AMERICAN JOURNAL of PHYSICS

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Citation: American Journal of Physics **23**, 111 (1955); doi: 10.1119/1.1933912
View online: http://dx.doi.org/10.1119/1.1933912
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/23/2?ver=pdfcov
Published by the American Association of Physics Teachers

Rectangular Potential Well Problem in Quantum Mechanics

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(Received June 3, 1954)

The one-dimensional rectangular potential well problem is one of the standard examples used in courses to illustrate quantum-mechanical properties. It has also been of value in the study of nuclear energy states. However, it is necessary to resort to graphical or numerical procedures at the last if one desires explicit values for the energy levels. It is the purpose of this article to point out a simpler and more perspicuous method of graphical solution than those used at the present time.

WE shall use the commonly accepted notation for the Schrödinger equation in this discussion. The potential is represented by

$$V(x)=\begin{cases}-V_0 \text{ for } (-a\leq x\leq +a)\\ 0 \text{ otherwise.}\end{cases}$$

It is convenient to set the energy E equal to $-\epsilon$, since only the bound states are to be considered. Note that the range of ϵ is $(0\leq\epsilon\leq V_0)$. The Schrödinger equation then assumes the form

$$\left.\begin{aligned}&\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}(V_0-\epsilon)\psi=0 \quad (-a\leq x\leq +a)\\ &\frac{d^2\psi}{dx^2}-\frac{2m}{\hbar^2}\epsilon\psi=0 \quad (|x|\geq a).\end{aligned}\right\} \quad (1)$$

The solution of these equations is readily obtained and will not be discussed here. The imposition of continuity requirements on the solution leads to the equations

(even solutions)

$$\left[\frac{V_0-\epsilon}{\epsilon}\right]^{\frac{1}{2}}\tan\left[\frac{2ma^2}{\hbar^2}(V_0-\epsilon)\right]^{\frac{1}{2}}=+1 \quad (2a)$$

(odd solutions)

$$\left[\frac{V_0-\epsilon}{\epsilon}\right]^{\frac{1}{2}}\cot\left[\frac{2ma^2}{\hbar^2}(V_0-\epsilon)\right]^{\frac{1}{2}}=-1. \quad (2b)$$

These equations must ordinarily be handled graphically or numerically, if one wishes in any particular case to obtain explicit values for the energy levels.

GRAPHICAL SOLUTION

Both Bohm and Schiff[1] give methods (not the same) for extracting the roots of Eqs. (2). These references should be consulted for details. We propose to add here another method to the list. For this purpose, it is convenient to transform Eqs. (2) somewhat. Let

$$\alpha=a\left[\frac{2mV_0}{\hbar^2}\right]^{\frac{1}{2}}, \quad \eta=\alpha[1-\epsilon/V_0]^{\frac{1}{2}}.$$

Then, using well-known trigonometric identities,

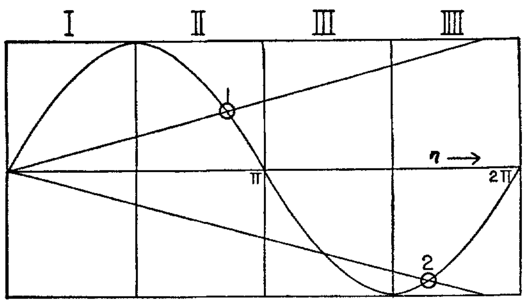

FIG. 1. Odd Solutions. Intersections 1 and 2 correspond to true roots of Eq. (3b); the intersection in quadrant III does not correspond to a true root.

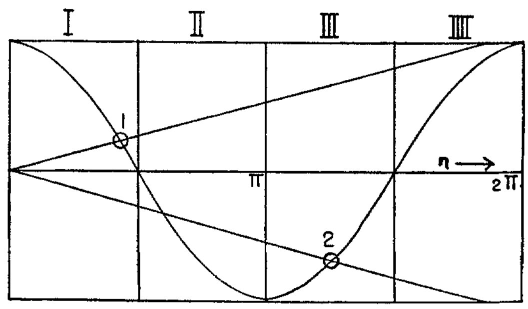

FIG. 2. Even Solutions. Intersections 1 and 2 correspond to true roots of Eq. (3a); the intersection in quadrant II does not correspond to a true root.

[1] See, for example, D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., New York, 1951), p. 247–255; or L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 36–40.

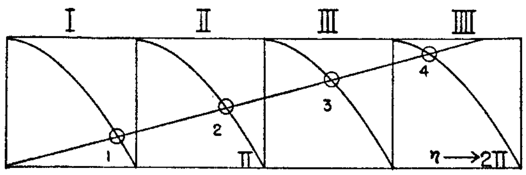

FIG. 3. All Solutions. Intersections 1 and 3 correspond to even roots; intersections 2 and 4 correspond to odd ones.

we obtain

$$\text{(even solutions)}\ \cos\eta = \pm\eta/\alpha \tag{3a}$$

$$\text{(odd solutions)}\ \sin\eta = \pm\eta/\alpha. \tag{3b}$$

For the even solutions the procedure is to plot $y_1 = \cos\eta$ *vs* η. Then plot $y_2 = \pm\eta/\alpha$ *vs* η (two straight lines). The roots may be found at the intersections. One restriction must be observed: from Eq. (2a) it is clear that real roots will occur only in the 1st, 3rd, 5th, etc., quadrants, in which regions the cosine is, respectively, (+), (−), (+), etc. Accordingly in the 1st, 5th, 9th, etc., quadrants, the roots are taken using $y_2 = +\eta/\alpha$; in the 3rd, 7th, etc., use $y_2 = -\eta/\alpha$. For the odd solutions, plot $y_1 = \sin\eta$ *vs* η and $y_2 = \pm\eta/\alpha$. Again the intersections give the roots. A similar observation to that given above requires that we use $y_2 = +\eta/\alpha$ in the 2nd, 6th, etc., quadrants and $y_2 = -\eta/\alpha$ in the 4th, 8th, etc., quadrants. The remaining quadrants in each case do not yield true roots. Note that since $(0 \leq \epsilon \leq V_0)$, it follows that $(0 \leq \eta \leq \alpha)$ and $(0 \leq y_2 \leq +1)$ or $(-1 \leq y_2 \leq 0)$. The foregoing analysis results in an elegantly simple graphical procedure. At the risk of seeming repetitious we

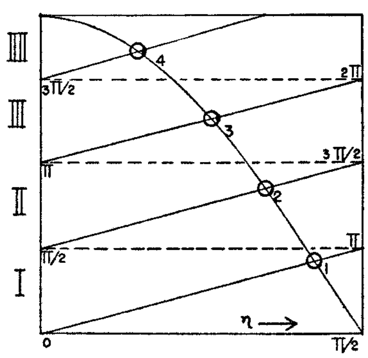

FIG. 4. All Solutions. This is the same as Fig. 3 with all the quadrants superposed. The numbers and labels are unchanged.

summarize these results.

$$\text{(even)}: \text{Plot } y_1 = \cos\eta$$
$$y_2 = \pm\eta/\alpha$$

(straight lines from the origin to the points $(\eta = \alpha,\ y_2 = +1)$, $(\eta = \alpha,\ y_2 = -1)$. Now observe that

$0 \leq \eta \leq \pi/2$: use (+) sign,
$\pi/2 \leq \eta \leq \pi$: no solution,
$\pi \leq \eta \leq 3\pi/2$: use (−) sign,
$3\pi/2 \leq \eta \leq 2\pi$: no solution,
etc.

$$\text{(odd)}: \text{Plot } y_1 = \sin\eta$$
$$y_2 = \pm\eta/\alpha.$$

Observe that

$0 \leq \eta \leq \pi/2$: no solution,
$\pi/2 \leq \eta \leq \pi$: use (+) sign,
$\pi \leq \eta \leq 3\pi/2$: no solution,
$3\pi/2 \leq \eta \leq 2\pi$: use (−) sign,
etc.

The method is illustrated in Figs. 1 and 2.

It is obvious from the drawings that further simplication is possible. We observe that separate sine and cosine curves are not essential. The straight lines of negative slope are also unnecessary. Accordingly, we combine the two systems as shown in Fig. 3.

A final pruning may be made on Fig. 3, which increases the accuracy of the plot. The repetition of the cosine curve in Fig. 3 can be dispensed with as shown in Fig. 4.

If we let η_k be the value of η measured along the abscissa which corresponds to the kth root, then the energy levels are given by

$$\epsilon_k = V_0\left(1 - \frac{\eta_k^2}{\alpha^2}\right). \tag{4}$$

Alternatively, the energy may be measured from the bottom of the well, in which case

$$\epsilon_k' = V_0 - \epsilon_k = V_0\eta_k^2/\alpha^2 = \hbar^2\eta_k^2/2ma^2. \tag{5}$$

Although we have used four figures in this discussion, it should be noted that only one is required in actual computation. This may be either Fig. 3, Fig. 4 or a suitable hybrid. The advantages of this method over the ones found in most textbooks may be summarized as follows.

1. The functions to be plotted are of considerably simpler character.

2. The cosine plot is the same for all wells, and for particles of different masses. The details of the well and the particle appear only in the constant

$$\alpha = a[2mV_0/\hbar^2]^{\frac{1}{2}},$$

and this affects only the slope of the straight line. This feature makes it easy to discuss the dependence of the energy levels on the well parameters with only one graph.

3. It is easy to make rough sketches of the graphs which will exhibit the essential qualitative features of the problem.

It has been the experience of the author and the students in his classes that the time saved in working typical problems using this procedure is by itself well worth such additional effort as may be required to digest the preliminary analysis on the character of the roots.

The Hydrostatic Paradox: Phases I and II

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(Received January 11, 1954)

The term hydrostatic paradox is likely to suggest the classical experiment of "Pascal's vases." In a demonstration with this apparatus the observer's attention is centered upon the vertical direction, specifically the common vertical depth of liquid in the different branches of the container, with their varying cross sections and volumes. But the term properly has a broader connotation. Fully as striking is the hydrostatic paradox seen in the fact that the horizontal thrust of the water against a dam is independent of the volume of water impounded, and dependent only on its depth at the barrier. To differentiate sharply between these two aspects of hydrostatic paradox, the terms "Phase I" and "Phase II" are proffered. The two aspects are discussed in the paper. A simple projection cell is described for demonstrating Phase II in a direct manner, corresponding to the demonstration of Phase I with Pascal's vases. Considered also is the involvement of hydrostatic paradox with Archimedes' principle, and in capillary tubes.

INTRODUCTION

The horizontal thrust on a dam by the water behind it depends only on its *depth* at the contact face. It does not at all depend on the volume impounded, except as that volume, for a given contour of bed of the reservoir, determines depth at the barrier.

Being paradoxical, this horizontal effect belongs under the general subject of hydrostatic paradox. Although it is due to the behavior of static liquid pressure, it is at the same time different from the paradox as usually demonstrated with "Pascal's vases" (see below), where the effect has to do with the vertical direction.

In demonstrations therefore it seems desirable to particularize as to terms. The writer suggests: for the vertical effect, "Hydrostatic Paradox, Phase I"; and for the horizontal, "Hydrostatic Paradox, Phase II." In each of these situations the effect is independent of horizontal dimension, and of the shape of vertical cross section below the free surface of the static liquid. In both of them the paradoxical character rests on the observer's natural tendency to regard volume, or weight, instead of pressure, which depends directly on depth and only incidentally on volume.

A main purpose in this paper is to describe simple equipment for showing Phase II. However, for a broader consideration of hydrostatic paradox in general, and for drawing a comparison, Phase I will be discussed briefly before Phase II is treated in more detail. Following these discussions we shall consider hydrostatic paradox involved with Archimedes' principle and then with capillarity.