation of the S matrix between unperturbed states (VI). It turns out that all these steps are nicely illustrated by the forced oscillator problem. Moreover, the perturbation expansion can be summed to all orders for an arbitrary force, thereby allowing a complete solution.

Since this specialized problem is primarily of interest for the insight it offers into general methods, it seems worthwhile to display our work as an illustrative example for students of field theory. The forced oscillator initially in the ground state is in effect studied by field theory methods in investigations of the infrared divergence in quantum-electrodynamics.[2] However, these discussions are by no means self-contained, but assume knowledge of much material with which an inexperienced (or forgetful) reader may feel uncomfortable. The discussion here proceeds from first principles to final results without omitting useful review material.

II. THE CLASSICAL OSCILLATOR

We consider an undamped, one-dimensional harmonic oscillator acted upon by a spatially

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[1] An orthodox quantum-mechanical solution to this problem has been obtained by E. H. Kerner, Can. J. Phys. 36, 371 (1958). He obtains the exact wavefunctions after transforming the Schrödinger equation into a separable form. His treatment furnishes considerable insight into the similarities between the classical and quantum-mechanical oscillators.
[2] See, for example, A. I. Akhiezer and V. B. Berestetsky, "Quantum Electrodynamics," AEC-tr-2876, Sec. 32, and references therein. This reference may also be consulted for more detailed and general discussions of the field theoretical concepts which we use.

uniform (no x dependence) external force $F(t)$. Throughout this paper we limit ourselves to force functions $F(t)$ for which the Fourier transform, Eq. (7), exists. Thus $F(t)$ vanishes in the limits $t \to \pm\infty$; we may regard the system initially and finally as an unperturbed oscillator. The equation of motion is given by

$$\ddot{x}+\omega^2 x=(1/m)F(t), \tag{1}$$

where m is the mass and ω is the natural frequency of the oscillator. We may rewrite (1) as

$$\dot{z}-i\omega z=(1/m)F(t), \tag{2}$$

where

$$z=\dot{x}+i\omega x. \tag{3}$$

Let us now assume that $F(t)$ is nonzero only for an interval $T_1 \leq t \leq T_2$ and that the oscillator is originally at rest ($x=\dot{x}=z=0$ for $t \leq T_1$). Multiplying both sides of (2) by $\exp(-i\omega t)$ and integrating, we obtain

$$\int_{T_1}^{T_2}(\dot{z}-i\omega z)e^{-i\omega t}dt=(1/m)\int_{T_1}^{T_2}F(t)e^{-i\omega t}dt. \tag{4}$$

Integration by parts of the term on the left gives

$$z(T_2)e^{-i\omega T_2}=(1/m)\int_{T_1}^{T_2}F(t)e^{-i\omega t}dt, \tag{5}$$

where we have used the boundary condition that z vanishes for $t \leq T_1$. Since $F(t)$ vanishes outside the range of integration in (5), we may extend the limits of integration to $\pm\infty$. Then (5) may be written as

$$z(T_2)e^{-i\omega T_2}=(1/m)f(\omega), \tag{6}$$

where $f(\omega)$ is the Fourier transform of $F(t)$ evaluated at the natural frequency of the oscillator,

$$f(\omega)\equiv\int_{-\infty}^{\infty}F(t)e^{-i\omega t}dt. \tag{7}$$

For times outside the interval $T_1 \leq t \leq T_2$, the energy E of the oscillator is given by

$$E=(m\dot{x}^2/2)+(m\omega^2x^2/2)=(m/2)|z|^2. \tag{8}$$

Since the oscillator initially has zero energy, the energy ΔE transferred to the oscillator by $F(t)$ is $E(T_2)$, or from (6)

$$\Delta E=(1/2m)|f(\omega)|^2. \tag{9}$$

Energy transfer only occurs at the resonant frequency ω, as is seen by observing that only the ωth component of the Fourier transform of $F(t)$ enters in (9). The result expressed in (9) remains valid whenever $F(t)$ possesses a Fourier transform, even if the force does not vanish identically for t outside the time interval T_1 to T_2.

III. REVIEW OF THE INTERACTION REPRESENTATION AND THE THE S MATRIX

Before turning specifically to a study of the quantum oscillator, we briefly review some aspects of the interaction representation which lead to the "S-matrix" formalism. Schrödinger's time-dependent equation for a system characterized by a Hamiltonian H_0 with no explicit time dependence and acted on by an external force obtained from a potential $V_s(x,t)$[3] is

$$(H_0+V_s)|\psi(x,t)\rangle_s=i\hbar(\partial/\partial t)|\psi(x,t)\rangle_s. \tag{10}$$

We introduce the interaction representation by defining

$$|\psi\rangle_I=e^{iH_0t/\hbar}|\psi\rangle_s. \tag{11}$$

On substitution of (11) into (10) we obtain

$$V_I(t)|\psi(t)\rangle_I=i\hbar(\partial/\partial t)|\psi(t)\rangle_I, \tag{12}$$

where $V_I(t)$ is defined by the relation

$$V_I(t)=e^{iH_0t/\hbar}V_s(t)e^{-iH_0t/\hbar}. \tag{13}$$

Equation (12) can be formally integrated to give

$$|\psi(t)\rangle_I=|\psi(T)\rangle_I-i/\hbar\int_T^t dt'\,V_I(t')|\psi(t')\rangle_I. \tag{14}$$

A formal solution of (14) is given by iteration:

$$|\psi(t)\rangle_I=\left\{1+(-i/\hbar)\int_T^t dt_1 V_I(t_1)+(-i/\hbar)^2\times\int_T^t dt_1\int_T^{t_1}dt_2 V_I(t_1)V_I(t_2)+\cdots\right\}\times|\psi(T)\rangle_I. \tag{15}$$

[3] For convenience we suppress explicit reference to spatial dependence. The subscript s designates operators and state vectors in the Schrödinger representation. We use a mixed notation for the state functions to emphasize the applicability of the following discussion both to "ordinary" quantum mechanics and to second-quantized formulations.

Unfortunately, the integration variables in (15) appear unsymmetrically. It would be more convenient if the upper limits for all integrations were the same. This alteration in (15) can be accomplished by the introduction of a "time-ordered product."

Let us define the time-ordering operator T (first introduced by Dyson) as that operator which rearranges a product of time-dependent operators such that functions of later times always stand to the left of functions of earlier times. Thus, if $A(t_1)$, $B(t_2)$, and $C(t_3)$ are time-dependent operators, we obtain

$$\begin{aligned} T[A(t_1)B(t_2)C(t_3)] &= A(t_1)B(t_2)C(t_3) \quad \text{when} \quad t_1>t_2>t_3, \\ &\quad C(t_3)B(t_2)A(t_1) \quad \text{when} \quad t_3>t_2>t_1, \\ &\quad B(t_2)A(t_1)C(t_3) \quad \text{when} \quad t_2>t_1>t_3, \end{aligned} \tag{16}$$

and so on, where three other combinations are possible. Clearly, products of commuting operators are not affected by the time ordering operation.

We are now able to see the validity of the following identity:

$$\int_T^t dt_1 \int_T^{t_1} dt_2 \cdots \int_T^{t_{n-1}} dt_n V_I(t_1)\cdots V_I(t_n) = \frac{1}{n!}\int_T^t dt_1 \int_T^t dt_2 \cdots \int_T^t dt_n \times T[V_I(t_1)\cdots V_I(t_n)]. \tag{17}$$

First of all, we note that the factors of V_I appearing on the left side of (17) are already time ordered because of the different upper limits of integration. All times appearing on the right side may be permuted arbitrarily, because of the definition of the T product, without changing the value of the integral. Since there are $n!$ possible permutations of the factors in the T product, the given integral is $n!$ times the contribution of an integration over any one specific time ordering of the integrand.

Using (17) we may then write (15) in the desired form,

$$|\psi(t)\rangle_I = \left\{ \sum_{n=0}^{\infty} (-i/\hbar)^n \frac{1}{n!} \int_T^t dt_1 \cdots \int_T^t dt_n \times T[V_I(t_1)\cdots V_I(t_n)] \right\} |\psi(T)\rangle_I. \tag{18}$$

In scattering theory, we usually begin with the system in an eigenstate of H_0 at very early times ($t \to -\infty$), when the particle to be scattered has not yet come within range of the force, or before the interaction V_I has been "turned on." Likewise as t becomes large the system again goes into an eigenstate of H_0. For such a physical situation the interaction representation state function given by (11) may be expected to be well-defined at $t=\pm\infty$. Hence we can define the "S matrix" as that operator which transforms the state at $-\infty$ into the state at $+\infty$:

$$|\psi(+\infty)\rangle_I = S|\psi(-\infty)\rangle_I, \tag{19}$$

where from (18)

$$S = \sum_{n=0}^{\infty} (-i/\hbar)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \times T[V_I(t_1)\cdots V_I(t_n)]. \tag{20}$$

The transition amplitude from a state $|\psi_n(-\infty)\rangle$ to another state $|\psi_m(+\infty)\rangle$ is given by

$$S_{mn} = \langle \psi_m(+\infty)|S|\psi_n(-\infty)\rangle \tag{21}$$

(where an integration over space variables is implied by the bracket notation). What is commonly called the "S matrix" should more properly be called the "S operator," which then gives rise to the matrix element of (21).

Usually the series expansion of S given by (20) is useful in calculations only when V_I is a small perturbation. Then only the lowest-order term with a nonvanishing contribution to (21) need be evaluated. Moreover, difficulties in principle often arise in field theory when higher-order contributions to (21) are found to diverge. When divergences occur, meaningful results can still be obtained in some theories by redefining the masses and coupling constants and introducing "renormalization constants." Theories with divergences that cannot be "defined away" or

renormalized (e.g., four Fermion beta-decay theory) may nonetheless yield in lowest-order numbers in striking agreement with experiment, but perhaps these results should be viewed more as the fruits of revelation than of reason. None of these difficulties arise, however, in our example of the harmonic oscillator in an external field, like that of Sec. II. No higher-order divergences arise, and we are able to sum the series of (20) without any restriction on the magnitude of the perturbation. Thus an exact solution is obtained to the quantum mechanical analog of the problem treated classically in Sec. II.

IV. THE QUANTUM OSCILLATOR

The Hamiltonian H_0 for the unperturbed harmonic oscillator of Sec. II is given in the Schrödinger representation by

$$H_0=(p^2/2m)+\tfrac{1}{2}m\omega^2x^2, \tag{22}$$

where

$$p=(\hbar/i)(d/dx). \tag{23}$$

If the operators a and $a^\dagger$ are defined by

$$a=(xm\omega+ip)(2m\omega\hbar)^{-\frac{1}{2}} \tag{24a}$$

and

$$a^\dagger=(xm\omega-ip)(2m\omega\hbar)^{-\frac{1}{2}}, \tag{24b}$$

then we can rewrite H_0 as

$$H_0=\hbar\omega(a^\dagger a+\tfrac{1}{2}). \tag{25}$$

From (23) and (24), we have

$$[a,a^\dagger]=1. \tag{26}$$

We denote by $|n,t\rangle_s$ the nth solution to the harmonic oscillator wave equation,

$$H_0|n,t\rangle_s=i\hbar(\partial/\partial t)|n,t\rangle_s.$$

In terms of elementary functions, one finds

$$|n,t\rangle_s=(\alpha/\pi^{\frac{1}{2}}2^n n!)^{\frac{1}{2}}H_n(\alpha x)\times\exp(-\tfrac{1}{2}\alpha^2x^2)e^{-i(n+\frac{1}{2})\omega t}, \tag{27}$$

where H_n is the nth-order Hermite polynomial, and

$$\alpha^2=m\omega/\hbar.$$

It is convenient to make the time dependence of $|n,t\rangle_s$ explicit by writing

$$|n,t\rangle_s=e^{-i\omega(n+\frac{1}{2})t}|n\rangle, \tag{28}$$

where $|n\rangle$ is the solution of the (time-independent) equation

$$H_0|n\rangle=\hbar\omega(n+\tfrac{1}{2})|n\rangle. \tag{29}$$

From (25), (26), and (28) one obtains the relations

$$a^\dagger|n\rangle=(n+1)^{\frac{1}{2}}|n+1\rangle,$$
$$(a^\dagger)^n|0\rangle=(n!)^{\frac{1}{2}}|n\rangle, \tag{30}$$

$$a|n\rangle=(n)^{\frac{1}{2}}|n-1\rangle,$$
$$a|0\rangle=0, \tag{31}$$

which allow the interpretation of $a^\dagger$ and a as creation and destruction operators, respectively. Hence, the formalism we are using has the characteristics of a simple case of "second quantization" as used in the quantum theory of the photon field. We want to re-emphasize, however, that *our* state vectors $|n\rangle$ are in fact elementary functions which are "raised" or "lowered" by the algebraic and differential operations given in Eqs. (23), (24), (30), and (31). Thus, all that follows could easily be written in terms of Hermite polynomials and weighting functions in x space. However, insight as well as elegance would be lost by such a procedure as would our analogy to field theory calculations.

We now seek the solutions of (10) when H_0 is given by (25) and V_s is given by

$$V_s=-xF(t)=-(a+a^\dagger)(\sqrt{2}\alpha)^{-1}F(t), \tag{32}$$

which classically corresponds to the introduction of the force $F(t)$ acting on the oscillator.

In the interaction representation, the states $|n,t\rangle_I$ of the unperturbed oscillator are given according to (11) and (28) by

$$|n,t\rangle_I=|n\rangle.$$

To find V_I we need to evaluate $a_I(t)$, defined by

$$a_I(t)\equiv e^{iH_0t/\hbar}ae^{-iH_0t/\hbar}.$$

At $t=0$, $a_I(t)=a$, and

$$(da_I(t)/dt)=(i/\hbar)e^{iH_0t/\hbar}[H_0,a]e^{-iH_0t/\hbar}$$
$$=-i\omega a_I(t),$$

using (25) and (26). Hence we have

$$a_I(t)=ae^{-i\omega t}.$$

From (13) and (32), we then obtain

$$V_I(t)=-F(t)(\sqrt{2}\alpha)^{-1}(ae^{-i\omega t}+a^\dagger e^{i\omega t}). \tag{33}$$

We now calculate the probability that an

oscillator that was in the ground state before the force $F(t)$ is applied is in the jth excited state after application of the force has ceased. We require the matrix element[4]

$$\langle j|\psi(\infty)\rangle$$

when $|\psi(-\infty)\rangle=|0\rangle$. Using the definition of the S matrix [Eq. (19)] and its expansion in terms of V_I [Eq. (20)], we obtain

$$\langle j|S|0\rangle=\sum_{n=0}^{\infty}\frac{i^n}{n!(\sqrt{2}\alpha\hbar)^n}\left\langle j\left|\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_n\right.\right.$$
$$\left.\left.\times T\left\{\prod_{k=1}^{n}F(t_k)(ae^{-i\omega t_k}+a^\dagger e^{i\omega t_k})\right\}\right|0\right\rangle. \quad (34)$$

It is now convenient to apply Wick's reordering theorem[5] to evaluate the matrix element of the T product in (34).

V. THE REORDERING THEOREM

For brevity, let $(ae^{-i\omega t_k}+a^\dagger e^{i\omega t_k})$ be denoted simply by its time index k. The product of an arbitrary number of such factors is then denoted by $kl\cdots n$. We define the *normal product* $N(kl\cdots n)$ by the following prescription: Regard a and $a^\dagger$ as though they were commuting operators and then form $N(kl\cdots n)$ by multiplying the factors $kl\cdots n$, always writing each a to the right of every $a^\dagger$. Thus

$$kl=a^2e^{-i\omega(t_k+t_l)}+(a^\dagger)^2e^{i\omega(t_k+t_l)}$$
$$+a^\dagger ae^{i\omega(t_k-t_l)}+aa^\dagger e^{i\omega(t_l-t_k)},$$
$$N(kl)=a^2e^{-i\omega(t_k+t_l)}+(a^\dagger)^2e^{i\omega(t_k+kl)}$$
$$+a^\dagger ae^{i\omega(t_k-t_l)}+a^\dagger ae^{i\omega(t_l-t_k)}.$$

Normal products are invariant under permutation of indices. A *contraction* of two factors denoted here by $C(kl)$ is defined as the difference between the T and N products,

$$C(kl)\equiv T(kl)-N(kl).$$

The contraction is also invariant under interchange of the two indices. It is easily seen in our case that

$$C(kl)=e^{-i\omega|t_k-t_l|}. \quad (35)$$

In general, the contraction is always a "C number" (rather than an operator). Wick's reordering theorem as applied here states that the time-ordered product $T(1,2\cdots n)$ may be expanded as follows[6]:

$$T(1,2\cdots n)$$
$$=\sum_{(p)}{}'\{C(1,2)C(3,4)\cdots C(n-1,n)+C(1,2)$$
$$\times C(3,4)\cdots C(n-3,n-2)N(n-1,n)+\cdots$$
$$+N(1,2\cdots n)\} \quad \text{(even } n\text{)}, \quad (36)$$
$$T(1,2\cdots n)$$
$$=\sum_{(p)}{}'\{C(1,2)C(3,4)\cdots C(n-2,n-1)N(n)$$
$$+C(1,2)C(3,4)\cdots C(n-4,n-3)$$
$$\times N(n-2,n-1,n)+\cdots$$
$$+N(1,2\cdots n)\} \quad \text{(odd } n\text{)}.$$

The sum $\sum'$ is to be taken over all permutations of $1, 2\cdots n$; the prime indicates that only the terms of each succeeding permutation which are not identically equivalent to terms already present are to be retained. Thus permutations which only interchange indices within contraction symbols and/or within normal products are not included. Also excluded are permutations which change only the order in which the contractions are written.

For two factors the ordering theorem is obvious by definition of $C(1,2)$. For three factors, the expansion of $T(1,2,3)$ is given by (36) as

$$T(1,2,3)=C(1,2)N(3)+C(1,3)N(2)$$
$$+C(2,3)N(1)+N(1,2,3).$$

That the above equation holds may be easily checked by evaluating both sides for the time order $t_1>t_2>t_3$. Wick[5] gives a general proof for n factors; the difficulties are largely notational. In the case of the forced oscillator here considered, in which the $F(t_k)$'s are not operators, it may in fact be shown[7] that the time ordering can be omitted from (34) and use of Wick's theorem may thus be avoided. We prefer to illustrate the convenience of the reordering theorem, since its use

[4] The notation $\langle j|$ indicates the Hermitian conjugate of $|j\rangle$, while the combined bracket notation $\langle j|k\rangle$ implies integration over all space coordinates. Hence the orthonormality condition is given by $\langle j|k\rangle=\delta_{jk}$.

[5] G. C. Wick, Phys. Rev. **80**, 268 (1950).

[6] Since the factors $F(t_k)$ in (34) are c numbers, they may be factored out of the T product and need not concern us in this section.

[7] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

is more typical in the general field theoretical calculations we desire to parallel.

VI. EVALUATION OF THE TRANSITION PROBABILITY

The choice of the ground state, $|0\rangle$, as our initial state makes the application of Wick's theorem to (34) particularly simple. Here, we discuss some of the steps which lead from (34) and (36) to the following result:

$$\langle j|S|0\rangle=(j!)^{\frac{1}{2}}\sum_{\substack{n=j\\(n-j\text{ even})}}^{\infty}\frac{i^n}{n!(\sqrt{2}\alpha\hbar)^n}\int_{-\infty}^{\infty}dt_1\cdots\times\int_{-\infty}^{\infty}dt_n\left\{\prod_{k=1}^{n}F(t_k)\right\}C(1,2)\cdots\times C(n-j-1,n-j)e^{i\omega t_{n-j+1}}\times e^{i\omega t_{n-j+2}}\cdots e^{i\omega t_n}K_{nj},\quad(37)$$

where

$$K_{nj}=n!/2^{(n-j)/2}[(n-j)/2]!j!.\quad(38)$$

We observe that:

(a) The only terms from (36) with nonvanishing matrix elements between $\langle j|$ and $|0\rangle$ are those containing a normal product of j factors, since no other normal-ordered terms can raise the state $|0\rangle$ to the state $|j\rangle$. Hence the first $j-1$ terms of the sum in (34) must vanish. The factor $(j!)^{\frac{1}{2}}$ arises from the relation $(a^\dagger)^j|0\rangle=(j!)^{\frac{1}{2}}|j\rangle$ [Eq. (30)].

(b) In view of (a), terms from the Wick expansion for $n>j$ can contribute to (34) only if $n-j$ factors have been contracted out. Hence the sum is restricted to $n-j$ even.

(c) From each contributing normal product there arise j factors of $e^{i\omega t_k}$, which are the coefficients of $(a^\dagger)^j$.

(d) The factor K_{nj} accounts for the possible permutations in the Wick expansion. Since only inequivalent terms appear in the permutation sum of (36), the $n!$ possible orderings of the dummy integration variables must be divided by (1) $j!$, the number of equivalent permutations of the normal product of j factors of $(a^\dagger e^{i\omega t}+ae^{-i\omega t})$; (2) $2^{(n-j)/2}$, the number of ways of interchanging indices within $(n-j)/2$ contractions; (3) $[(n-j)/2]!$, the number of permutations which change only the order in which the $(n-j)/2$ contractions appear.

Some obvious cancellations and a change of summation index to $m=(n-j)/2$ lead to

$$\langle j|S|0\rangle=\frac{i^j}{(\sqrt{2}\alpha\hbar)^j(j!)^{\frac{1}{2}}}[f^*(\omega)]^j\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(4\alpha^2\hbar^2)^m}\frac{D^m}{m!},\quad(39)$$

where

$$D\equiv\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2e^{-i\omega|t_1-t_2|}F(t_1)F(t_2).\quad(40)$$

We have used (7) and expression (35) for the contraction to obtain concise expressions for the time integrations. Eq. (39) yields at once

$$\langle j|S|0\rangle=\frac{i^j}{(\sqrt{2}\alpha\hbar)^j(j!)^{\frac{1}{2}}}[f^*(\omega)]^j\exp(-D/4\alpha^2\hbar^2).\quad(41)$$

The probability that the oscillator is in the final state j is denoted by P_{j0}. On observing that

$$D+D^*=2|f(\omega)|^2,$$

we obtain from (41)

$$P_{j0}=|\langle j|S|0\rangle|^2=|f(\omega)|^{2j}/(2\alpha^2\hbar^2)^j j!\times\exp[-|f(\omega)|^2/2\alpha^2\hbar^2].\quad(42)$$

Transitions from the ground state involve only one process—absorption. Since unperturbed oscillator levels are evenly spaced, the $0\rightarrow j$ transition may be viewed as an absorption of j quanta, each of energy $\hbar\omega$. If consecutive absorptions are statistically independent, a Poisson distribution is to be expected. The Poisson character of (42) is emphasized if it is rewritten as

$$P_{j0}=(x^j/j!)e^{-x},\quad(42a)$$

where

$$x=|f(\omega)|^2/2\alpha^2\hbar^2.\quad(42b)$$

The mean energy transfer to the quantum oscillator is given by

$$\Delta E=\hbar\omega\sum_j jP_{j0}=|f(\omega)|^2/2m,\quad(43)$$

in agreement with the classical result (9). Using (42b) and (43), we see that $\Delta E/\hbar\omega=x$, so that x may be identified with the mean number of quanta absorbed by the oscillator.

The matrix element $\langle j|S|k\rangle$ for $k\neq 0$ can be found by using (20) for the S matrix and the procedure of Sec. VI. (See Appendix.) The situation is more complicated, since terms involving factors $(a^\dagger)^{j-(k-l)}a^l$ (in the case $j\geqslant k$) contribute to the matrix element for all $l\leqslant k$ for the nth term of the S-matrix series (when $n\geqslant k+j$). Thus the final result, corresponding to (42) is given by a (finite) series arising from the sum over l. From a detailed consideration, we obtain for $j\geqslant k$

$$|\langle j|S|k\rangle|^2=(k!/j!)x^{j-k}e^{-x}[L_k^{j-k}(x)]^2,$$

where x is given by (42b), and $L_k^{j-k}(x)$ is the associated Laguerre polynomial.[8] When $k>j$, we must interchange j and k on the right-hand side of the above equation for $|\langle j|S|k\rangle|^2$.

By use of identities in reference 8, it can be verified that the sum of transitions probabilities over all j is unity. Furthermore, the mean energy transfer can be calculated. It is found to be independent of the initial state and is given by (43).

We have not found it necessary to introduce explicitly configuration space wavefunctions (appropriately weighted Hermite polynomials) or to perform spatial integrations in order to calculate matrix elements. Our discussion is thus an extension of the well-known algebraic operator techniques that lead to a simple solution of the unperturbed oscillator.

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APPENDIX. TRANSITIONS FROM EXCITED STATES

Once the analysis of (34) by the methods of Sec. VI is understood, the calculation of $\langle j|S|k\rangle$ for arbitrary j and k becomes straightforward, though careful attention must be paid to the details. Our discussion outlines the procedure without attempting to cover every step. It may, therefore, prove difficult for the reader to follow the paragraphs leading to (52) without some work on his own.

When $|0\rangle$ is replaced by $|k\rangle$ in (34) and the expansion (36) is introduced, we obtain for the case $j\geqslant k$ the double series:

$$\begin{aligned}\langle j|S|k\rangle=&\sum_{n=0}^{\infty}{}'\frac{i^n}{(\sqrt{2}\alpha\hbar)^n n!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_n\\&\times\sum_{l=0}^{\tilde{l}}C(1,2)\cdots C(2m-2l-1,\,2m-2l)\\&\times F(t_{2m-2l+1})e^{i\omega t_{2m-2l+1}}\cdots F(t_{n-l})\\&\times e^{i\omega t_{n-l}}F(t_{n-l+1})e^{-i\omega t_{n-l+1}}\\&\times F(t_n)e^{-i\omega t_n}[(k!j!)^{\frac{1}{2}}/(k-l)!]K_{jknl}.\end{aligned}\tag{44}$$

The prime on the summation indicates that only terms for which $n-(j-k)$ is nonnegative and even are kept. We further define

$$m=(n-j+k)/2,\tag{45}$$

$$\tilde{l}=\text{minimum }(k,m),\tag{46}$$

$$\begin{aligned}K_{jknl}=&\,n![(m-l)!2^{m-l}(j-k+2l)!]^{-1}\\&\times\binom{j-k+2l}{l}.\end{aligned}\tag{47}$$

The factor $(k!j!)^{\frac{1}{2}}/(k-l)!$ arises according to (30) and (31) from the terms

$$(a^\dagger)^{j-(k-l)}a^l|k\rangle=(k!)^{\frac{1}{2}}(j!)^{\frac{1}{2}}/(k-l)!\,|j\rangle,\tag{48}$$

which are factors in the only terms of the Wick expansion of $S|k\rangle$ that do not vanish when bracketed with $\langle j|$. The factor K_{jknl} is obtained from the same kind of analysis that led to (38), except for the binomial coefficient, which appears as the coefficient of $(a^\dagger)^{j-(k-l)}a^l$ in the expansion of $(a^\dagger e^{i\omega t}+ae^{-i\omega t})^{j-k+2l}$.

It is now convenient to select m as the summation index instead of n and to introduce $f(\omega)$ and D by (7), (35), and (40), thereby obtaining

$$\begin{aligned}&\langle j|S|k\rangle\\&=\frac{(k!j!)^{\frac{1}{2}}}{(\sqrt{2}\alpha\hbar)^{j-k}}i^{j-k}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2\alpha^2\hbar^2)^m}\\&\times\sum_{l=0}^{\tilde{l}}D^{m-l}[f^*(\omega)]^{j-(k-l)}\cdot[f(\omega)]^l\\&\times[(m-l)!2^{m-l}(j-k+l)!l!(k-l)!]^{-1}.\end{aligned}\tag{49}$$

[8] Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, Sec. 10.12.

We now interchange summations, observing that

$$\sum_{m=0}^{\infty}\sum_{l=0}^{\tilde{l}}=\sum_{l=0}^{k}\sum_{m=l}^{\infty}. \tag{50}$$

A shift of index, $m'=m-l$, then permits one summation to be carried out, and leads readily to the result,

$$\langle j|S|k\rangle=(k!j!)^{\frac{1}{2}}i^{j-k}[f^*(\omega)]^{j-k}\exp(-D/4\alpha^2\hbar^2)\times(\sqrt{2}\alpha\hbar)^{k-j}\sum_{l=0}^{k}\frac{(-1)^l}{(2\alpha^2\hbar^2)^l}\times\frac{|f(\omega)|^{2l}}{[j-(k-l)]!l!(k-l)!}. \tag{51}$$

The transition probability for $j\geqslant k$ is then given by

$$P_{jk}=|\langle j|S|k\rangle|^2=k!j!x^{j-k}e^{-x}\times\left\{\sum_{l=0}^{k}\frac{(-x)^l}{[j-(k-l)]!l!(k-l)!}\right\}^2, \tag{52}$$

where

$$x=|f(\omega)|^2/2\alpha^2\hbar^2. \tag{53}$$

For $j<k$ it can be checked that (52) holds after the interchange of j and k on the right side. P_{jk} is conveniently given in terms of Laguerre polynomials by

$$P_{jk}=(k!/j!)x^{j-k}e^{-x}[L_k{}^{j-k}(x)]^2, \quad j\geqslant k \tag{54a}$$

$$=(j!/k!)x^{k-j}e^{-x}[L_j{}^{k-j}(x)]^2, \quad j<k, \tag{54b}$$

where

$$L_n{}^{\alpha}(x)=(e^x x^{-\alpha}/n!)(d^n/dx^n)(e^{-x}x^{n+\alpha})=\sum_{m=0}^{n}\binom{n+\alpha}{n-m}\frac{(-x)^m}{m!}. \tag{55}$$

Obtaining the mean energy transfer,

$$\Delta E_k=\sum_j P_{jk}(j-k)\hbar\omega,$$

from (54) is facilitated by use of the identity[8]

$$[L_k{}^{j-k}(x)]^2=\frac{j!}{k!}\sum_{l=0}^{k}\frac{x^{2l}L_{k-l}{}^{j-k+2l}(2x)}{(j-k+l)!l!}. \tag{56}$$

Using (56) in (54) we obtain

$$\Delta E_k/\hbar\omega=\sum_{m=0}^{\infty}e^{-x}mx^m\sum_{l=0}^{k}\frac{x^{2l}L_{k-l}{}^{m+2l}(2x)}{(m+l)!l!}-\sum_{m=1}^{k}e^{-x}mx^m\sum_{l=0}^{k-m}\frac{x^{2l}L_{k-m-l}{}^{m+2l}(2x)}{(m+l)!l!}, \tag{57}$$

by setting $m=j-k$ in (54a) and $m=k-j$ in (54b). By a further change of indices:

$$l\rightarrow l+m$$

$$m\rightarrow -m,$$

the second summation may be combined with the first to give

$$\Delta E_k e^x/\hbar\omega=\sum_{l=0}^{k}\sum_{m=-l}^{\infty}\frac{mx^{m+2l}L_{k-l}{}^{m+2l}(2x)}{l!(l+m)!}. \tag{58}$$

A further shift of index, $m\rightarrow m+l$, gives

$$\Delta E_k e^x/\hbar\omega=\sum_{l=0}^{k}\sum_{m=0}^{\infty}\frac{(m-l)x^m x^l L_{k-l}{}^{m+l}(2x)}{l!m!},$$

which by (55) may be written

$$\Delta E_k e^x/\hbar\omega=e^{2x}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\frac{(m-l)}{l!m!2^{m+l}(k-l)!}\times\frac{d^{k-l}}{du^{k-l}}(e^{-u}u^{k+m}), \tag{59}$$

where $u=2x$.

On observing that

$$\sum_{m=0}^{\infty}\frac{(m-l)(u/2)^m}{m!}=e^{u/2}(u/2-l),$$

we obtain from (59)

$$\Delta E_k e^{-x}/\hbar\omega=\sum_{l=0}^{k}\frac{2^{-l-1}}{l!(k-l)!}\frac{d^{k-l}}{du^{k-l}}(e^{-u/2}u^{k+1})-\sum_{l=0}^{k}\frac{2^{-l}}{(l-1)!(k-l)!}\frac{d^{k-l}}{du^{k-l}}(e^{-u/2}u^k). \tag{60}$$

By use of the binomial expansion and a shift of index in the second sum, it is easy to obtain

$$\Delta E_k e^{-x}/\hbar\omega = \frac{1}{k!}\left(1+\frac{d}{dx}\right)^k (e^{-x}x^{k+1}) - \frac{1}{(k-1)!}\left(1+\frac{d}{dx}\right)^{k-1}(e^{-x}x^k). \quad (61)$$

This expression is simple to evaluate because

$$\left(1+\frac{d}{dx}\right)(e^{-x}x^k) = ke^{-x}x^{k-1}.$$

We readily find that

$$\Delta E_k = \hbar\omega x = |f|^2/2m, \quad (62)$$

which is independent of k and equal to the mean energy transfer for transitions from the ground state.

Faraday's Law and Ampere's Law

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It is suggested that the origin of the induced electromotive force in a stationary circuit, given by Faraday's law of induction, could be made clearer to a beginning student if emphasis were placed in introductory courses on the geometry of the induced electric field set up by a time-varying magnetic field instead of on the line integral of this field around a closed path, or the emf in the path. If the Faraday law is written as $\oint \mathbf{E}\cdot d\mathbf{s} = -\dot{\Phi}$, instead of $\mathcal{E} = -\dot{\Phi}$, it has the same form as Ampere's law, $\oint \mathbf{B}\cdot d\mathbf{s} = \mu_0 I$, and both laws are simply the integral forms of the Maxwell equations, curl $\mathbf{E} = -\dot{\mathbf{B}}$, curl $\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$.

IT scarcely seems necessary to point out the importance, even to an elementary student, of a clear understanding of the nature of the induced electric field associated with a time-varying magnetic field. In most introductory courses, however, we emphasize instead the more complicated concept of induced electromotive force; that is, of the line integral of the induced electric field around a closed path, and the nature and even the existence of the induced electric field, itself, remains a mystery to most students. Three references serve as illustrations.

In a review[1] by Hugh C. Wolfe of a recent text in Electricity and Magnetism,[2] there appears the following remark: "The author suggests that the student who is baffled by induced emf in a wire looped around a uniformly wound toroid in which the current varies—because B is zero where the wire is, no matter how the current varies—may feel better if he knows that the vector potential is not zero and its time derivative gives the induced electric field."

We will all agree that many students are baffled by this question, particularly, if they have been brought up to believe that every induced emf results from the "cutting" of lines of induction; but, it is small solace to a beginning student to be told that he has only to calculate the vector potential, and that its time derivative will give the induced electric field.

Also, in an article[3] by Vladislav Bevc, on "The Electromagnetic field of a Ferromagnetic Transformer," the author says: "The emf induced in a loop around the transformer core can be calculated by Faraday's law. Now, if a student that has been presented with such a model wishes to calculate the electric field $\mathbf{E}$ at a point just outside the core (where the windings can be placed) using Maxwell's equations, he is faced with the dilemma of how to use the field equations where no magnetic induction exists···. These difficulties are resolved if we recall the fact that dynamic electromagnetic fields must be solutions of the wave equation derived from Maxwell's equations."

[1] Hugh C. Wolfe, Am. J. Phys. 28, 681 (1960).
[2] William T. Scott, *The Physics of Electricity and Magnetism* (John Wiley & Sons, Inc., New York, 1959).
[3] Vladislav Bevc, Am. J. Phys. 28, 637 (1960).