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the points where the branch is connected, and in Fig. 6 it is greater. While the potential difference between the junctions is still 5.0 volts in each case, the secondary emf was 4.0 volts in Fig. 5 and 6.0 volts in Fig. 6. In the first case it is evident that a current, small relative to that in the main line, flows clockwise through the branch charging the secondary source of emf. In Fig. 6 current from the primary source flows only in the main line while the secondary emf discharges by circulating a current counterclockwise in the branch.

Diagrams such as these are helpful in illustrating for students the abstract concept of potential and can well be employed in the laboratory as variants on the usual study performed by the students themselves.

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[2] B. Brinker, Am. J. Phys. 18, 318(T) (1950).

Some Exact Solutions of the Time-Dependent Schroedinger Equation

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Exact solutions of the time-dependent wave equation are given for the one-dimensional motion of a particle acted on by an external force and for a forced oscillator. The solutions chosen correspond to the same time dependence for the classical momentum and for the quantum-mechanical space average of the momentum.

INTRODUCTION

If a system is subject to some external disturbance, the quantum-mechanical behavior is described by the time-dependent Schroedinger equation[1]

$$H\psi = i\hbar(\partial\psi/\partial t). \tag{1}$$

In general, this equation can be solved only by successive approximations; however, there are instances in which exact solutions can be obtained. For simplicity, only the one-dimensional motion of a single particle is considered. The cases discussed in this paper are: (A) particle subject to a time-dependent force and (B) harmonic oscillator with a sinusoidal driving force.

A. PARTICLE ACTED ON BY A TIME-DEPENDENT FORCE

If a particle is acted on by a force which is an arbitrary function of the time but is independent of position, the Hamiltonian is

$$H = (p^2/2m) - xF, \tag{2}$$

where F is the force. This choice of the Hamiltonian is easily verified from Hamilton's equations,

$$dx/dt = \partial H/\partial p = p/m, \tag{3}$$

$$dp/dt = -(\partial H/\partial x) = F. \tag{4}$$

If Eq. (3) is differentiated again, the result is Newton's second law,

$$md^2x/dt^2 = F. \tag{5}$$

The classical expression for the momentum is

$$p = mdx/dt = \int F dt + \text{const.} \tag{6}$$

The time-dependent Schroedinger equation is obtained by substituting $-i\hbar(\partial/\partial x)$ for p in Eq. (2). For this case the quantum-mechanical equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - xF\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{7}$$

In order to solve Eq. (7) let

$$\psi = e^{ikx}T, \tag{8}$$

[1] David Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1951), p. 192.

where k and T are functions of the time. The substitution of Eq. (6) into Eq. (7) leads to

$$(\hbar^2k^2/2m)T-xFT=-(dk/dt)\hbar xT+i\hbar dT/dt. \quad (9)$$

This equation must hold for all x and t. This is possible only if

$$dk/dt=F/\hbar, \quad (10)$$

and

$$dT/dt=-(i\hbar k^2/2m)T. \quad (11)$$

The integration of these equations is straightforward if the time dependence of F is known:

$$k=(1/\hbar)\int Fdt+C, \quad (12)$$

$$\ln T=-(i\hbar/2m)\int k^2dt, \quad (13)$$

where C is a constant.

The function e^{ikx} represents a state in which the momentum and kinetic energy are known with complete certainty since it is an eigenfunction for both the momentum operator $-i\hbar(\partial/\partial x)$ and the kinetic-energy operator $(-\hbar^2/2m)(\partial^2/\partial x^2)$. The function T is not acted on by these operators. The momentum and kinetic energy are functions of time through the time dependence of k. The position of the particle has an infinite uncertainty for the state function $e^{ikx}T$.

The observable momentum $\bar{p}$ is given by the equation $-i\hbar(\partial/\partial x)(e^{ikx})=\bar{p}e^{ikx}$:

$$\bar{p}=k\hbar=\int Fdt+\text{const.} \quad (14)$$

This expression is seen to be in agreement with the classical result, Eq. (6).

It is possible to form a wave pocket in the following way.[2] A general solution is

$$\Psi(x,t)=\int_{-\infty}^{\infty}G(C)e^{ikx}T(C)dC. \quad (15)$$

$G(C)$ can be determined from the Fourier transform of Ψ. If Ψ is specified at the time $t=0$ and T is taken to be unity at $t=0$,

$$G(C)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx. \quad (16)$$

[2] Margenau and Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1943), p. 380.

A particular example is the constant force. Consider a gravitational force acting in the negative x direction; $F=-mg$. We obtain

$$k=-(mgt/\hbar)+C,$$
$$\ln T=-\frac{i\hbar}{2m}\left(\frac{m^2g^2}{3\hbar^2}t^3-\frac{mg}{\hbar}Ct^2+C^2t\right). \quad (17)$$

The state function is

$$\psi(x,t)=e^{i[-(mgt/\hbar)+C]x}T. \quad (18)$$

The momentum is

$$\bar{p}=k\hbar=-mgt+C\hbar. \quad (19)$$

This result has a simple physical interpretation. $C\hbar$ is the initial momentum at $t=0$. The effect of the gravitational force is to cause the momentum to change linearly with time.

B. HARMONIC OSCILLATOR WITH A SINUSOIDAL DRIVING FORCE

The Hamiltonian is

$$H=(p^2/2m)+\tfrac{1}{2}m\omega_0^2x^2-F_0x\sin\omega t, \quad (20)$$

where ω_0 is the angular frequency of the free oscillator, and ω is the angular frequency of the driving force. F_0 is the amplitude of the driving force. Any complications due to resonance will be avoided by assuming $\omega\neq\omega_0$. The Hamiltonian given in Eq. (20) is a proper choice if Hamilton's equations lead to the expected equation of motion,

$$dx/dt=\partial H/\partial p=p/m,$$
$$dp/dt=-\partial H/\partial x=-m\omega_0^2x+F_0\sin\omega t=md^2x/dt^2. \quad (21)$$

Equation (21) is the well-known formula for an oscillator acted on by a sinusoidal driving force.

The classical expression for the momentum is

$$m\frac{dx}{dt}=\alpha\sin\omega_0t+\beta\cos\omega_0t-\frac{F_0\omega\cos\omega t}{\omega^2-\omega_0^2}, \quad (22)$$

where α and β are arbitrary constants determined by the initial conditions. The Schroedinger equation for the forced oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+(\tfrac{1}{2}m\omega_0^2x^2-F_0x\sin\omega t)\psi=i\hbar\frac{\partial\psi}{\partial t}. \quad (23)$$

The lowest energy state for the free oscillator is represented by the function $\exp(-\frac{1}{2}a^2x^2)$ with $a^2=m\omega_0/\hbar$. Equation (23) may be solved in a straightforward manner with the analog of the function representing the lowest state of the free oscillator. Assume a solution of the following type:

$$\psi=\exp(-\tfrac{1}{2}a^2x^2)e^{ikx}T, \quad (24)$$

where k and T are functions of the time. The substitution of (24) into (23) yields

$$-(\hbar^2/2m)[(-a^2x+ik)^2-a^2]T + (\tfrac{1}{2}m\omega_0^2x^2-F_0x\sin\omega t)T = i\hbar[i(dk/dt)xT+(dT/dt)]. \quad (25)$$

This equation must be satisfied for all x and T; accordingly,

$$\begin{aligned} dk/dt+i\omega_0k&=(F_0/\hbar)\sin\omega t,\\ i\hbar dT/dt&=(\hbar^2/2m)[(m\omega_0/\hbar)+k^2]T. \end{aligned} \quad (26)$$

The solution of Eq. (26) is

$$k=Ae^{-i\omega_0t}+[F_0/\hbar(\omega^2-\omega_0^2)] \times(i\omega_0\sin\omega t-\omega\cos\omega t). \quad (27)$$

A is an arbitrary constant and may be complex. The expression for $\ln T$ consists of a sum of exponentials but is not given since it does not need to be known to obtain the momentum or kinetic energy.

The momentum operator $-i\hbar(\partial/\partial x)$ yields

$$-i\hbar(d/dx)\exp(ikx-\tfrac{1}{2}a^2x^2) =\hbar(k+ia^2x)\exp(ikx-\tfrac{1}{2}a^2x^2).$$

The space average of this operator is

$$\frac{\hbar\int_{-\infty}^{\infty}(k+ia^2x)\exp[i(k-k^*)x-a^2x^2]dx}{\int_{-\infty}^{\infty}\exp[i(k-k^*)x-a^2x^2]dx} =\tfrac{1}{2}\hbar(k+k^*). \quad (28)$$

If A in Eq. (27) is written in terms of its real part A_r and imaginary part A_i, the space average of the momentum is

$$\bar{p}=\hbar(A_r\cos\omega_0t+A_i\sin\omega_0t) -[F_0\omega\cos\omega t/(\omega^2-\omega_0^2)]. \quad (29)$$

The quantum-mechanical expression for the momentum has the same form as the classical expression.

Simple Derivation of the Clebsch-Gordan Coefficients

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The three-dimensional angular momentum operator **J** takes a simple form when the operand is a scalar function $f(\xi,\eta)$ of the two complex variables ξ, η; so do the angular momentum eigenstates $f_m{}^j(\xi,\eta)$. Exploitation of this circumstance leads to a simple derivation of Racah's formula for the Clebsch-Gordan (vector addition) coefficients.

I. ANGULAR MOMENTUM OPERATORS AND EIGENSTATES

THE essential property of the angular momentum operators J_x, J_y, J_z is their commutation rule $\mathbf{J}\times\mathbf{J}=i\mathbf{J}$. This property is faithfully reproduced if we take[1]

$$J_x=-\frac{1}{2}\left(\eta\frac{\partial}{\partial\xi}+\xi\frac{\partial}{\partial\eta}\right),\quad J_y=\frac{i}{2}\left(\eta\frac{\partial}{\partial\xi}-\xi\frac{\partial}{\partial\eta}\right), \quad J_z=-\frac{1}{2}\left(\xi\frac{\partial}{\partial\xi}-\eta\frac{\partial}{\partial\eta}\right). \quad (1)$$

The operands of **J** here are understood to be scalar functions $f(\xi,\eta)$ of the two complex

[1] It is a familiar fact that unitary unimodular mappings of complex two dimensional space can be made to correspond to rotations of real three-dimensional space [see for example H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1950) pp. 109–118]. The form (1) for J_i can be inferred from the change $df=-i\epsilon J_if$ when f is subjected to the mapping corresponding to an infinitesimal rotation ϵ about the ith (x, y or z) coordinate axis.