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Henry Zatzkis

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Thomson Atom

Henry Zatzkis
Newark College of Engineering, Newark, New Jersey
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The historical development leading up to the Rutherford-Bohr model of the atom is briefly surveyed. The atomic model, suggested by J. J. Thomson before Rutherford's model, is investigated. The quantization shows that the bound states bear a marked resemblance to those of the hydrogen atom. However, the states are in general nondegenerate. Thus, the Thomson atom could not have survived the spectroscopic evidence even if no Rutherford scattering experiments had been performed.

HISTORICAL BACKGROUND

DURING the nineteenth century one of the principal goals of theoretical physics was the construction of a mechanical ether theory. This ether should not only explain the propagation of light, but also, as a by-product, give an account of the atomistic structure of matter. Kelvin was perhaps the main protagonist of these endeavors. Helmholtz', C. A. Bjerknes', and his own hydrodynamical researches led him to the concept of the ether as an ideal fluid and the atoms as vortices embedded in it. The indestructibility of the vortices would account for the conservation of matter, and the hydrodynamical actions between them would explain the interatomic forces. The atoms thus consisted of the same substance as the ether but were at the same time different from it in structure, just as the knot in a string is different from the rest of the string but still of the same material.

With the advent of Faraday's and Maxwell's electrodynamics, the mechanical models were gradually discarded. Instead, the aim now became to explain mechanics in terms of electrodynamics. Probably the first atomic model, held together by electric forces exclusively, was suggested by J. J. Thomson. He developed his ideas on this subject in two papers in the *Philosophical Magazine*, in December, 1903, and March, 1904. The basic assumption was that the atom consisted of a homogeneous, positively charged sphere in which were imbedded smaller negative charges, the whole system being electrically neutral. Thomson calculated the possible arrangements and motions of the negative charges which rendered the configurations stable. Thus Thomson needed no additional hypotheses to explain why electromagnetic radiation of the accelerated charges inside the positive nucleus would not lead eventually to a collapse of such an atomic system.

Subsequently Thomson's students, H. Geiger and E. Marsden,[1] carried out a series of careful measurements on the scattering of α particles in matter. In 1910 Thomson attempted to give a theoretical explanation of their findings.[2]

He[3] assumed "multiple scattering" to be the cause of the observed results, i.e., the deviations of the α particles from their original paths were assumed to be the resultants of a large number of small random deviations, the average deviation due to passing through one atom being only a rather small fraction of a degree.

The observations, however, showed also large deviations up to angles of 150° away from the incident direction. Although the number of these deviations was small, it was much too large to be explained solely on the basis of pure randomness. This anomaly induced Rutherford to propose his theory of "single scattering"[4] according to which the large deviations are due to single collisions with atoms and not to the summation of a large number of small deviations. In order to account for these large deviations, an intense field in the atom was required, and this in turn led Rutherford to propose his well-known theory of the planetary atom.

[1] Proc. Roy. Soc. (London) **A82**, 495 (1909), and **A83**, 492 (1910).
[2] J. J. Thomson, Proc. Cambridge Phil. Soc. **15**, 465 (1910).
[3] See, e.g., A. Sommerfeld, *Atombau and Spektrallinien* (Friedrich Vieweg and Sohn, Braunschweig, Germany, 1919), first edition, p. 62; or H. A. Wilson, *Modern Physics* (Blackie and Son Limited, London, 1941), third edition, p. 221.
[4] Phil. Mag. **21**, 669 (1911).

About the electrodynamic stability of his atomic model, Rutherford remained silent. It is conceivable that Rutherford's model would not have gained such ready acceptance had it not been for Bohr's synthesis of the planetary features and Planck's quantum theory. The brilliant success in the explanation of the spectrum of the hydrogen atom removed all doubts.

It was this particular aspect that prompted the author to investigate the spectrum of the simplest type of a Thomson atom (one single negative charge imbedded in the positive nucleus) by quantizing it. It turns out that the lowest energy levels of the hydrogen atom are missing, and furthermore that all states are nondegenerate and thus unable to explain the observed multiplet structure of the spectral lines. Thus, even in the absence of any scattering experiments, spectroscopic evidence alone would have been sufficient to rule out a structure of the atom along the lines of ideas suggested by Thomson. Yet, Thomson's model was the immediate forerunner of Rutherford's model, and in fact inspired it. Thus, it will retain an important position in the history of the evolution of our present physical concepts. As far as the writer is aware, no previous attempt has been made to quantize the Thomson atom.

MATHEMATICAL TREATMENT

Before Rutherford proposed his atomic model, J. J. Thomson, as noted above, made the suggestion that the atom may consist of a uniformly charged positive sphere with an equal negative point charge oscillating in and about the positive sphere. It is interesting to discover that the Thomson atom could not have survived the spectroscopic evidence even if no scattering experiments had been performed. Some of the principal features of the Thomson model can be demonstrated by solving the Schrodinger equation for a potential well corresponding to this view of the atom.

If the charges are denoted by $\pm e$ and the radius of the nucleus by R, the potential function becomes in this case

$$V=\frac{e^2}{2R^3}r^2-\frac{3e^2}{2R} \quad r\leq R, \qquad V=-e^2/r \quad r\geq R, \tag{1}$$

where r is the radial distance from the center of the nucleus.

Schrodinger Equation and its Solution

When the Schrodinger equation is written down in spherical coordinates and the angular momentum operator squared is replaced by $l(l+1)\hbar$, we obtain, for the radial wave equation,

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\frac{2M}{\hbar^2}[E-V(r)]f-\frac{l(l+1)}{r^2}f=0, \tag{2}$$

where M is the mass of the negative charge and $l=0, 1, 2, 3\cdots$. The cases $l=0$ and $l\neq 0$ have to be considered separately since they behave quite differently.

Case $l=0$

For the case $l=0$ the differential equation reduces to

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\frac{2M}{\hbar^2}[E-V(r)]f=0. \tag{3}$$

In the region $0<r<R$, this equation reads

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\left[\frac{2ME}{\hbar^2}+\frac{3Me^2}{\hbar^2R}-\frac{Me^2}{\hbar^2R^3}r^2\right]f=0. \tag{4}$$

If the following definitions are made,

$$\begin{aligned} &g(r)=rf(r),\\ &E'=E+(3e^2/R),\\ &\omega^2=e^2/MR^3=\text{the classical angular frequency in the region where the potential is harmonic,}\end{aligned} \tag{5}$$

we obtain

$$\frac{d^2g}{dr^2}+\left[\frac{2ME'}{\hbar^2}-\frac{M^2\omega^2r^2}{\hbar^2}\right]g=0. \tag{6}$$

This equation is identical with that of the linear harmonic oscillator. Its general solution[5] can be written down at once and we obtain

$$f(r)=\frac{\exp(-\lambda r^2/2)}{r}\times[C_1F(a;\tfrac{1}{2};\lambda r^2)+C_2\lambda^{\frac{1}{2}}rF(a+\tfrac{1}{2};\tfrac{3}{2};\lambda r^2)],$$

[5] See, e.g., S. Flugge and H. Marshall, *Rechenmethoden der Quantentheorie* (Springer-Verlag, Berlin, 1952), second edition, Part 1, p. 62.

where

$$\lambda = M\omega/\hbar,$$

$$a = \frac{1}{4} - \frac{E'}{2\hbar\omega},$$

$$F(a;b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + + \cdots$$

$$= \text{the hyperconfluent function.} \qquad (7)$$

The constants C_1 and C_2 are arbitrary constants of integration. Despite the singularity at the origin the function is square integrable and is therefore admissible.

In the region $R \leq r \leq \infty$ the radial differential equation is

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \frac{2M}{\hbar^2}\left[E + \frac{e^2}{r}\right]f = 0. \qquad (8)$$

The only normalizable solution[6] is

$$f(r) = e^{-\gamma r}F(1-\tau;2;2\gamma r)$$
$$\tau = Me^2/\gamma\hbar \qquad (9)$$
$$\gamma^2 = -2ME/\hbar^2, \quad \gamma > 0.$$

The other linearly independent solution tends to ∞ as $r \to \infty$ in any case and cannot be normalized. Even this solution above will diverge asymptotically as the function $e^{\gamma r}$ unless the function $F(1-\tau;2;2\gamma r)$ becomes a polynomial and this will be the case if

$$1 - \tau = -n_r \quad (n_r = 0, 1, 2, 3 \cdots). \qquad (10)$$

The above condition yields the quantized energy values

$$E_n = -Me^4/2\hbar^2 n^2,$$

where $n = n_r + 1$ and

$$f = e^{-\gamma r}F(1-n;2;2\gamma r). \qquad (11)$$

In other words the energy levels are the same as those of the hydrogen atom, except that a finite lower number is cut off since E_n must satisfy the inequality

$$-\frac{3}{2}\frac{e^2}{R} \leq E_n. \qquad (12)$$

[6] See reference 5, p. 108.

If the expression for E_n is substituted in this equality one obtains

$$n \geq \frac{e}{\hbar}(MR/3)^{\frac{1}{2}}. \qquad (13)$$

If R is chosen of the order of 10^{-12} the left side of the inequality is of the order 10^{-2} so that all integer values are allowed; if R is of the order 10^{-8} the left side is of order unity so that the number of levels cut off in this case is small.

We must finally determine the constants C_1 and C_2 from the two boundary conditions so that the radial function and its derivative must be continuous at $r = R$. In this connection the property of the hyperconfluent function

$$(d/dz)F(a;b;z) = (a/b)F(a+1;b+1;z) \qquad (14)$$

will be very helpful. If the following abbreviations are introduced,

$$\begin{aligned} A &\equiv F(a;\tfrac{1}{2};\lambda R^2), \\ B &\equiv F(a+\tfrac{1}{2};\tfrac{3}{2};\lambda R^2), \\ D &\equiv F(1-n;2;2\gamma R), \\ N &\equiv F(a+1;\tfrac{3}{2};\lambda R^2), \\ S &\equiv F(a+\tfrac{3}{2};\tfrac{5}{2};\lambda R^2), \\ T &\equiv F(2-n;3;2\gamma R), \end{aligned} \qquad (15)$$

the boundary conditions then can be expressed as

$$\exp(-\lambda R^2/2)\left[C_1\frac{A}{R} + C_2\lambda^{\frac{1}{2}}B\right] = e^{-\gamma R}D$$

and

$$\exp\left(-\frac{\lambda R^2}{2}\right)\left\{C_1\left[4\lambda a N - A\left(\frac{1}{R^2}+\lambda\right)\right]\right.$$
$$\left. + C_2\lambda^{\frac{3}{2}}R\left(2S\frac{(2a+1)}{3} - B\right)\right\}$$
$$= \gamma e^{-\gamma R}[(1-n)T - D]. \qquad (16)$$

These equations can now be solved for C_1 and C_2 and we will have thus constructed the unnormalized wave functions for the case $l=0$. To normalize these functions would require numerical values for the hyperconfluent functions. Since these are not tabulated, only a numerical integration would yield the normalization factor. The solution for C_1 and C_2 can be easily written down as

$$C_1=U/V, \quad C_2=W/V, \tag{17}$$

where

$$U=\lambda^{\frac{1}{2}}\exp\left(+\frac{\lambda R^2}{2}-\gamma R\right)\times\begin{vmatrix} D, & B \\ \gamma[(1-n)T-D], & \lambda R\left(\frac{2(2a+1)}{3}S-B\right)\end{vmatrix} \tag{18}$$

$$W=\exp\left(+\frac{\lambda R^2}{2}-\gamma R\right)\times\begin{vmatrix} A/R, & D \\ 4\lambda aN-A\left(\frac{1}{R^2}+\lambda\right), & \gamma[(1-n)T-D\end{vmatrix} \tag{19}$$

$$V=\lambda^{\frac{1}{2}}\left[2AS\lambda\frac{(2a+1)}{3}-4B\lambda aN+\frac{AB}{R^2}\right]. \tag{20}$$

Case $l\neq 0$

In this case the differential equation for the region $0\leq r\leq R$ is

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\left[\frac{2ME'}{\hbar^2}-\frac{M^2\omega^2r^2}{\hbar^2}-\frac{l(l+1)}{r^2}\right]f=0; \tag{21}$$

the two independent solutions behave near $r=0$ like r^l and $r^{-(l+1)}$. Only the first solution can be normalized. The complete solution for the region $0\leq r\leq R$ can be written

$$f(r)=\exp\left(-\frac{\lambda r^2}{2}\right)\times\left[C_1r^{-(l+1)}F\left(\frac{-l+\frac{1}{2}-\mu}{2};\tfrac{1}{2}-l;\lambda r^2\right)+C_2\lambda^{\frac{1}{2}}r^lF\left(\frac{l+\frac{3}{2}-\mu}{2};l+\tfrac{3}{2};\lambda r^2\right)\right],$$

where

$$\mu=E'/\hbar\omega. \tag{22}$$

The first term in this expression becomes so strongly singular near $r=0$ that it is no longer square integrable. Therefore C_1 must be put equal to zero. But then we have only one constant C_1 at our disposal, and this is not sufficient to match boundary conditions at $r=R$. From this we must conclude that no wave function with $l\neq 0$ exists which can satisfy the necessary conditions. Since the term $l(l+1)/r^2$ in the differential equation corresponds classically to a centrifugal force, one can see that the negative charge can only oscillate linearly through the center. Any centrifugal force will make the system unstable. The Thomson atom would show no Zeeman splitting in a magnetic field and would lead to incorrect occupation numbers for the main energy levels in the atoms.

CONCLUSION

The Thomson atom has the same energy states as that of the hydrogen atom except that the lowest state may not be given by $n=1$ and depends on the radius of the spherical positive charge distribution. Furthermore all states are nondegenerate since no states with $l\neq 0$ are allowed. The description of the nucleus according to Thomson is very incomplete so much so that even the point nucleus gives a more valid account of the extra nuclear behavior of the electron.

The atomic physicist has had to resign himself to the fact that his science is but a link in the infinite chain of man's argument with nature, and that it cannot simply speak of nature "in itself." *Science always presupposes the existence of man and, as Bohr has said, we must become conscious of the fact that we are not merely observers but also actors on the stage of life.*—W. HEISENBERG, *The Physicist's Conception of Nature.*